Optimal Stirring & Maximal Mixing

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National Science Foundation
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Awards PHY-0855335 & PHY-1205219

J. Fluid Mech. (2011), vol. 675, pp. 465–476.
doi:10.1017/S0022112011000292

Optimal stirring strategies for passive scalar mixing

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JOURNAL OF MATHEMATICAL PHYSICS **53**, 115611 (2012)

Optimal mixing and optimal stirring for fixed energy, fixed power, or fixed palenstrophy flows

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Mixing via incompressible fluid flows:

shearing & straining (stretching & folding)
but no direct transport, dilution or concentration

Mathematical model

Given a flow field $\vec{u}(\vec{x},t)$ with $\nabla\cdot\vec{u}=0,$ consider ...

Differential equation: $$\frac{d\vec{X}(t)}{dt}=\vec{u}(\vec{X},t)$$

- $\boldsymbol{X}(t)$ is **passive tracer particle position**

Advection equation: $$\partial_t\rho+\vec{u}\cdot\nabla\rho=0$$

- $\rho(\boldsymbol{x},t)$ is **passive scalar concentration**

From here on $\vec{x}\in[0,L]^d$ w/periodic boundary conditions.

Primary question we propose to address here:

Given initial tracer distribution $\rho_0(\vec{x})$,
what incompressible flow $\vec{u}(\vec{x},t)$
is the *optimal* stirring?

Secondary questions:

What is *optimal*?

What constraints on $\vec{u}(\vec{x},t)$?

Measures of mixing

Note:

Given $\partial_t \rho + \vec{u} \cdot \nabla \rho = 0$ with $\rho(\vec{x},0) = \rho_0(\vec{x})$,

$$\left\langle \rho(\cdot,t) \right\rangle \equiv \frac{1}{L^d} \int_{[0,L]^d} \rho(\vec{x},t)\, d\vec{x} = \left\langle \rho_0 \right\rangle$$

$$\left\langle \left(\rho(\cdot,t) - \left\langle \rho \right\rangle \right)^2 \right\rangle = \left\langle \left(\rho_0 - \left\langle \rho_0 \right\rangle \right)^2 \right\rangle$$

Definition:

$$\partial_t \theta + \vec{u} \cdot \nabla \theta = 0 \quad \text{with} \quad \theta(\vec{x},0) = \theta_0(\vec{x}) = \rho_0(\vec{x}) - \left\langle \rho_0 \right\rangle$$

so that

$$\left\langle \theta(\cdot,t) \right\rangle = 0, \ \left\langle \theta(\cdot,t)^2 \right\rangle = \left\langle \theta_0^2 \right\rangle, \ \ldots, \ \left\| \theta(\cdot,t) \right\|_{L^\infty} = \left\| \theta_0 \right\|_{L^\infty}$$

Measures of mixing

Definition:

$\vec{u}(\vec{x},t)$ is *mixing* if, for every $g(\vec{x}) \in L^2([0,L]^d)$,

$$\lim_{t\to\infty} \int_{[0,L]^d} g(\vec{x})\, \rho(\vec{x},t)\, d\vec{x} = \langle \rho_0 \rangle \int_{[0,L]^d} g(\vec{x})\, d\vec{x}$$

... that is, if $\lim_{t\to\infty} \int_{[0,L]^d} g(\vec{x})\, \theta(\vec{x},t)\, d\vec{x} = 0$

$$\Updownarrow$$

$\theta(\vec{x},t) \underset{t\to\infty}{\longrightarrow} 0$ *weakly* in L^2.

Measures of mixing

Fact:

$$\theta(\vec{x},t) \underset{t\to\infty}{\longrightarrow} 0 \textit{ weakly} \text{ in } L^2$$

$$\Updownarrow$$

$$\|\theta(\cdot,t)\|_{L^2} \text{ uniformly bounded \& } \|\theta(\cdot,t)\|_{H^{-a}} \underset{t\to\infty}{\longrightarrow} 0$$

$$\text{where } \|\theta\|_{H^{-a}} \equiv \sqrt{\sum_{\vec{k}\neq 0} \frac{\left|\hat{\theta}_{\vec{k}}\right|^2}{k^{2a}}} \text{ with } a > 0.$$

$\therefore$ H^{-a} norm can serve a *mix-norm*.

Measures of mixing

Fact:

$$\theta(\vec{x},t) \underset{t\to\infty}{\longrightarrow} 0 \text{ \textit{weakly} in } L^2$$

$$\Updownarrow$$

$$\|\theta(\cdot,t)\|_{L^2} \text{ uniformly bounded \& } \|\theta(\cdot,t)\|_{H^{-a}} \underset{t\to\infty}{\longrightarrow} 0$$

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$$\downarrow$$

$\therefore\ \dot{H}^{-a}$ norm can serve a *mix-norm*.

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A multiscale measure for mixing

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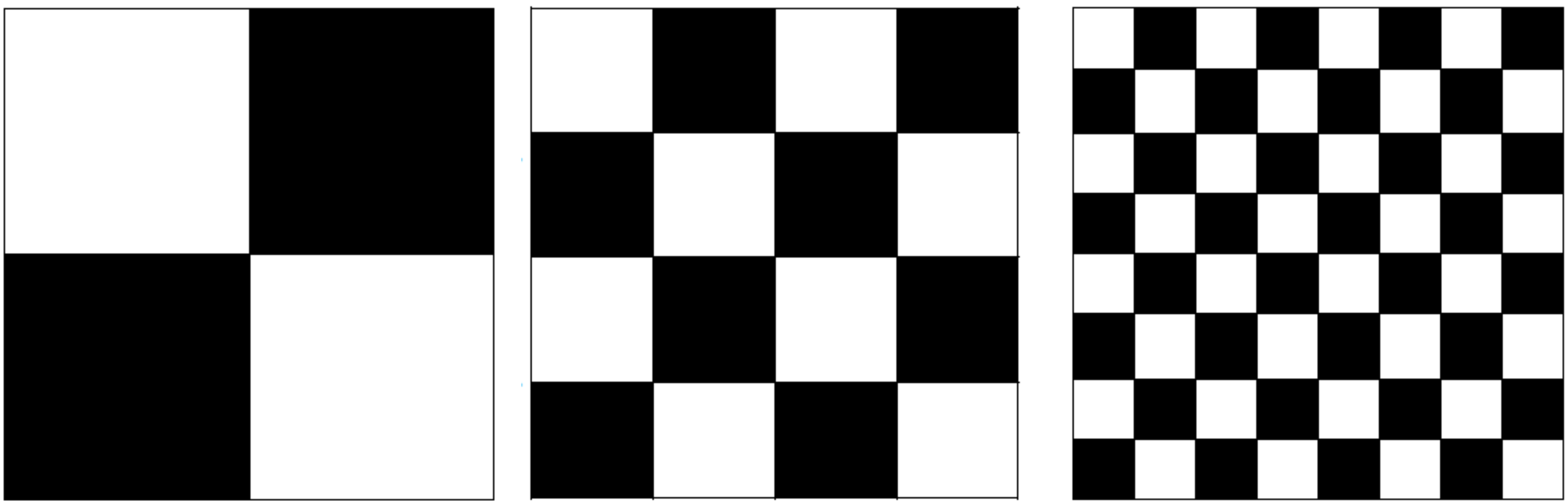
… introduced and used the $H^{-1/2}$ norm as a *mix - norm*.

But H^{-1} norm in perhaps more "natural" ...

$$\|\theta\|_{H^{-1}} / \|\theta\|_{L^2} = \text{"integral" length scale}$$



mix-norm $\|\theta_0\|_{H^{-1}}$ $\quad \Rightarrow$ mix-norm $= \frac{1}{2}\|\theta_0\|_{H^{-1}}$ $\quad \Rightarrow$ mix-norm $= \frac{1}{4}\|\theta_0\|_{H^{-1}}$

Stirring strategies

- Constraints must be imposed on the available flows in order to formulate an optimization problem.

- Natural focus is on bounded instantaneous kinetic energy proportional to the velocity field's L^2 norm:

$$\left\|\vec{u}(\cdot,t)\right\|_{L^2}^2 = L^d U^2 \text{ with constant RMS velocity } U$$

- … or bounded instantaneous power, for Newtonian fluids proportional to the H^1 norm of the velocity:

$$\left\|\vec{\nabla}\vec{u}(\cdot,t)\right\|_{L^2}^2 = \frac{L^d}{\tau^2} \text{ with constant RMS rate of strain } \frac{1}{\tau}$$

Stirring strategies

Given either constraint two natural questions are:

(I) What flow minimizes the mix-norm evaluated at a specified *final* time $t_{final} > 0$?

(II) What flow maximizes the *instantaneous* decay rate of the mixing measure?

