Ley lines in turbulent suspensions?

Michael Wilkinson and John Grant

Mathematics and Statistics,
Open University,
Milton Keynes

Overview

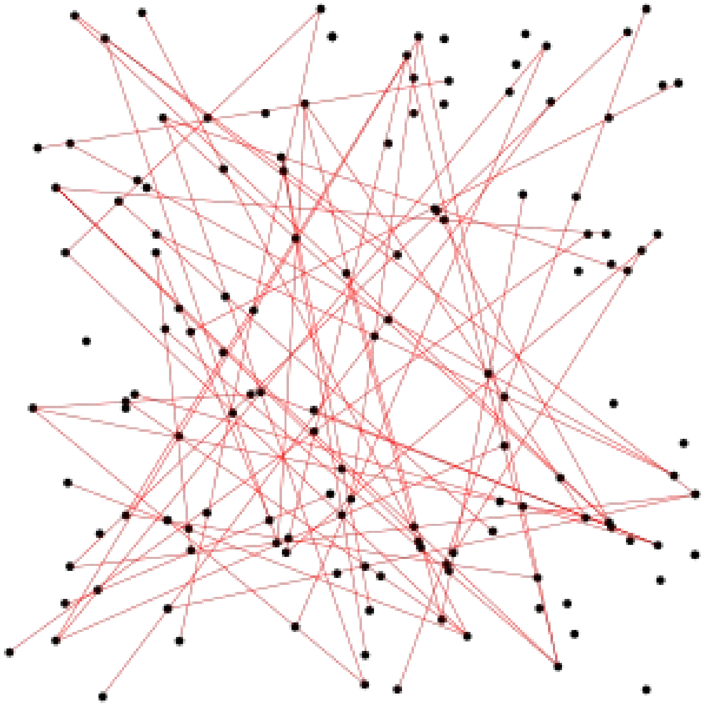
- Particles in turbulent flow exhibit fractal clustering: the mass in a ball of size ϵ is:

$$\mathcal{N}(\epsilon) \sim \epsilon^{D}$$

- What about shapes? How does the distribution of shapes of triangles formed by three particles in ball depend upon its size?
- There is a phase transition: below a critical dimension highly acute triangles become more common.
- Shape questions for fractals are a neglected topic. I believe they will have wide ranging applications.

Ley lines

These are supposed alignments of ancient monuments and/or geographical features, proposed by Alfred Watkins. Later it was realised that near alignments are easily seen in random scatters, e.g.



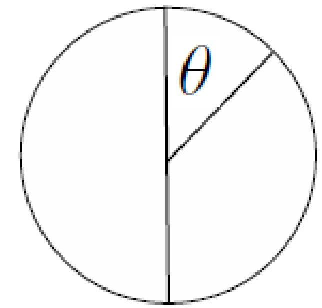
Images: Malvern hills and alignments in a random scatter, from Wikipedia.

D. G. Kendall studied the distribution of triangle shapes drawn from a random scatter, and R. Atkinson showed a set of 'ley lines' based on locations of public telephones.

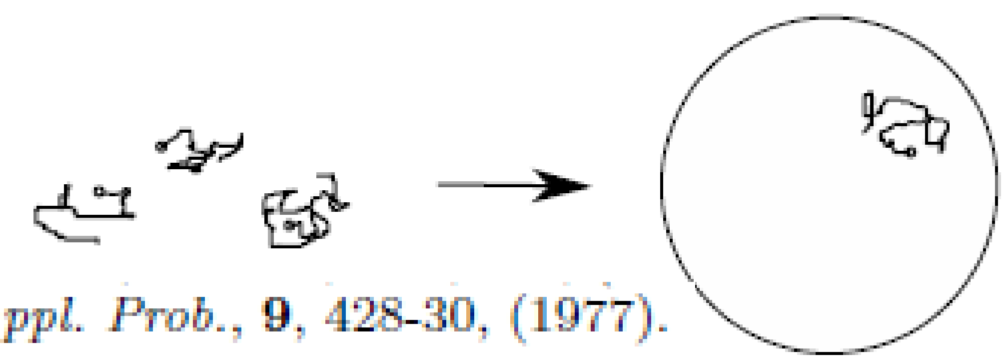
Kendall's sphere

Describing the shape of a triangle requires two parameters. D. G. Kendall used a spherical surface: equilaterals at the poles, co-linear triplets on the equator.

$$(z, \phi)\ , \quad z = \cos\theta \qquad z = \frac{2\mathcal{A}}{\sqrt{3}\mathcal{R}^2}$$



Kendall stated that if three points undergo Brownian motion on the plane, their triangle-shape point undergoes Brownian motion on the sphere.

