Turbulent pair dispersion as a continuous-time random walk

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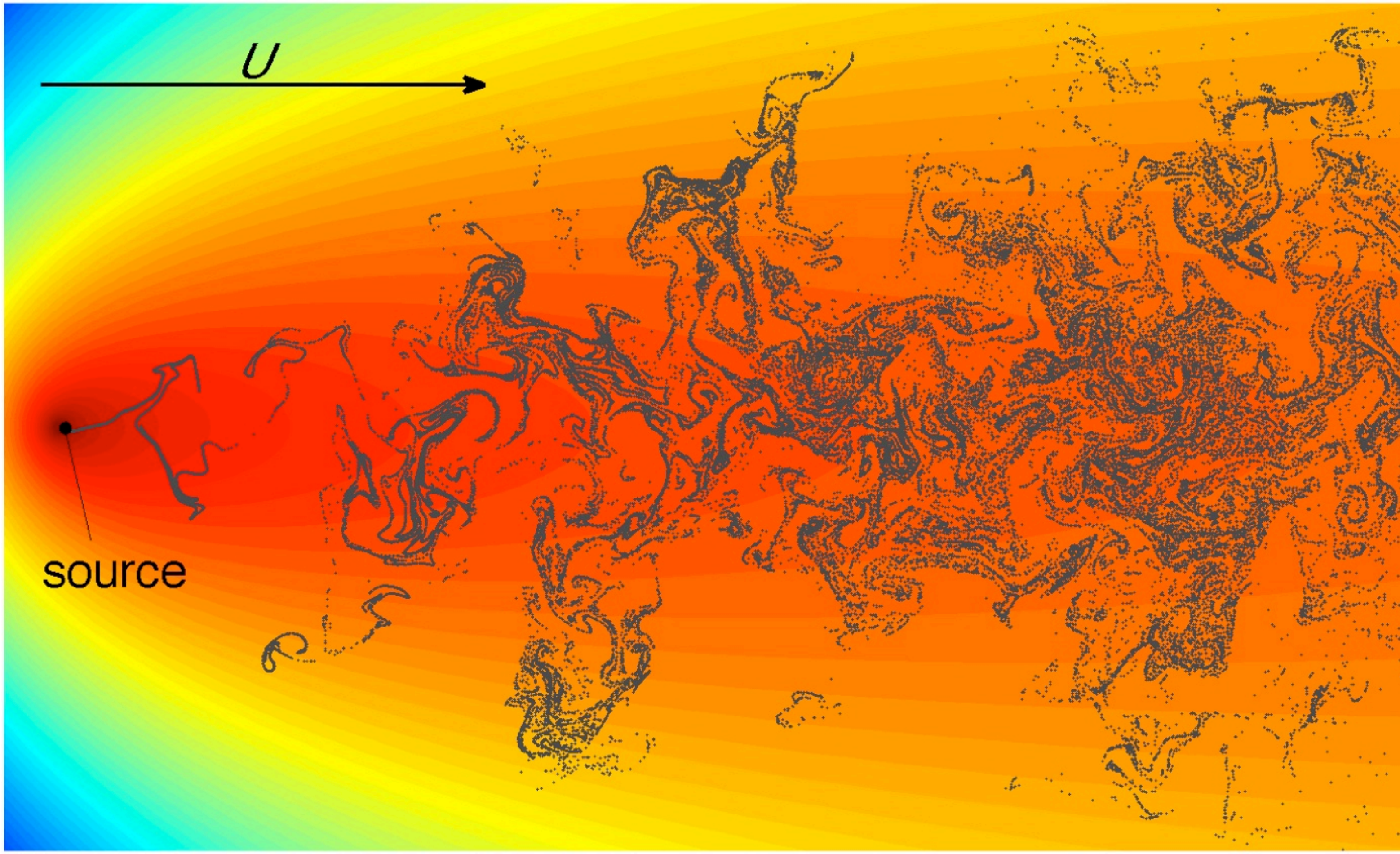
joint work with **Simon Thalabard** and **Giorgio Krstulovic**

arXiv:1405.7315

Fluctuations in turbulent transport

- Averaged concentration usually described by eddy diffusivity



- Spatial correlations relates to relative motion of tracers

Fluctuations and relative dispersion

- Tracers = characteristics of the advection equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t) = \boldsymbol{u}(\boldsymbol{X}(t),t) + \sqrt{2\kappa}\,\boldsymbol{\eta}(t) \;\Rightarrow\; \theta(\boldsymbol{x},t) = \langle \theta_0(\boldsymbol{X}(0)) \mid \boldsymbol{X}(t) = \boldsymbol{x}\rangle_\kappa$$