(I) is an optimal control problem …

J. Fluid Mech. (2007), vol. 580, pp. 261–281.
doi:10.1017/S0022112007005332 Printed in the United Kingdom

Optimal control of mixing in Stokes fluid flows

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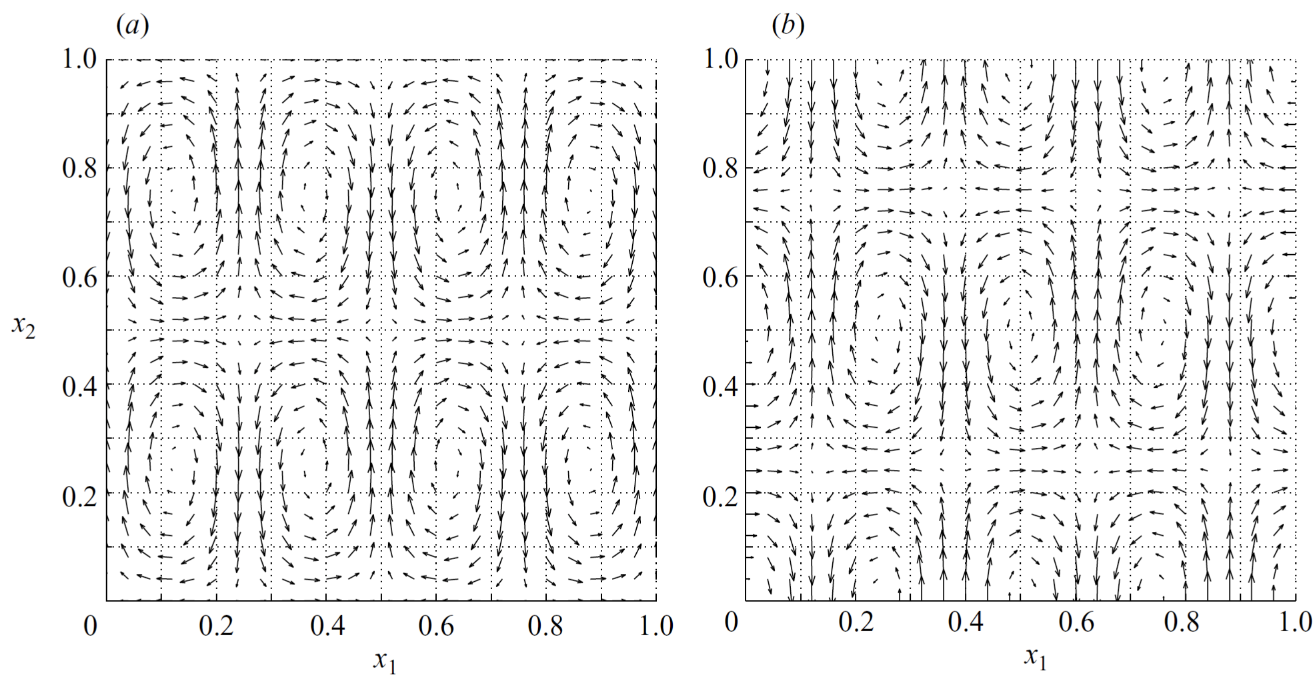
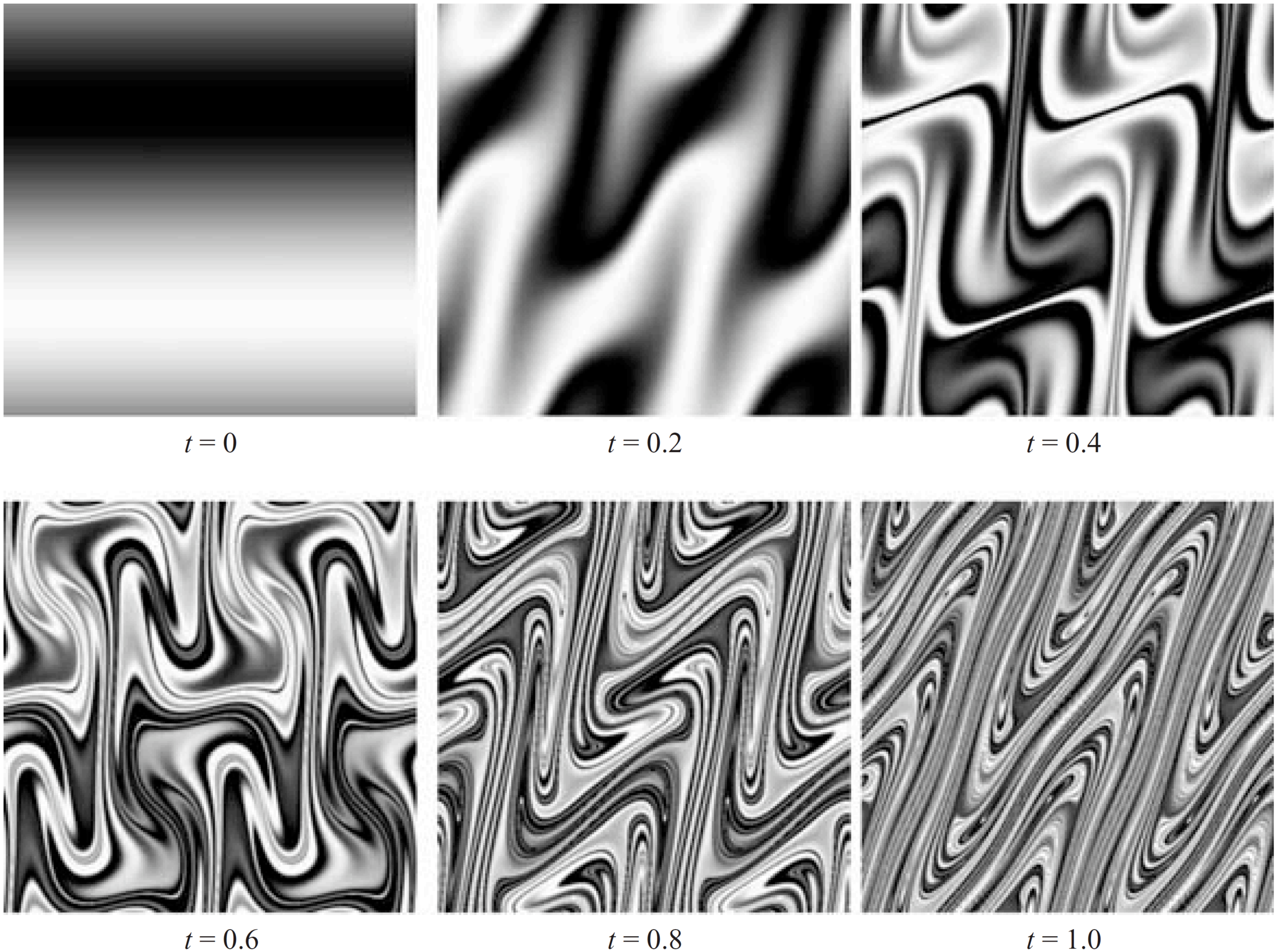
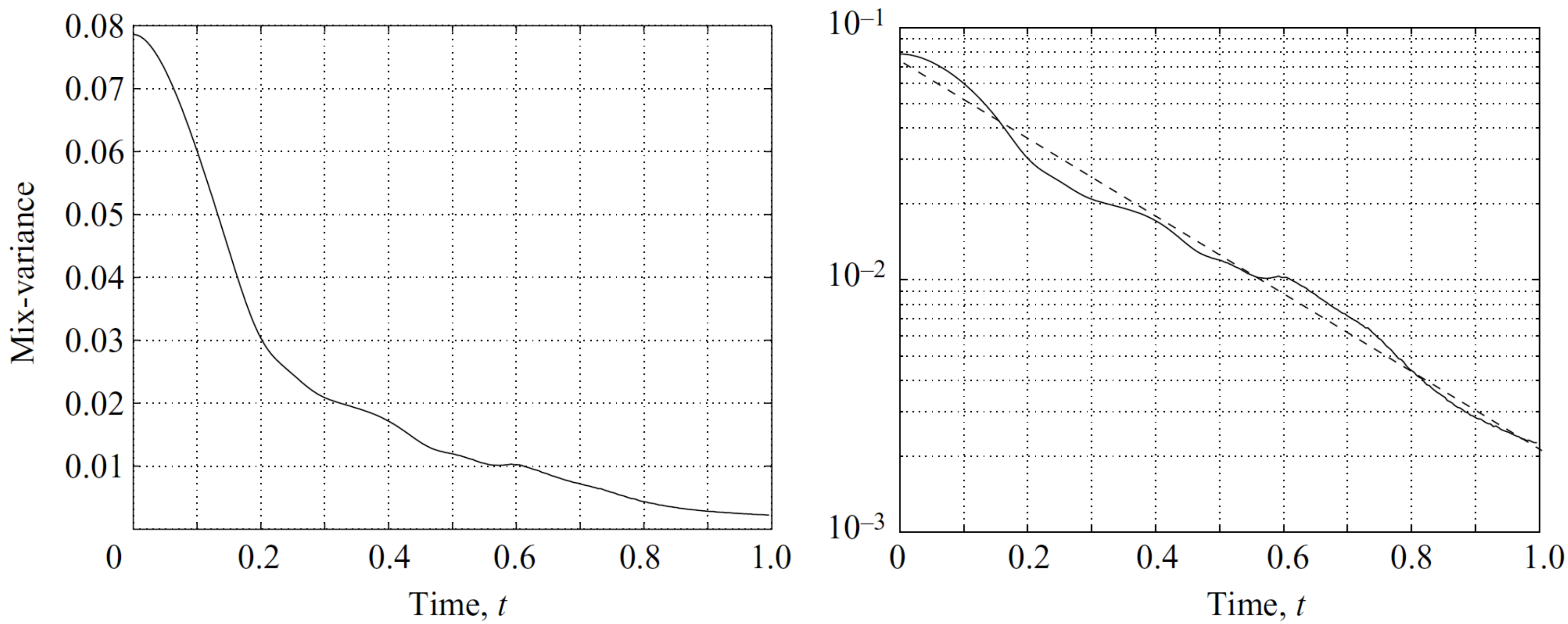



$t = 0$
$t = 0.2$
$t = 0.4$
$t = 0.6$
$t = 0.8$
$t = 1.0$



$$\text{Mix-variance} = \|\theta\|^2_{H^{-1/2}}$$

Stirring strategies

Given either constraint two natural questions are:

(I) What flow minimizes the mix-norm evaluated at a specified *final* time $t_{final} > 0$?

(II) What flow maximizes the *instantaneous* decay rate of the mixing measure?

(I) is an optimal control problem…

(II) is what we address …

Stirring strategies

Choose flow, subject to flow constraint, to extremize

$$\frac{d}{dt}\left\|\theta(\cdot,t)\right\|_{H^{-1}}^{2}$$

$$\left(\Delta^{-1}\theta\right)(\vec{x},t) \equiv -\sum_{\vec{k}\neq 0}\frac{\hat{\theta}_{\vec{k}}(t)}{k^{2}}\,e^{i\vec{k}\cdot\vec{x}}$$

An analysis aside ...