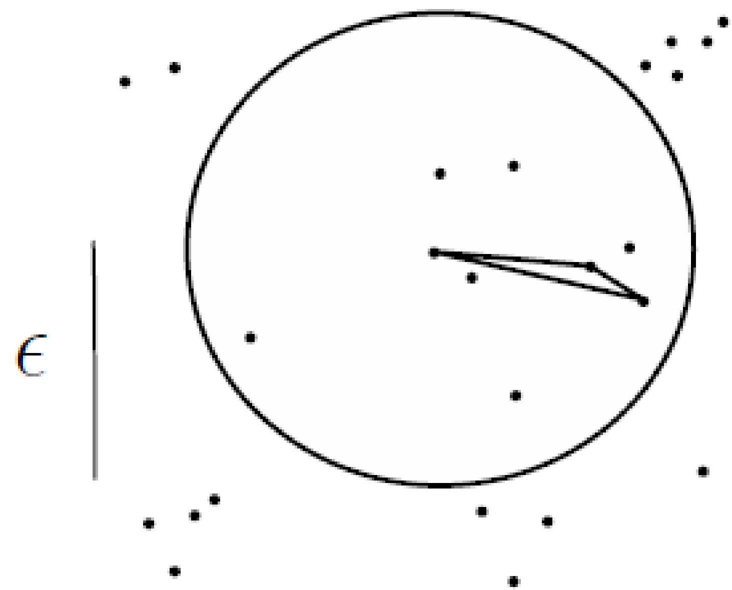


D. G. Kendall, *Adv. Appl. Prob.*, **9**, 428-30, (1977).

This gives the shape distribution for triplets: $P(z) = \frac{1}{2}$

Triangles in fractals

Most human activity is not randomly scattered. A fractal distribution is a better model. Perhaps Kendall should have asked: '*For a fractal distribution of points, how does the triangle shape distribution depend upon the size of the ball?*'



We considered this question for the distribution of acute triangles: do these become more common as the ball shrinks? That is: do ley lines exist in fractals?

A numerical experiment

We used random advection in a compressible flow, two-dimensional, with short correlation time:

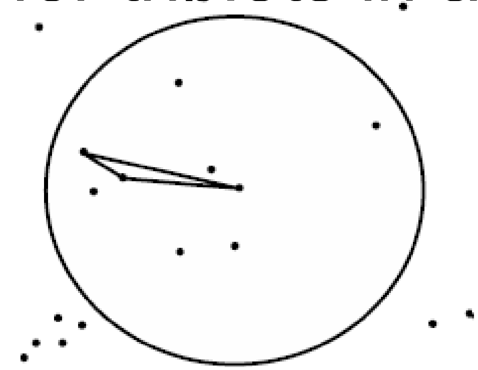
$$\dot{\boldsymbol{r}} = \boldsymbol{u}(\boldsymbol{r}, t) \qquad \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \boldsymbol{u}_n(\boldsymbol{x}_n)\sqrt{\delta t}$$

$$\boldsymbol{u}_n = (\partial_y \psi_n + \beta \partial_x \chi_n, -\partial_x \psi_n + \beta \partial_y \chi_n)$$

We evaluated the shape distribution for triplets in a ball.

$$z = \frac{2\mathcal{A}}{\sqrt{3}\mathcal{R}^2}$$

$$\mathcal{R}^2 = \frac{1}{3}\left[(\delta \boldsymbol{r}_1)^2 + (\delta \boldsymbol{r}_2)^2 + (\delta \boldsymbol{r}_1 - \delta \boldsymbol{r}_2)^2\right]$$



We find the shape distribution is independent of the ball size, until the dimension drops below a critical value. Then we see two power-laws:

$$P(z) \sim z^{-\alpha_1}\,, \quad z \to 0 \qquad P(z) \sim z^{-\alpha_2}\,, \quad 1 \gg z \gg z_c(\epsilon)$$