Estimates on decay of the mix-norm ... for *fixed energy*:

$$\frac{d}{dt}\|\theta(\cdot,t)\|_{H^{-1}}^{2} = -2\int \vec{u}\cdot\left[\theta\,\vec{\nabla}\left(\Delta^{-1}\theta\right)\right]\,d\vec{x}$$

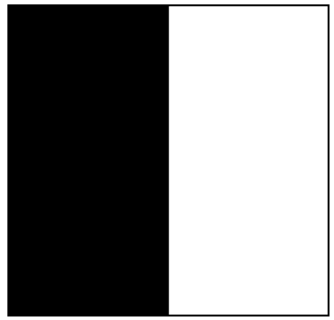
$$\geq -2\,U\,L^{d/2}\ \|\theta(\cdot,t)\|_{L^{\infty}}\,\|\theta(\cdot,t)\|_{H^{-1}}$$

$$\therefore\ \frac{d}{dt}\|\theta(\cdot,t)\|_{H^{-1}} \geq -\|\theta_0\|_{L^{\infty}}\ U\,L^{d/2}$$

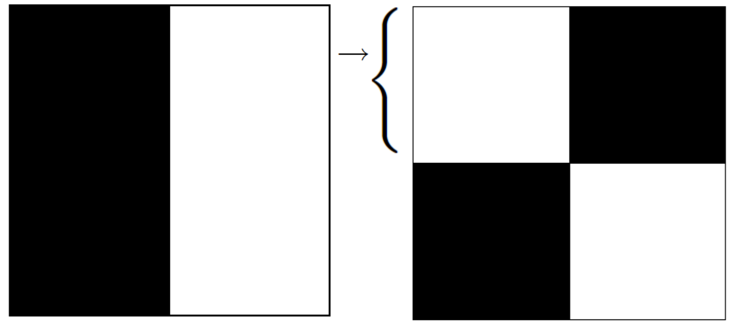
$$\Rightarrow\quad \|\theta(\cdot,t)\|_{H^{-1}} \geq \|\theta_0\|_{H^{-1}}\left(1-\frac{2\pi U t}{l_0}\right)\quad\text{where}\quad l_0 = 2\pi\frac{\left\langle\left|\vec{\nabla}\Delta^{-1}\theta_0\right|^2\right\rangle^{1/2}}{\|\theta_0\|_{L^{\infty}}}$$

... is it really possible to achieve *perfect mixing* in finite time***?***

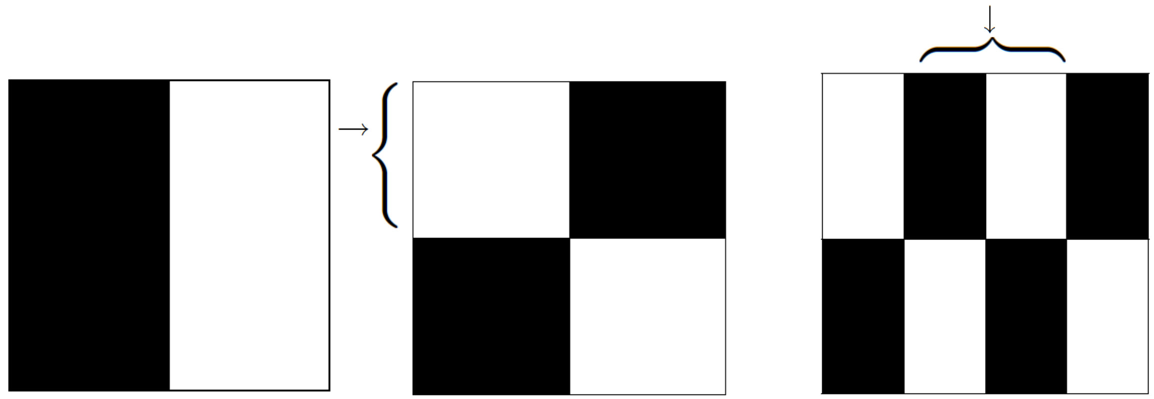
Yes!



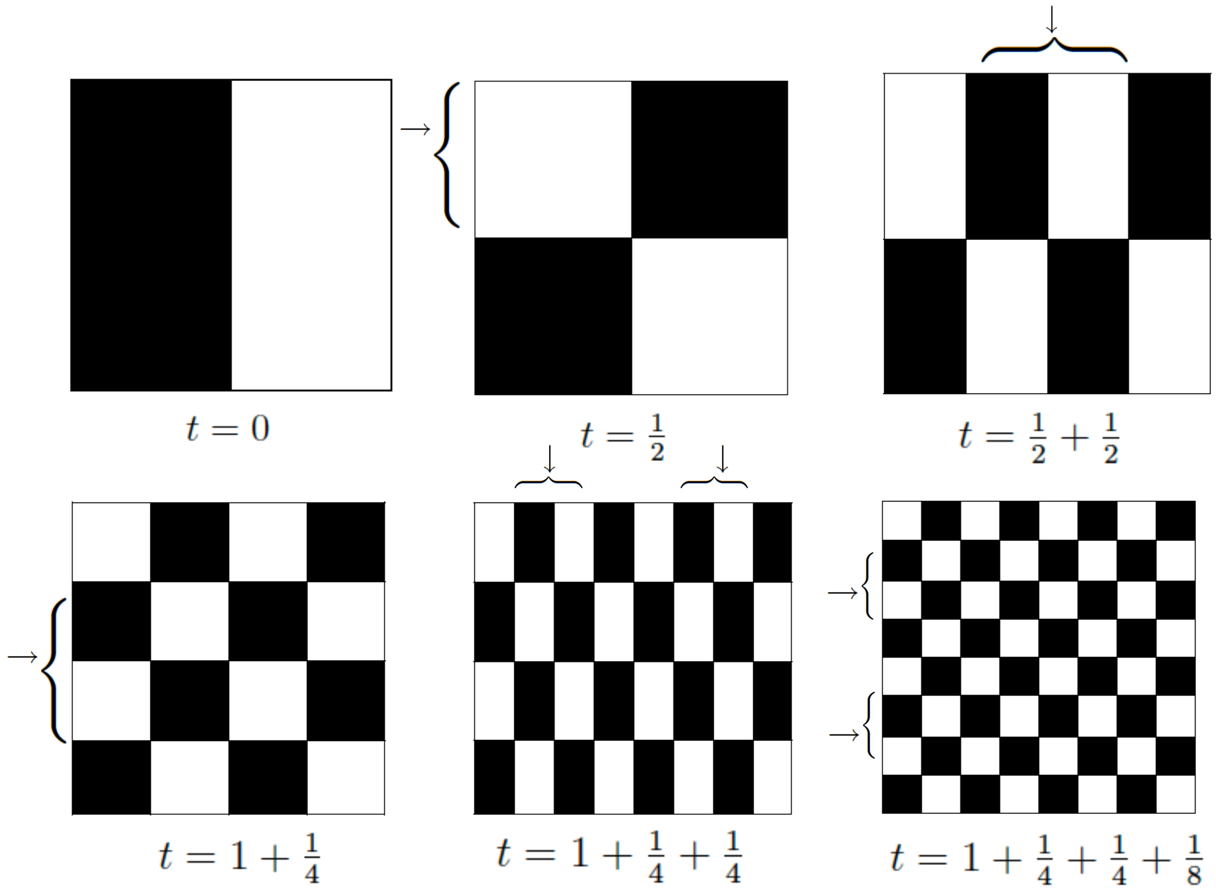
$t = 0$



$t = 0$ $\qquad$ $t = \frac{1}{2}$



$t = 0$ $\quad$ $t = \frac{1}{2}$ $\quad$ $t = \frac{1}{2} + \frac{1}{2}$


$t = 0$
$t = \frac{1}{2}$
$t = \frac{1}{2} + \frac{1}{2}$
$t = 1 + \frac{1}{4}$
$t = 1 + \frac{1}{4} + \frac{1}{4}$
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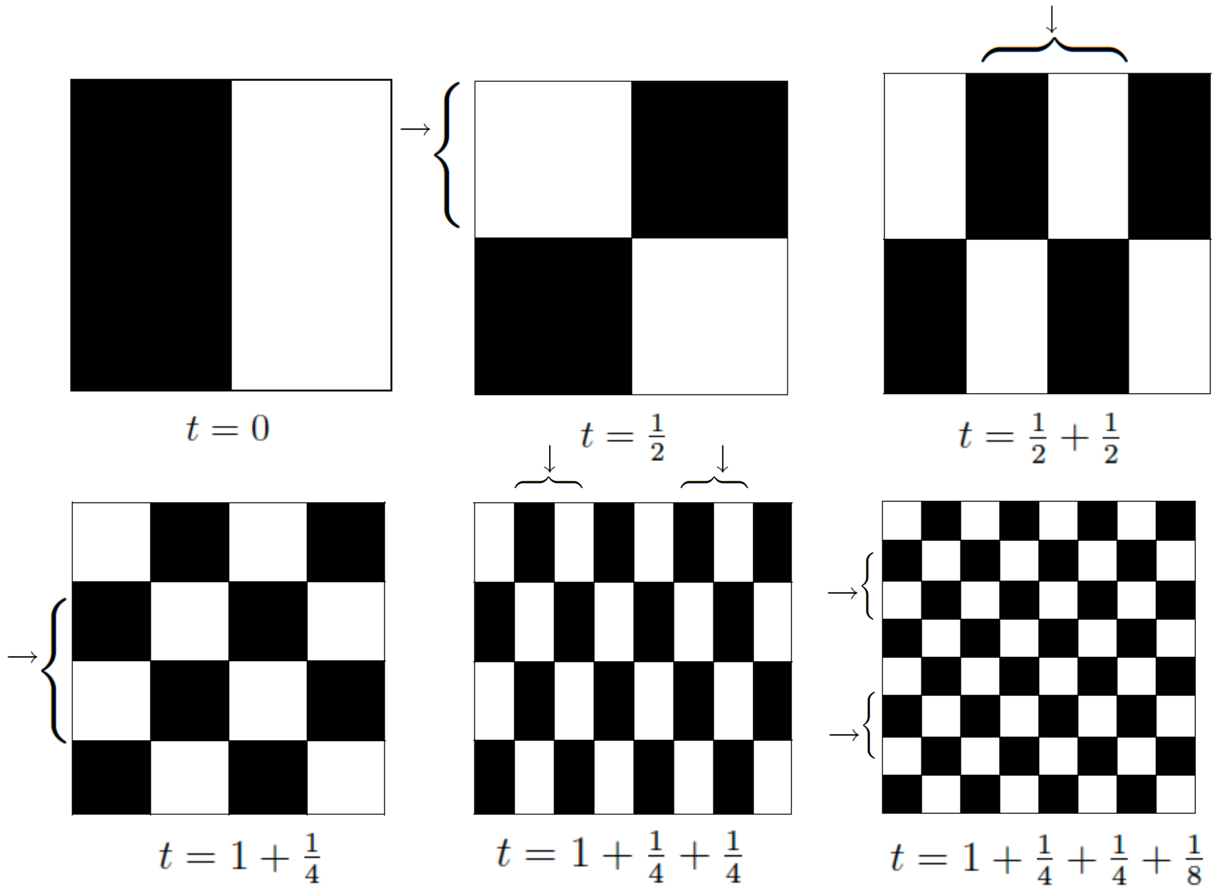


Time t	$1 - \frac{t}{2}$	Mix-Norm (N)
0	1	N_0
$\frac{1}{2}$	$\frac{3}{4}$	N_1
$\frac{1}{2} + \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}N_0$
$1 + \frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}N_1$
$1 + \frac{1}{4} + \frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}N_0$
$1 + \frac{1}{2} + \frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}N_1$
$1 + \frac{1}{2} + \frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}N_0$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{16}$	$\frac{3}{32}$	$\frac{1}{8}N_1$

An analysis aside ...