Phase transition of shape distribution

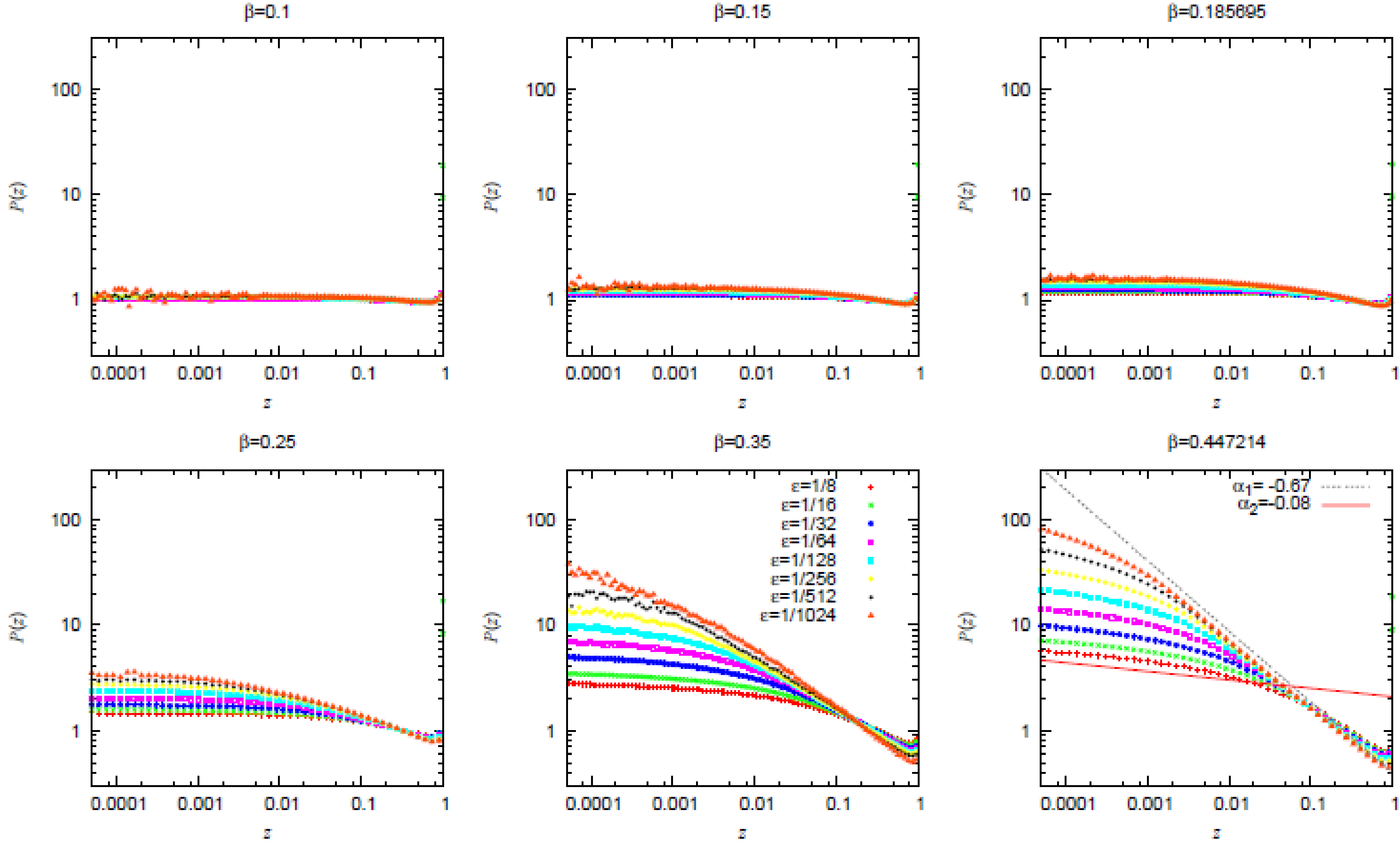
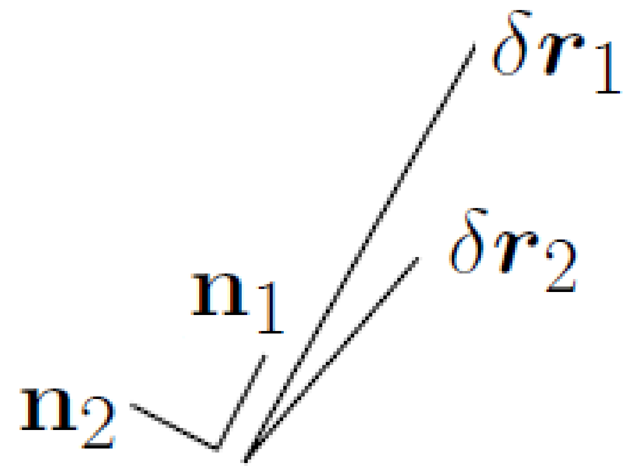


FIG. 1: Probability density $P(z)$ for various compressibilities β: $\beta_c = 1/\sqrt{29} = 0.185\ldots$ is our estimate of the critical compressibility, and $D_2 = 1$ when $\beta = 1/\sqrt{5} = 0.447\ldots$. Straight lines indicate estimates for α_1 and α_2 when $\beta = \beta_c$. Note that $P(z)$ is normalisable ($\alpha_1 > -1$) even at $\beta = 1/\sqrt{5}$, where $D_2 = 1$.

Motion of triangle parameters

Linerarise equation of motion:



$$\delta \dot{\boldsymbol{r}} = \mathbf{A}(t)\delta \boldsymbol{r} \ , \quad A_{ij}(t) = \frac{\partial u_i}{\partial r_j}(\boldsymbol{r}(t), t)$$

Represent shape of triplet of points by three parameters:

$$\delta \boldsymbol{r}_1 = R_1 \mathbf{n}_1 \ , \quad \delta \boldsymbol{r}_2 = R_2(\mathbf{n}_1 + \delta\varphi \mathbf{n}_2)$$

Equations of motion:

$$F_{ij} = \mathbf{n}_i \cdot \mathbf{A}\, \mathbf{n}_j \qquad \dot{\mathbf{n}}_1 \cdot \mathbf{n}_2 = F_{21}(t)$$

$$\frac{\dot{R}_1}{R_1} = F_{11}(t) \ , \quad \frac{\delta\dot{\varphi}}{\delta\varphi} = F_{22}(t) - F_{11}(t)$$

Logarthmic variables

New dynamical variables for triplet of points:

$$X_1 = -\ln \frac{R_1}{\xi} \ , \quad X_2 = -\ln \delta\varphi \ , \quad X_3 = \ln \left(\frac{R_1}{R_2} \right)$$

These have very simple equations of motion, for small acute triangles:

$$\dot{X}_i = \eta_i(t)$$

$$X_1 \gg 0$$

$$X_2 \gg 0$$

Their probability density obeys an advection-diffusion equation:

$$-v_i \partial_i P + \mathcal{D}_{ij} \partial_i \partial_j P = 0$$

$$v_i = \langle \dot{X}_i(t) \rangle$$

$$C_{ij}(t) = \langle [\dot{X}_i(t) - v_i][\dot{X}_j(0) - v_j] \rangle$$

$$\mathcal{D}_{ij} = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t \ C_{ij}(t) \ .$$

Power laws and boundary conditions

Translational invariance implies that advection-diffusion equation has exponential solutions:

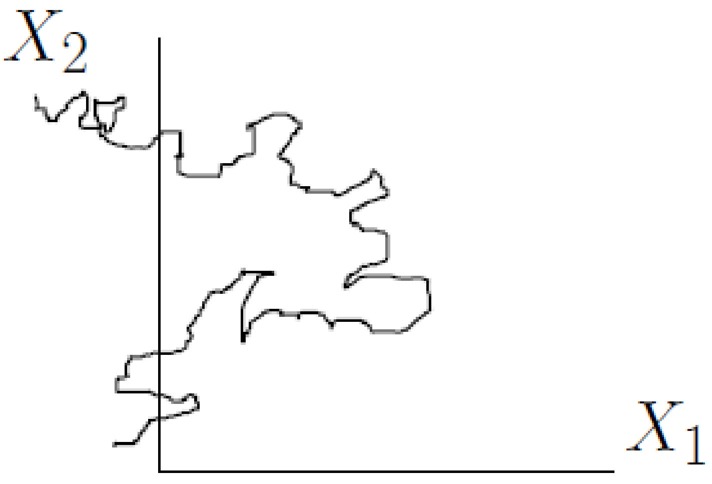
$$P(X_1, X_2, X_3) = \exp(\gamma_1 X_1 + \gamma_2 X_2) P_3(X_3)$$

Implying power-laws in original variables:

$$R_1^{\gamma_1 - 1} \qquad \gamma_1 = 2D_3 \qquad P(z) \sim z^{\alpha} \qquad \alpha = \gamma_2 - 1$$