Bounds on decay of the mix-norm ... for *fixed power*:

$$\frac{d}{dt}\left\|\theta(\cdot,t)\right\|_{H^{-1}}^{2} \;=\; -2\int\left(\Delta^{-1}\theta\right)\left[\vec{\nabla}\vec{u}\right]:\left(\vec{\nabla}\vec{\nabla}\Delta^{-1}\theta\right)d\vec{x}$$

It's possible to close the differential *in*equation in $d=2$ & 3:

$$d=2: \quad \left\|\Delta^{-1}\theta\right\|_{L^\infty} \le C_2\left\|\theta\right\|_{H^{-1}}\sqrt{1+\log\left(\frac{L\left\|\theta\right\|_{L^2}}{2\pi\left\|\theta\right\|_{H^{-1}}}\right)}$$

$$d=3: \quad \left\|\Delta^{-1}\theta\right\|_{L^\infty} \le C_3\left\|\theta\right\|_{H^{-1}}^{1/2}\left\|\theta\right\|_{L^2}^{1/2}$$

An analysis aside ...

Theorem: for fixed power in $d = 3$,

$$\left\|\theta(\cdot,t)\right\|_{H^{-1}} \geq \left\|\theta_0\right\|_{H^{-1}} \left(1 - \frac{t}{t_{mix}}\right)^{3/2}$$

$$\text{where} \quad t_{mix} = \tau \times \frac{2}{3C_3} \times \left(\frac{\ell_0}{2\pi L}\right)^{3/2}$$

$$\text{and} \quad \ell_0 = 2\pi \frac{\left\langle \left|\vec{\nabla}\Delta^{-1}\theta_0\right|^2 \right\rangle^{1/2}}{\left\langle \theta_0{}^2 \right\rangle^{1/2}} \leq L$$

An analysis aside ...

Theorem: for fixed power in $d = 2$,

$$\frac{d}{dt}\left\|\theta(\cdot,t)\right\|_{H^{-1}} \geq -\frac{C_2 L}{\tau}\left\|\theta_0\right\|_{L^2}\sqrt{1+\log\left(\frac{L\left\|\theta_0\right\|_{L^2}}{2\pi\left\|\theta(\cdot,t)\right\|_{H^{-1}}}\right)}$$

allowing for total mixing after time

$$t_{mix} = \tau \times \frac{e}{2\pi C_2} \times \int_{\log\frac{L}{\ell_0}}^{\infty} \frac{e^{-\zeta}}{\sqrt{\zeta}}\, d\zeta \underset{\ell_0 << L}{\sim} \tau \times \left(\frac{\ell_0}{L}\right) \times \left(\log\left[\frac{\ell_0}{L}\right]\right)^{-1/2}$$

An analysis aside ...

Note: if the rate of strain's *spatial L^∞ norm* has a uniformly bounded time average,

$$\frac{d}{dt}\left\|\theta(\cdot,t)\right\|_{H^{-1}}^{2} \quad = \quad 2\int\left(\vec{\nabla}\Delta^{-1}\theta\right)\cdot\left[\vec{\nabla}\vec{u}\right]\cdot\left(\vec{\nabla}\Delta^{-1}\theta\right)\,d\vec{x}$$

allowing for at most *exponential* decay of the mix-norm.

Stirring strategies

Choose flow, subject to flow constraint, to extremize

$$\frac{d}{dt}\|\theta(\cdot,t)\|^2_{H^{-1}} = 2\int\left(\vec{\nabla}\Delta^{-1}\theta\right)\cdot\left[\vec{\nabla}\vec{u}\right]\cdot\left(\vec{\nabla}\Delta^{-1}\theta\right)d\vec{x}$$

$$= -2\int\left(\Delta^{-1}\theta\right)\left[\vec{\nabla}\vec{u}\right]:\left(\vec{\nabla}\vec{\nabla}\Delta^{-1}\theta\right)d\vec{x}$$

$$= -2\int\theta\,\vec{u}\cdot\vec{\nabla}\left(\Delta^{-1}\theta\right)d\vec{x} = -2\int\vec{u}\cdot P\left[\theta\,\vec{\nabla}\left(\Delta^{-1}\theta\right)\right]d\vec{x}$$

$$\left(\Delta^{-1}\theta\right)(\vec{x},t) \equiv -\sum_{\vec{k}\neq 0}\frac{\hat{\theta}_{\vec{k}}(t)}{k^2}\,e^{i\vec{k}\cdot\vec{x}} \qquad P\left[\vec{v}\right] = \vec{v} - \vec{\nabla}\Delta^{-1}\left(\vec{\nabla}\cdot\vec{v}\right)$$

Stirring strategies

Then the *optimal flow* at each instant is

$$\vec{u}_e(\vec{x},t) = U \frac{P\left(\theta \vec{\nabla} \Delta^{-1} \theta\right)}{\left\langle \left| P\left(\theta \vec{\nabla} \Delta^{-1} \theta\right) \right|^2 \right\rangle^{1/2}}$$

or

$$\vec{u}_p(\vec{x},t) = \frac{1}{\tau} \frac{-\Delta^{-1} P\left(\theta \vec{\nabla} \Delta^{-1} \theta\right)}{\left\langle \left| \vec{\nabla} \Delta^{-1} P\left(\theta \vec{\nabla} \Delta^{-1} \theta\right) \right|^2 \right\rangle^{1/2}}$$

… as long as denominators don't vanish.

Stirring strategies

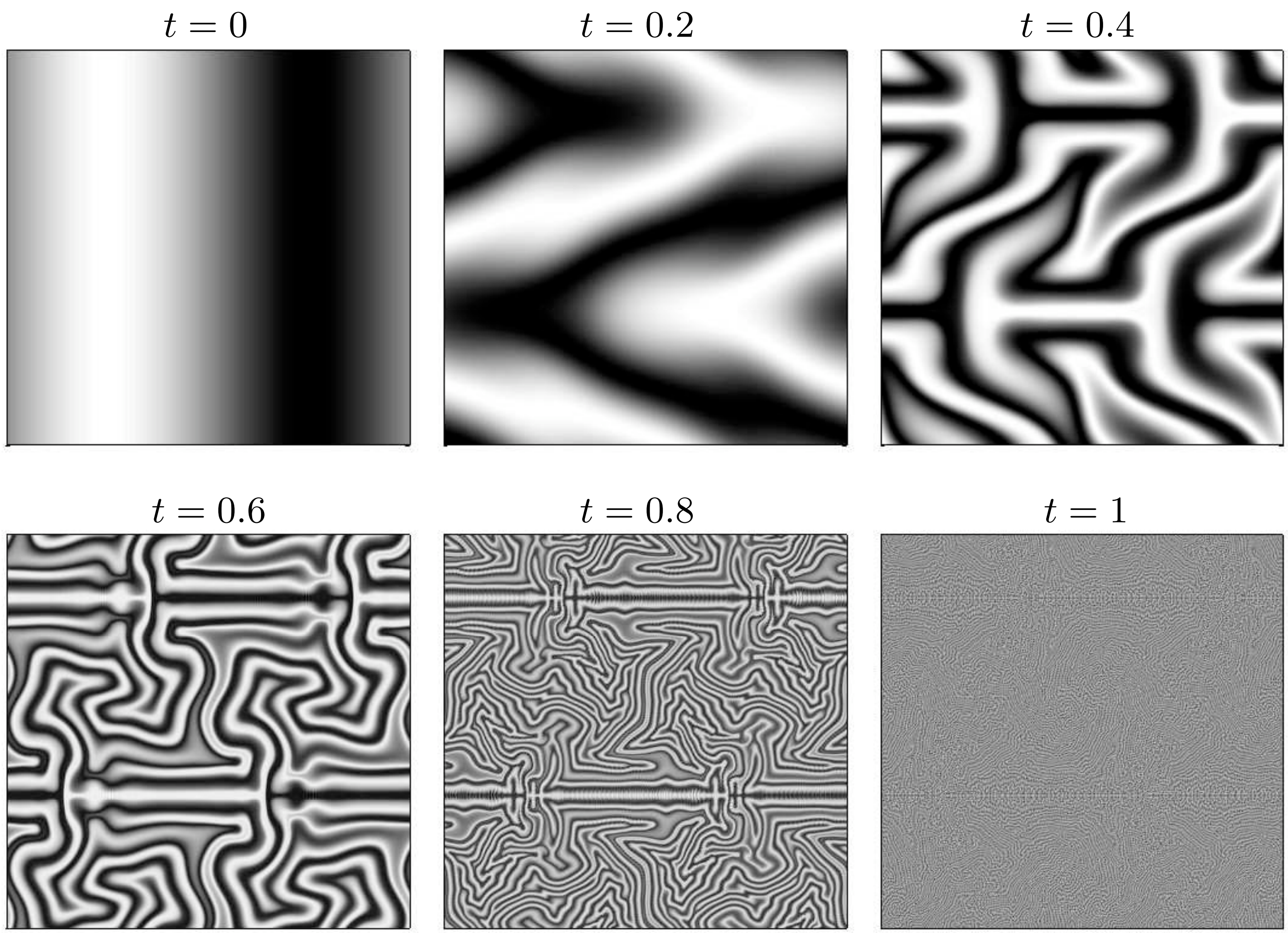
If/when $P\left(\theta\,\vec{\nabla}\Delta^{-1}\theta\right)=0$, choose $\vec{u}$ to minimize

$$\frac{d^2}{dt^2}\left\|\theta(\cdot,t)\right\|_{H^{-1}}^2 \quad = \quad 2\int\left[\vec{u}\cdot\vec{\nabla}\theta\left(\vec{\nabla}\Delta^{-1}\theta\right)\cdot\vec{u}-\vec{u}\cdot\vec{\nabla}\theta\Delta^{-1}\left(\vec{u}\cdot\vec{\nabla}\theta\right)\right]d\vec{x}$$

… an eigenvalue problem for the flow field.