Points representing triplets enter and leave domain representing small, acute triangles. Source density is

$$J(X_2) = \begin{cases} J_0 \exp(-X_2) & X_2 < 0 \\ 0 & X_2 \geq 0 \end{cases}$$



Point-source solution

For a continuous source, integrate propagator over time:

$$P(\boldsymbol{X}) = K \int_0^\infty \mathrm{d}t \; [4\pi \det(\mathbf{D}) t]^{-d/2} \exp\left[-S(\boldsymbol{X}, t)\right]$$

$$S(\boldsymbol{X}, t) = \frac{1}{4t} (\boldsymbol{X} - \boldsymbol{v}t) \cdot \mathrm{D}^{-1} (\boldsymbol{X} - \boldsymbol{v}t)$$

Stationary point:

$$\frac{\partial S}{\partial t}(\boldsymbol{X}, t^*) = 0 \qquad\qquad \boldsymbol{X} \cdot \mathrm{D}^{-1} \boldsymbol{X} - t^{*2} \boldsymbol{v} \cdot \mathrm{D}^{-1} \boldsymbol{v} = 0$$

Large-deviation function is a tilted cone:

$$P(\boldsymbol{X}) \sim \exp\left[-\Phi(\boldsymbol{X})\right]$$

$$\Phi_0(\boldsymbol{X}) = \frac{1}{2} \left[\sqrt{\boldsymbol{X} \cdot \mathrm{D}^{-1} \boldsymbol{X}} \sqrt{\boldsymbol{v} \cdot \mathrm{D}^{-1} \boldsymbol{v}} - \boldsymbol{X} \cdot \mathrm{D}^{-1} \boldsymbol{v} \right]$$

Distributed source

Triangles injected at the boundary of the diffusion region, at rate:

$$J(X_2) = \begin{cases} J_0 \exp(-X_2) & X_2 < 0 \\ 0 & X_2 \geq 0 \end{cases}$$

Propagator from distributed source:

$$P(X_1, X_2) = \int_0^\infty \mathrm{d}X_0 \;\; \exp(-X_0) \exp[-\Phi_0(X_1, X_2 - X_0)]$$

Perform integration by seeking a stationary point:

$$P(\boldsymbol{X}) = \exp[-\Phi(\boldsymbol{X})]$$

$$\Phi(X_1, X_2) = \Phi_0(X_1, X_2 - X^*) + X^*$$

$$0 = 1 - \frac{\partial \Phi_0}{\partial X_2}(X_1, X_2 - X^*)$$

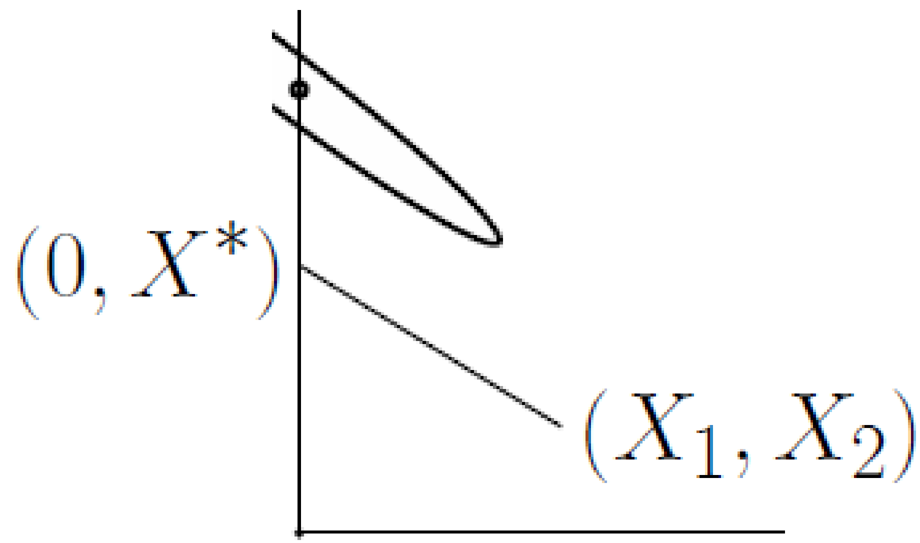
Stationary point obtained from:

$$X^* = X_2 - s(\beta) X_1$$

Origin of the phase transition

Stationary phase point depends upon a slope parameter:

$$X^* = X_2 - s(\beta) X_1$$



- Negative slope: stationary point exists. Small triangles are made by compressing nearly co-linear triplets along their axis.
- Positive slope: no stationary phase point. Small triangles are made by compressing typical-shape triangles in both directions.

Calculations for model flow

Drift velocities, diffusion coefficients:

$$v_1 = -(1-\beta^2)$$
$$v_2 = 2(1+\beta^2)$$

$$\mathcal{D}_{11} = \frac{1}{2}(1+3\beta^2)$$
$$\mathcal{D}_{12} = \mathcal{D}_{21} = -(1+\beta^2)$$
$$\mathcal{D}_{22} = 2(1+\beta^2)$$

Point source exponent:

$$\Phi_0(\boldsymbol{x}) = K\sqrt{X_2^2 + \Lambda(X_1^2 + X_1 X_2)} - X_1 - X_2$$

$$K = \frac{1+3\beta^2}{2\beta\sqrt{2(1+\beta^2)}}\,, \quad \Lambda = \frac{4(1+\beta^2)}{1+3\beta^2}$$

Slope function:

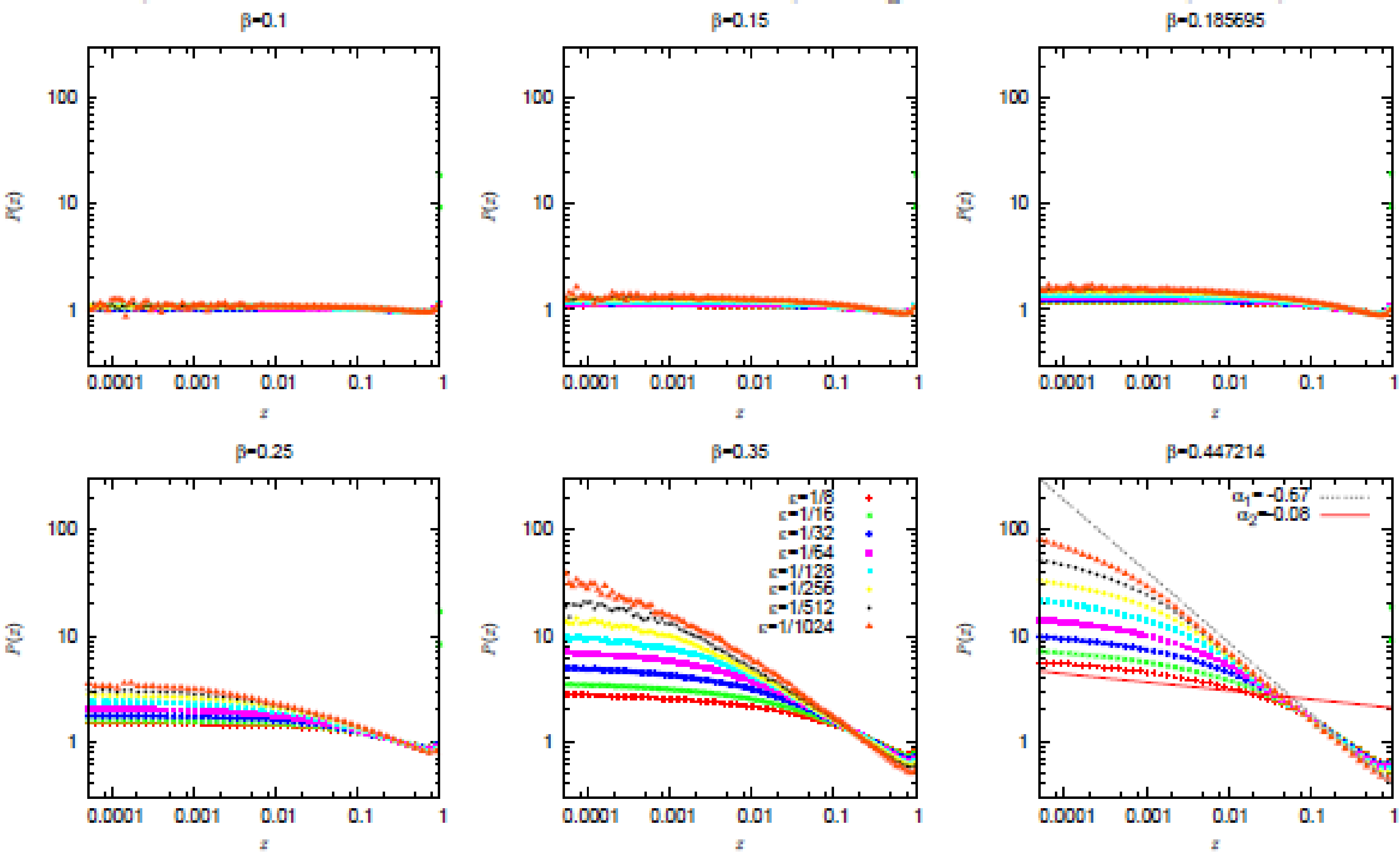
$$s(\beta) = \frac{2(1+\beta^2)}{1+3\beta^2}\left[\frac{8\beta^2}{\sqrt{1-26\beta^2-23\beta^4}} - 1\right]$$