Results

Computational tests (fixed power):



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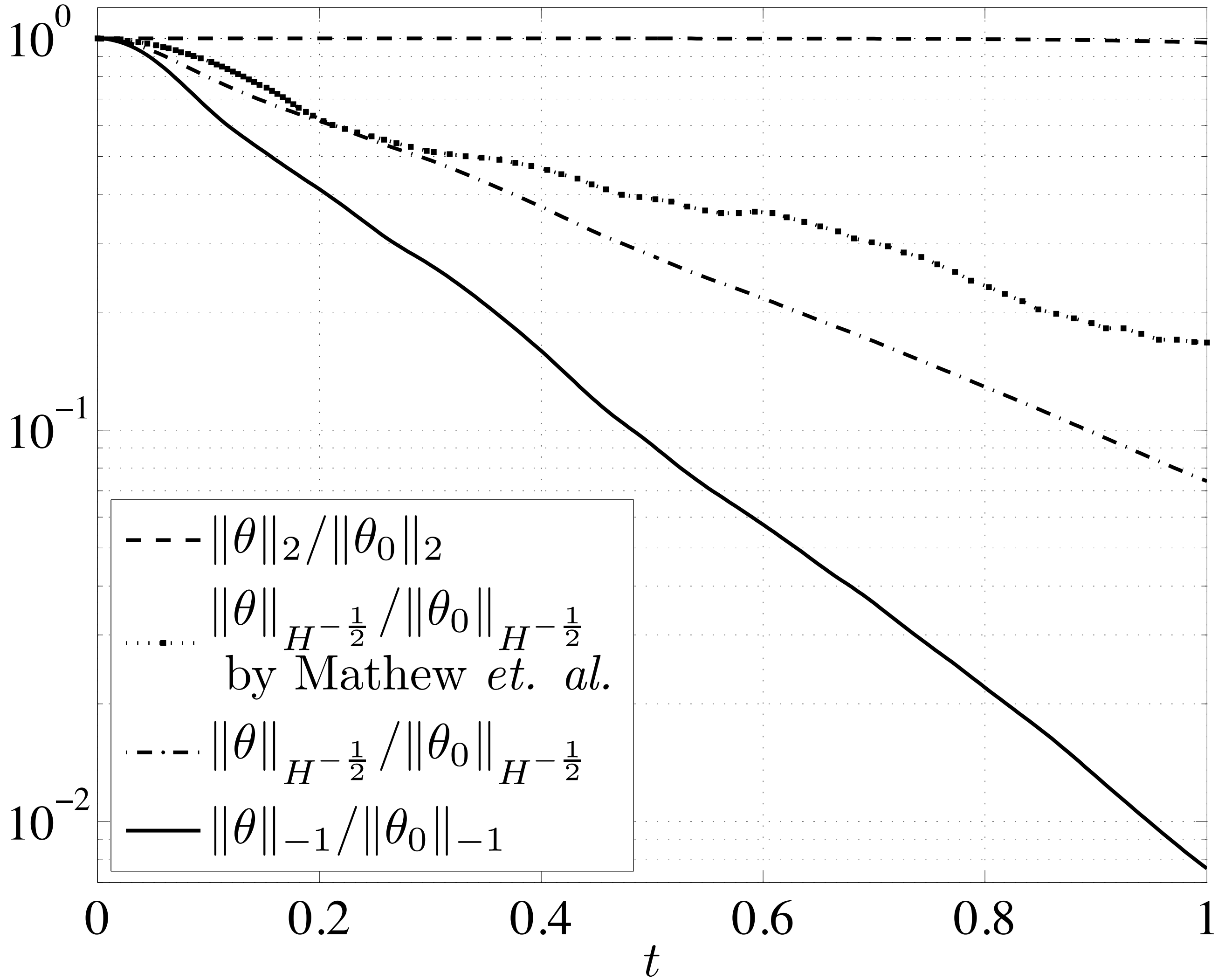

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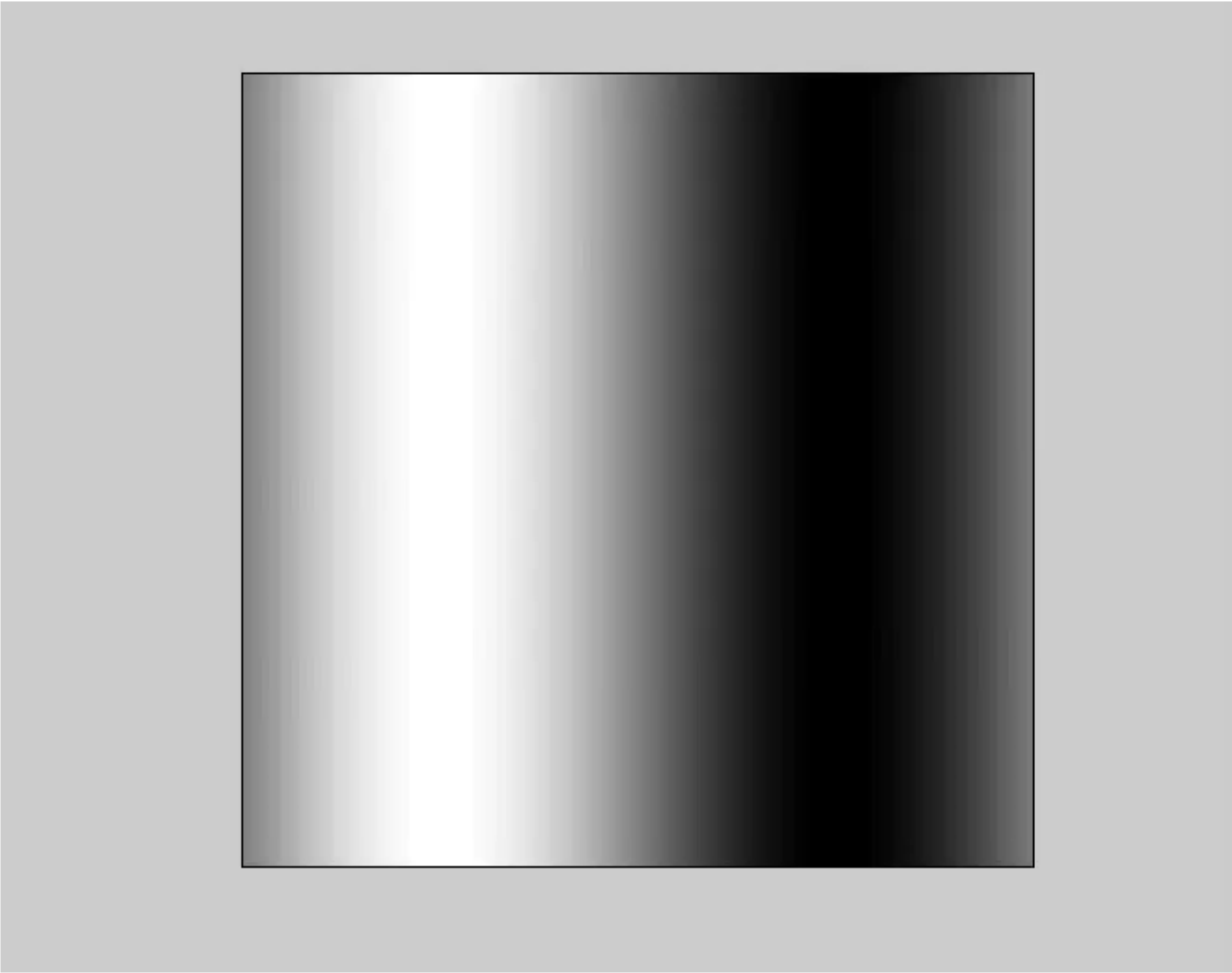

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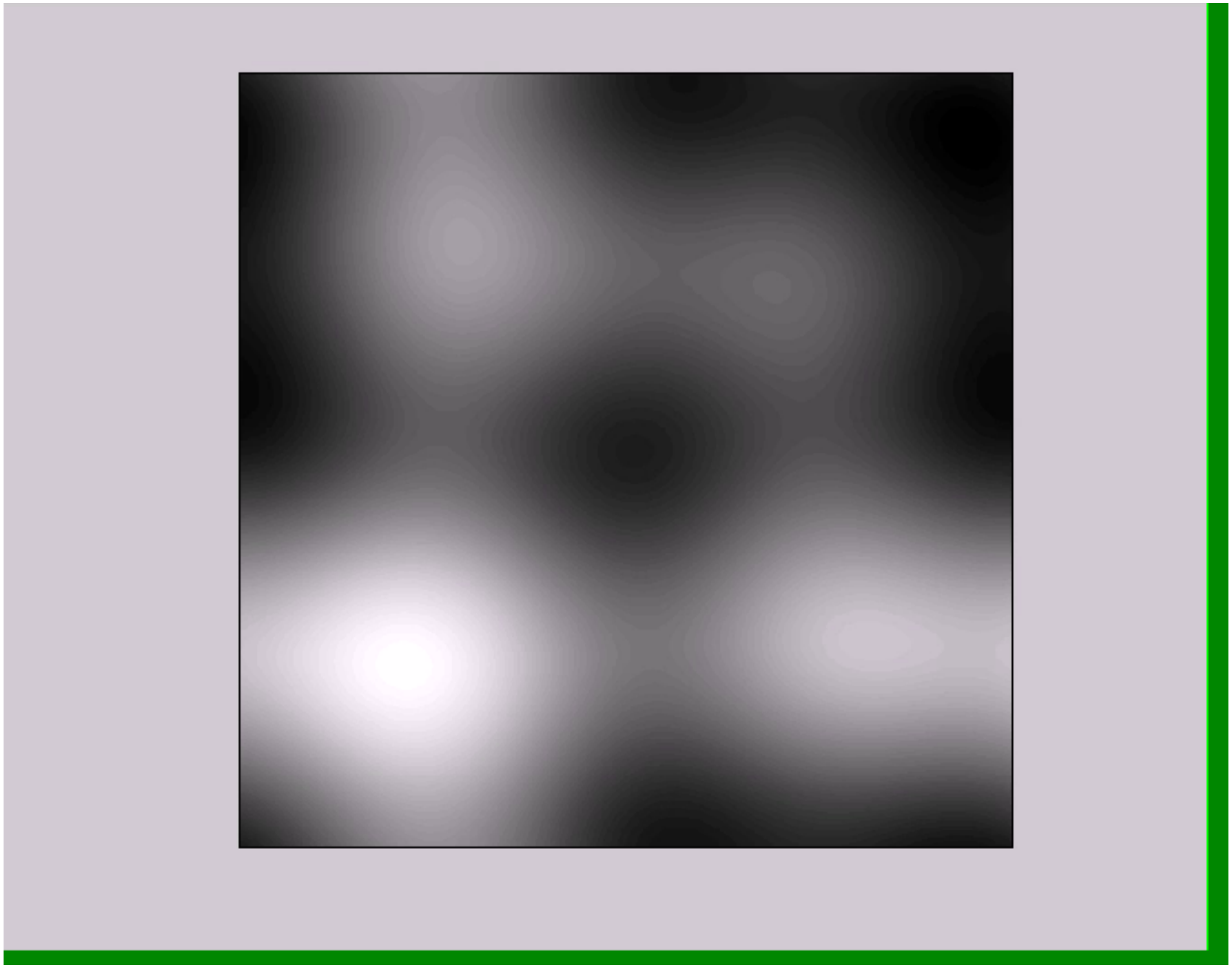

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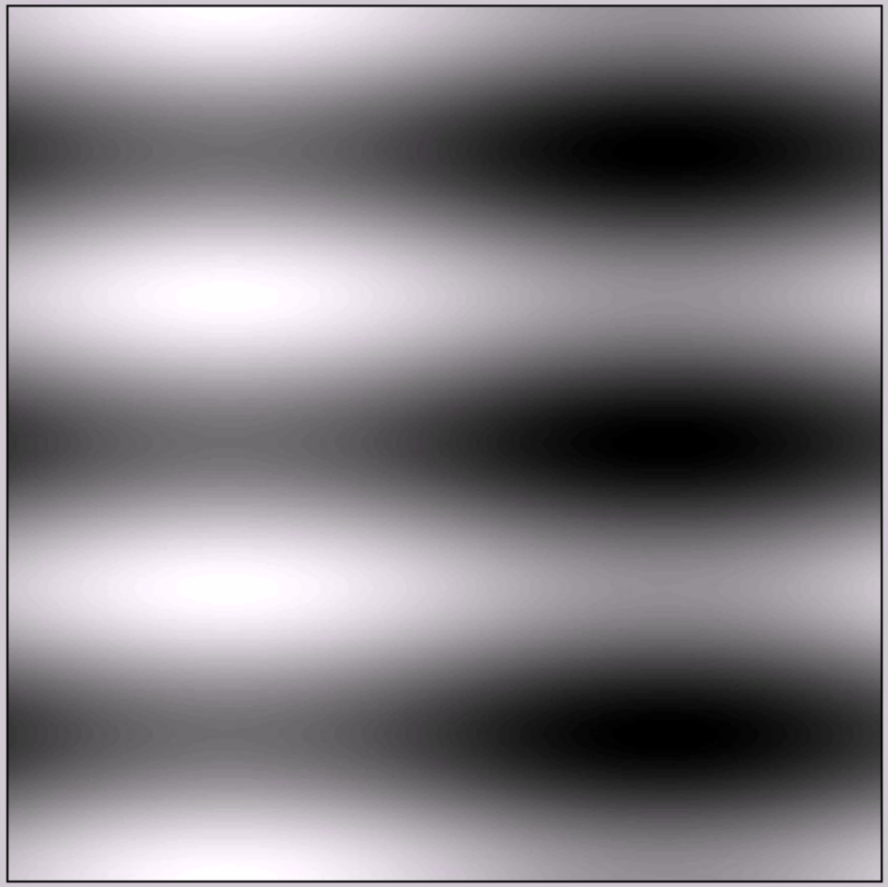

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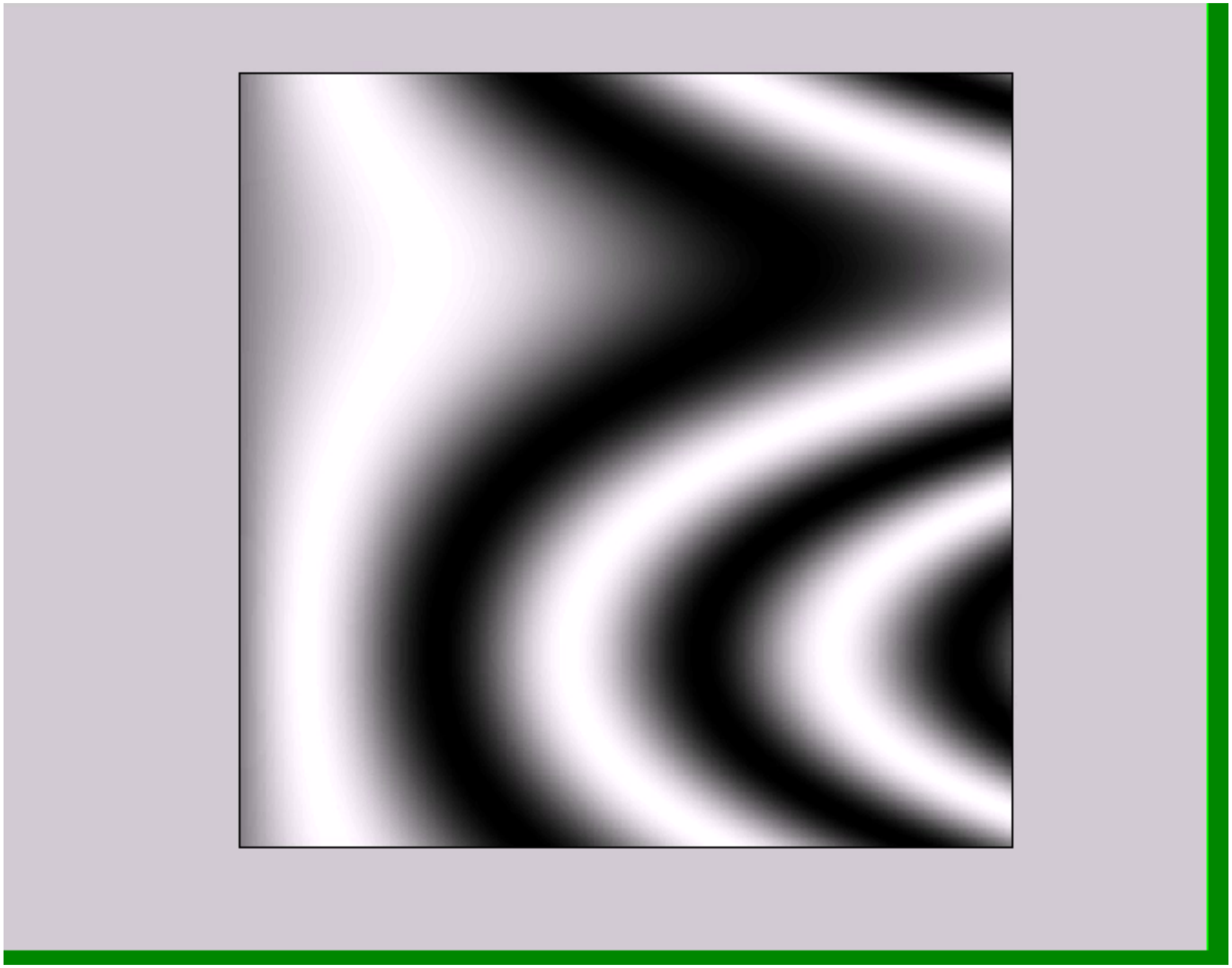

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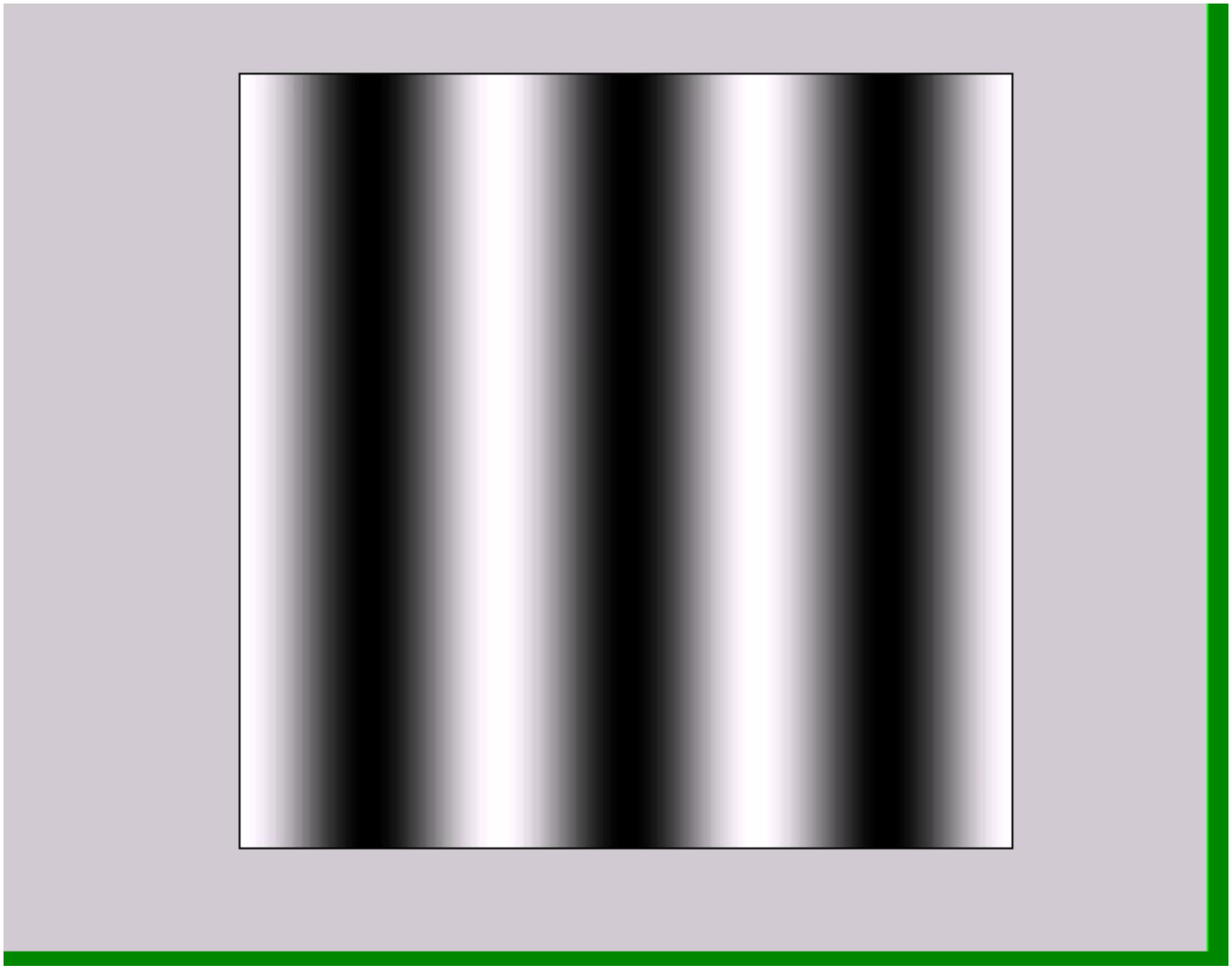