Test of prediction

Critical compressibility (zero slope): $\beta = 1/\sqrt{29} = 0.185\ldots$

Fractal dimension: $$D_2 = -\frac{v_1}{\mathcal{D}_{11}} = \frac{2(1-\beta^2)}{1+3\beta^2}$$

J. Bec, K. Gawedzki and P. Horvai, *Phys. Rev. Lett.*, **92**, 224501, (2004).



$\beta = 1/\sqrt{29}$
$D_2 = 7/4$

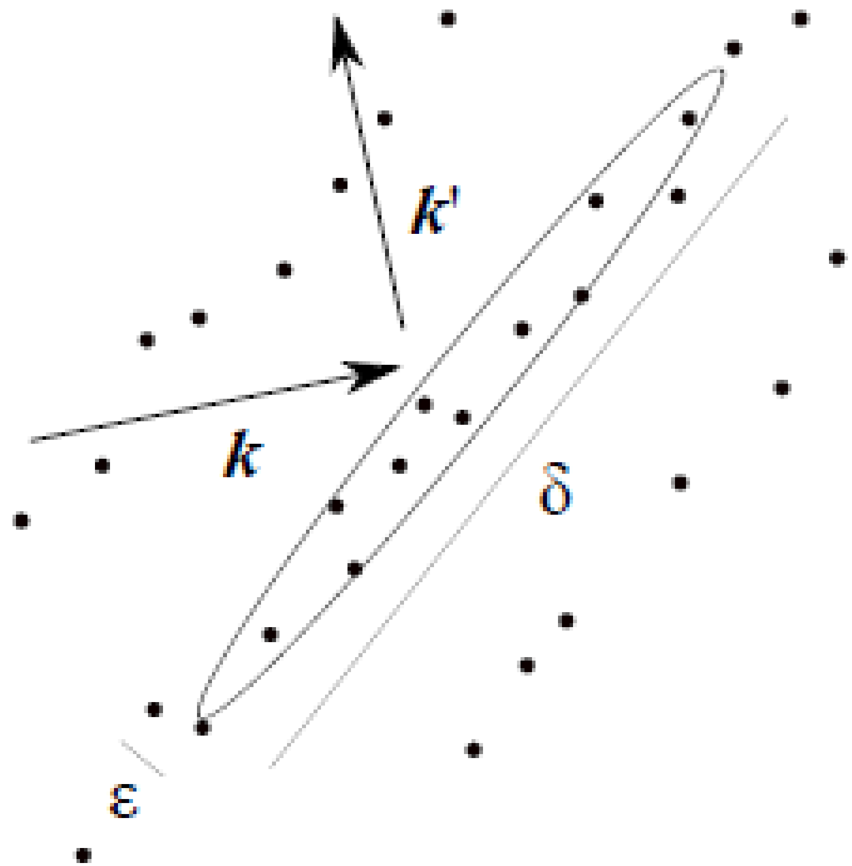
$\beta = 1/\sqrt{5}$
$D_2 = 1$

Coherent light scattering

Scattering amplitude:

$$a(\boldsymbol{K}) = a_0 \sum_{j=1}^{N} \exp[\mathrm{i}\boldsymbol{K} \cdot \boldsymbol{r}_j]$$

$$\boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{K}$$



Conclusion

The result: an unexpected phase transition in the distribution of triangle shapes: below a critical dimension, co-linearity is more prevalent as the ball size decreases.

The bigger picture: the massive literature on fractals emphasises mass. This is perhaps the first substantial result on shape-structure of fractals. There will be applications to light scattering, network connectivity, strengths of fractal aggregates...