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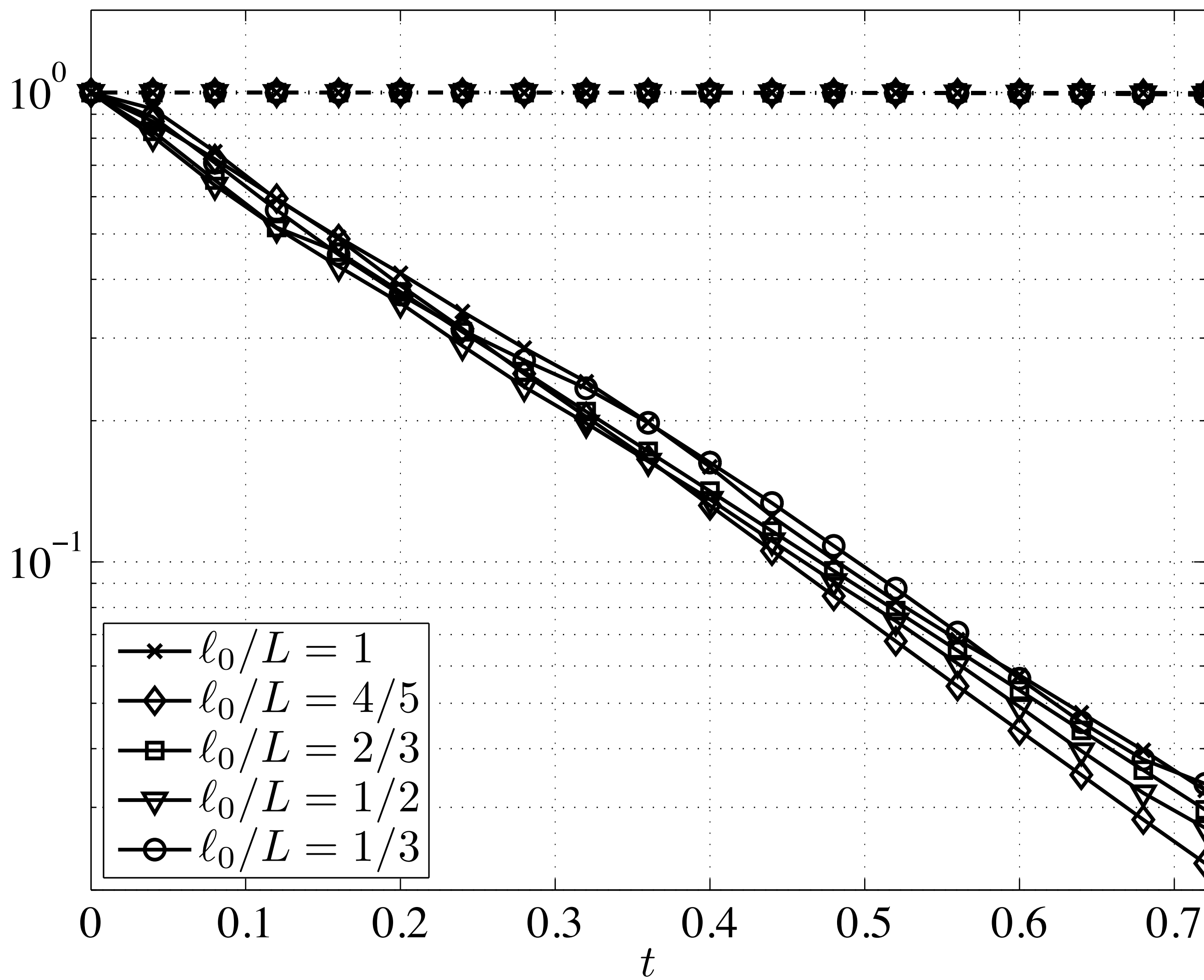

JOURNAL OF MATHEMATICAL PHYSICS **53**, 115611 (2012)

Optimal mixing and optimal stirring for fixed energy, fixed power, or fixed palenstrophy flows

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$$\mathcal{P} := L^{-2}\|\Delta \mathbf{u}\|_{L^2}^2 \quad \Longrightarrow \quad \left\|\theta(\cdot,t)\right\|_{H^{-1}} \le C_1 e^{-c_2 P t^2}$$

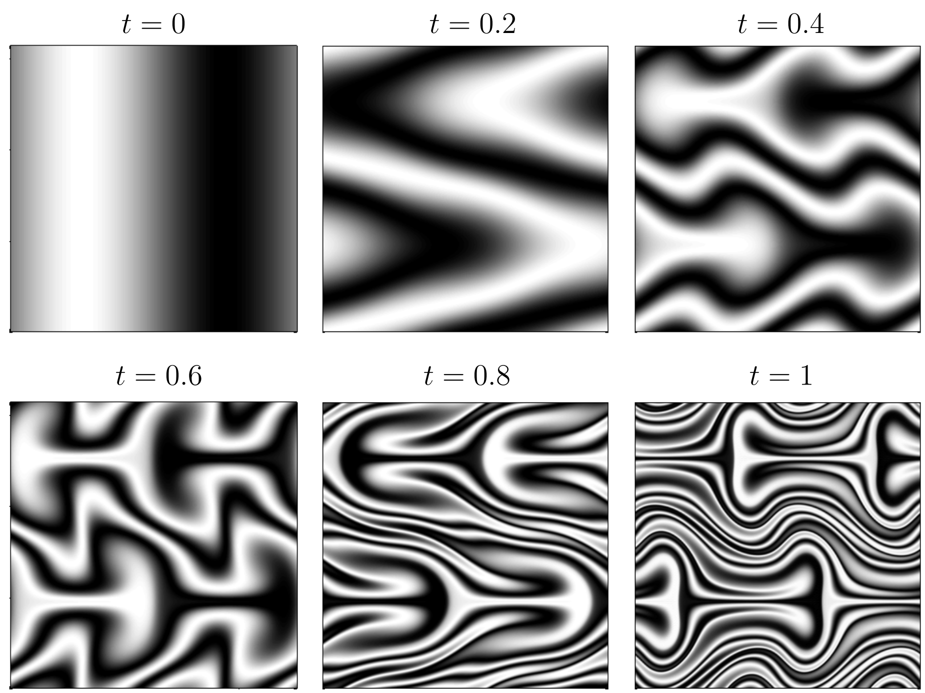

FIG. 4. Snapshots of the evolution of the scalar field with $\theta_0(x) = \sin x$ under the local fixed palenstrophy optimal mixer

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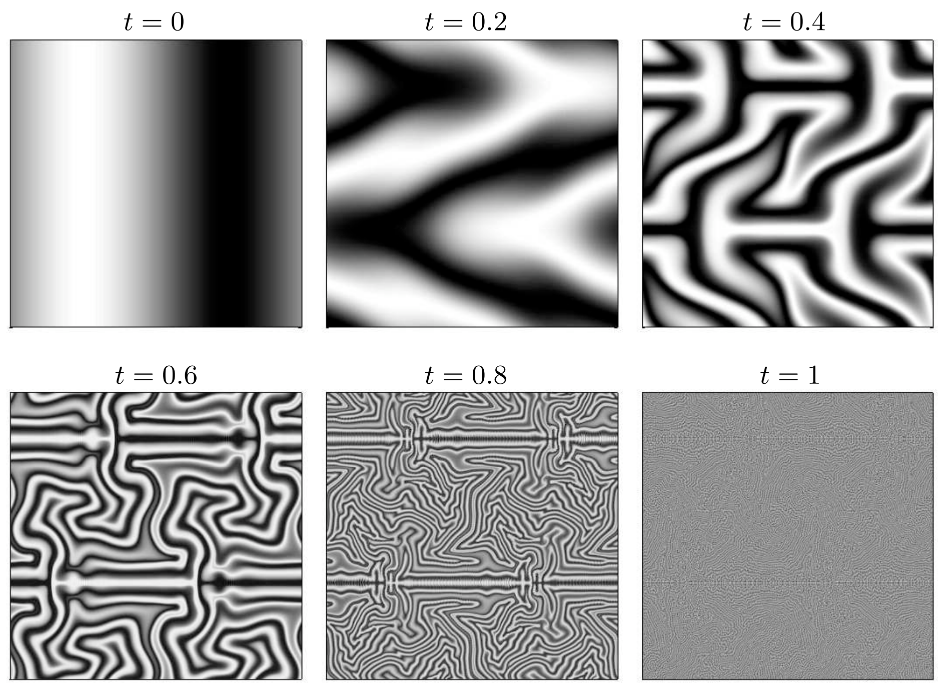

FIG. 4. Snapshots of the evolution of the scalar field with $\theta_0(x) = \sin x$ under the local fixed power ~~palenstrophy~~ optimal mixer

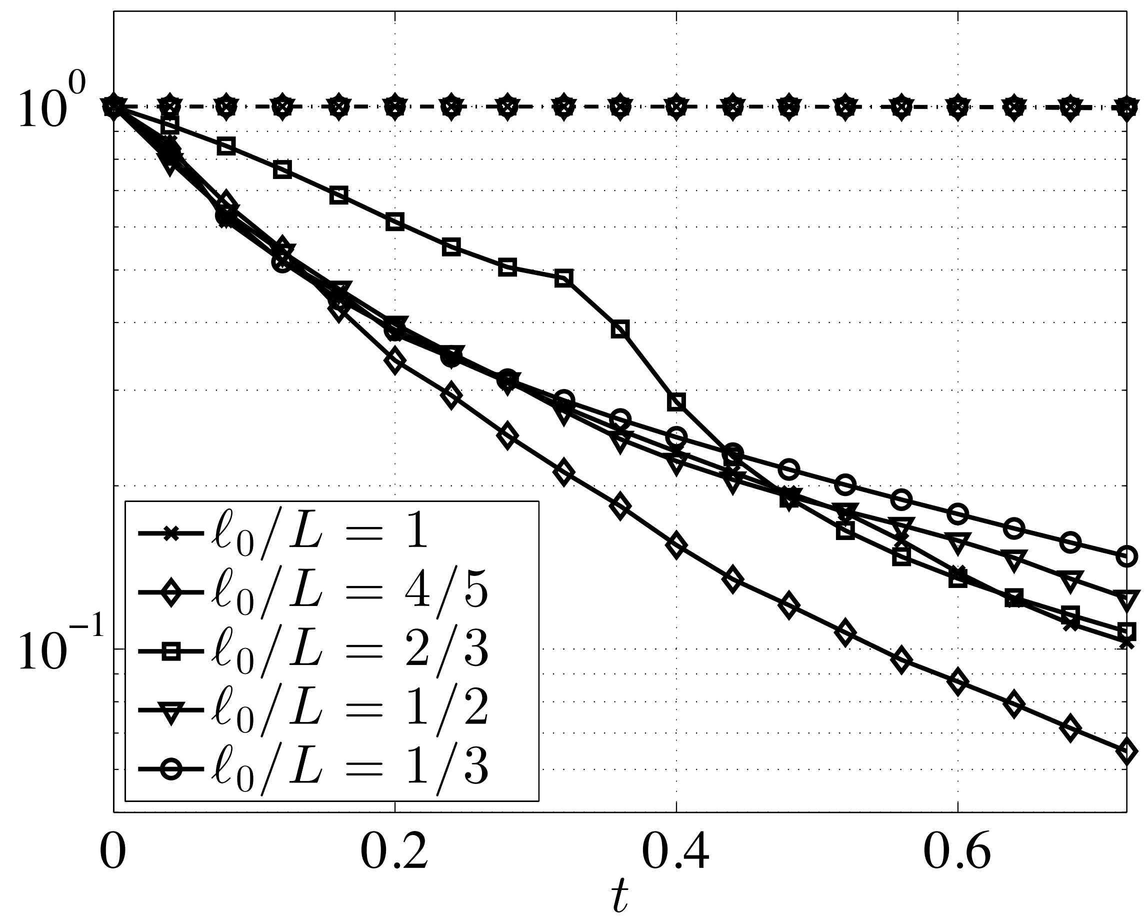

FIG. 3. Decay of the H^{-1} mix-norm for different initial data. See Lin *et al.*[1] for detailed description of the various initial scalar distributions.

Breaking News

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Nonlinearity **26** (2013) 3279–3289 doi:10.1088/0951-7715/26/12/3279

Maximal mixing by incompressible fluid flows

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Received 4 September 2013, in final form 7 September 2013
Published 21 November 2013
Online at stacks.iop.org/Non/26/3279

Recommended by B Eckhardt

Abstract
We consider a model for mixing binary viscous fluids under an incompressible flow. We prove the impossibility of perfect mixing in finite time for flows with finite viscous dissipation. As measures of mixedness we consider a Monge–Kantorovich–Rubinstein transportation distance and, more classically, the H^{-1} norm. We derive rigorous a priori lower bounds on these mixing norms which show that mixing cannot proceed faster than exponentially in time. The rate of the exponential decay is uniform in the initial data.

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Nonlinearity **26** (2013) 3279–3289 doi:10.1088/0951-7715/26/12/3279

Maximal mixing by incompressible fluid flows

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Theorem 2. *Let* $1 < p \leqslant \infty$. *There exists constants* $c, C > 0$ *depending on* p *and* d *only such that for every* $T > 0$

$$[\rho_0]_{BV} \|\rho(T,\,\cdot)\|_{\dot{H}^{-1}} \geqslant C \exp\left(-c \int_0^T \|\nabla u\|_{L^p}\,\mathrm{d}t\right).$$

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Nonlinearity **27** (2014) 973–985 doi:10.1088/0951-7715/27/5/973

Lower bounds on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows

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Received 10 October 2013, revised 12 February 2014
Accepted for publication 6 March 2014
Published 25 April 2014

Recommended by L Ryzhik

Abstract

Consider a diffusion-free passive scalar θ being mixed by an incompressible flow u on the torus $\mathbb{T}^d$. Our aim is to study how well this scalar can be mixed under an enstrophy constraint on the advecting velocity field. Our main result shows that the mix-norm ($\|\theta(t)\|_{H^{-1}}$) is bounded below by an exponential function of time. The exponential decay rate we obtain is not universal and depends on the size of the support of the initial data. We also perform numerical simulations and confirm that the numerically observed decay rate scales similarly to the rigorous lower bound, at least for a significant initial period of time. The main idea behind our proof is to use the recent work of Crippa and De Lellis (2008 *J. Reine Angew. Math.* **616** 15–46) making progress towards the resolution of Bressan's rearrangement cost conjecture.

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Nonlinearity **27** (2014) 973–985 doi:10.1088/0951-7715/27/5/973

Lower bounds on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows

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Theorem 1.1. *Let u be a smooth (time dependent) incompressible periodic vector field on the d-dimensional torus, and let θ solve (1.1) with periodic boundary conditions and L^∞ initial data θ_0. For any $p > 1$ and $\lambda \in (0, 1)$ there exists a length scale $r_0 = r_0(\theta_0, \lambda)$, an explicit constant $\varepsilon_0 = \varepsilon_0(\lambda, d)$, and a constant $c = c(d, p)$ such that*

$$\|\theta(t)\|_{H^{-1}} \geqslant \varepsilon_0 r_0^{d/2+1} \|\theta_0\|_{L^\infty} \exp\left(\frac{-c}{m(A_\lambda)^{1/p}} \int_0^t \|\nabla u(s)\|_{L^p} \, \mathrm{d}s \right). \tag{1.2}$$

Here A_λ is the super-level set $\{\theta_0 > \lambda \|\theta_0\|_{L^\infty}\}$.

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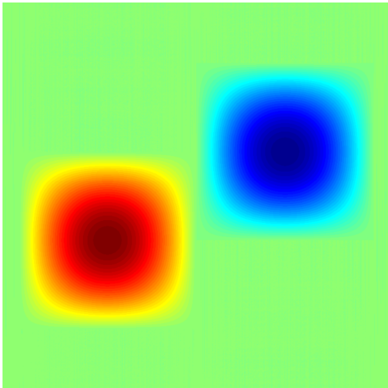
$$\theta_0'(x,y) = \begin{cases} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{2\pi (y+\frac{a}{8})}{a}\right) & \text{for } 0 < x < \frac{a}{2} \text{ and } \frac{-a}{8} < y < \frac{a}{2} - \frac{a}{8} \\ \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{2\pi (y-\frac{a}{8})}{a}\right) & \text{for } \frac{a}{2} < x < a \text{ and } \frac{a}{8} < y < \frac{a}{2} + \frac{a}{8} \\ 0 & \text{otherwise.} \end{cases}$$



(a) $t = 0$

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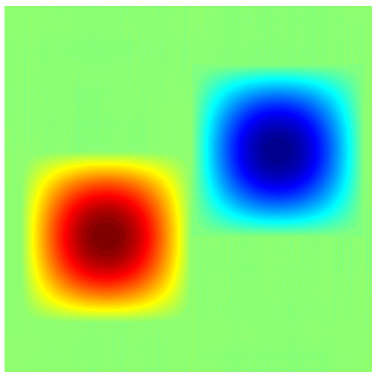
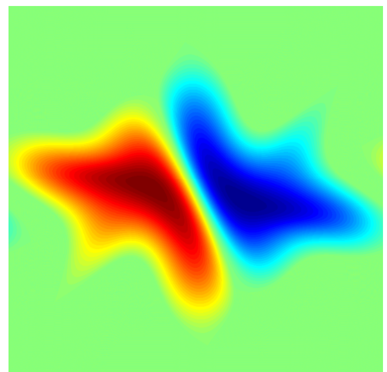
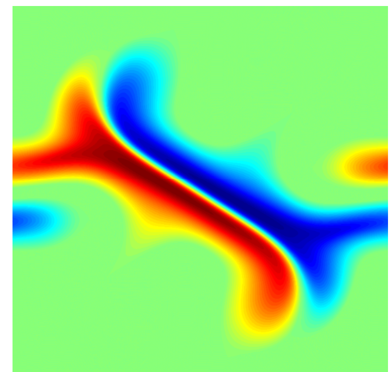


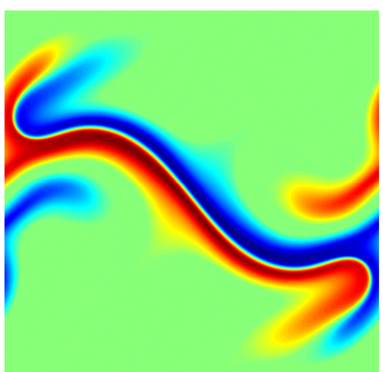
(a) $t = 0$



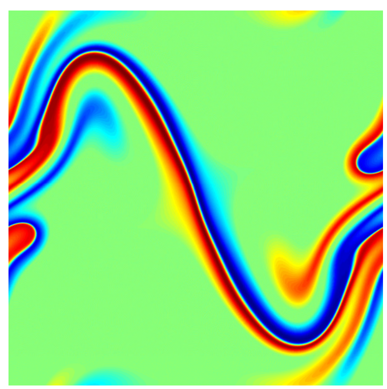
(b) $t = 1$



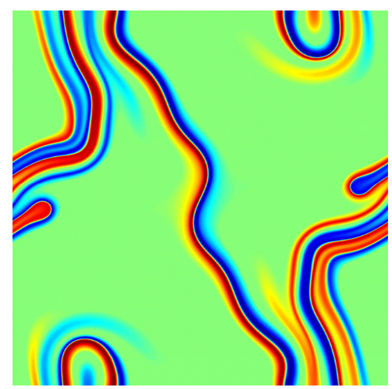
(c) $t = 2.05$



(d) $t = 3.1$



(e) $t = 4.15$



(f) $t = 5.19$

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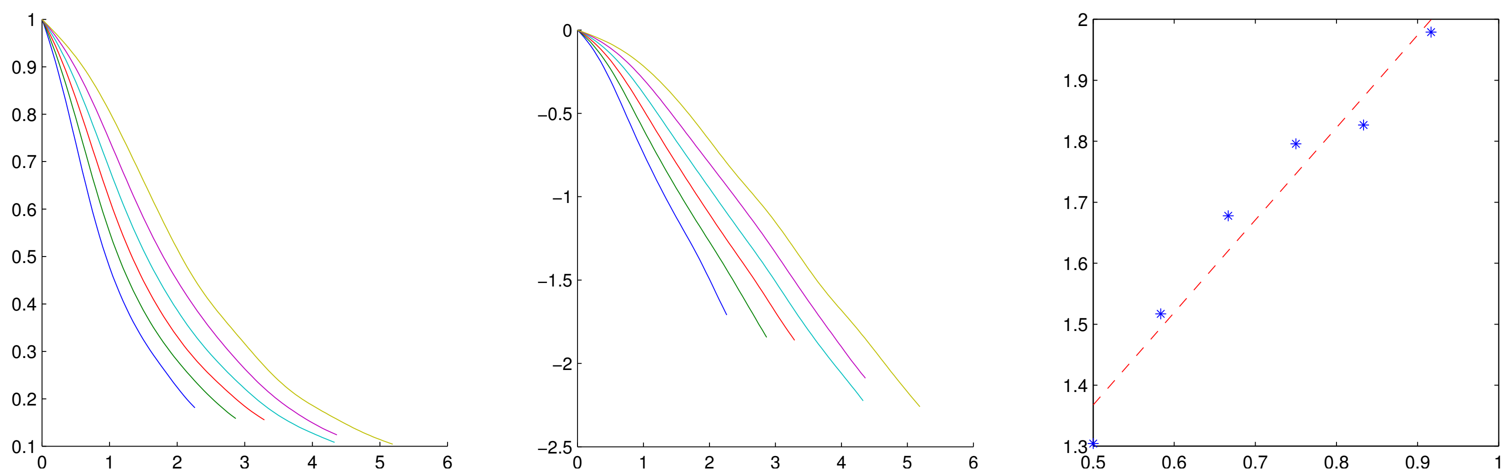

(a) $\|\theta(t)\|_{H^{-1}}$ vs t (b) $\ln\|\theta(t)\|_{H^{-1}}$ vs t (c) Exponential decay ~~rate~~.time

FIGURE 2. The mix norm of the scalar density (Figures (a) & (b)), and the negative reciprocal of the exponential decay rate vs a as a varies over $\{6/12, \ldots, 11/12\}$ (Figure (c)).

THANKS FOR YOUR ATTENTION!