- Spatial correlations of the concentration

$$\langle \theta(\boldsymbol{x}+\boldsymbol{r},t)\,\theta(\boldsymbol{x},t)\rangle = \iint \langle \theta_0(\boldsymbol{x}_1^0)\,\theta_0(\boldsymbol{x}_2^0)\rangle\, p_2(\boldsymbol{x}+\boldsymbol{r},\boldsymbol{x},t\,|\,\boldsymbol{x}_1^0,\boldsymbol{x}_2^0,0)\,\mathrm{d}\boldsymbol{x}_1^0\mathrm{d}\boldsymbol{x}_2^0$$

$p_2(\boldsymbol{x}_1,\boldsymbol{x}_2,t\,|\,\boldsymbol{x}_1^0,\boldsymbol{x}_2^0,0)$ = joint transition probability density of two tracers $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$

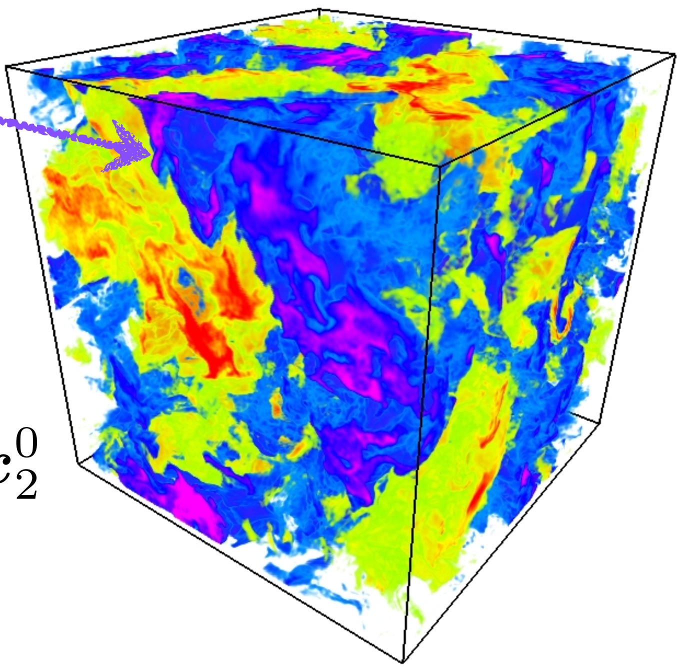
- Scalar dissipation anomaly

$$\varepsilon_\theta = -\kappa\langle(\nabla\theta)^2\rangle \to const$$

when $\kappa,\nu \to 0$ with fixed Pr

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\theta(\boldsymbol{x},t)^2\rangle = \iint \langle\theta_0(\boldsymbol{x}_1^0)\theta_0(\boldsymbol{x}_2^0)\rangle \times \partial_t p_2(\boldsymbol{x},\boldsymbol{x},t|\boldsymbol{x}_1^0,\boldsymbol{x}_2^0,0)\,\mathrm{d}\boldsymbol{x}_1^0\mathrm{d}\boldsymbol{x}_2^0$$

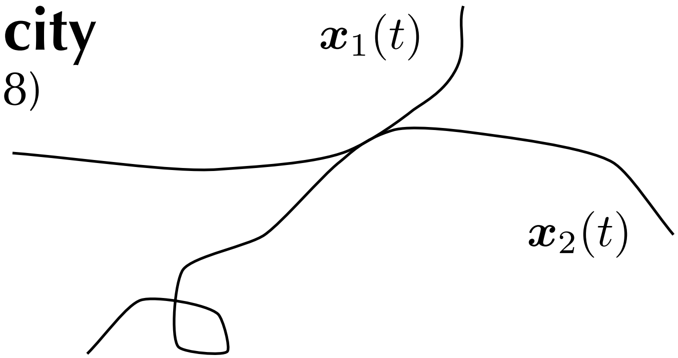
Fronts



Larchevêque & Lesieur, *J. Méc.* 1981
Nelkin & Kerr, *PoF* 1981 ; Thomson, *JFM* 1996

Turbulent dissipative anomaly

- **Generalized flows and spontaneous stochasticity**
  (Bernard *et al.*, *J. Stat. Phys.* 1998; Eyink, *Physica* D 2008)



$$|\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{u}(\boldsymbol{x}',t')|\sim|\boldsymbol{x}-\boldsymbol{x}'|^h$$

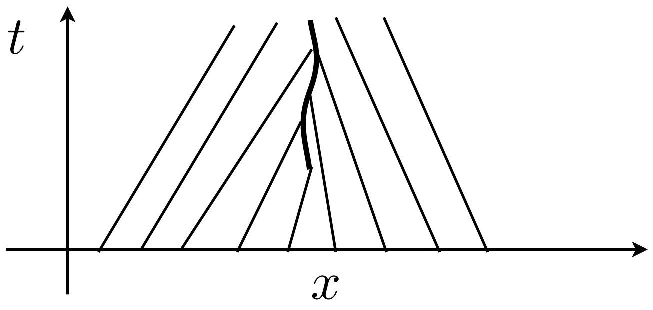
$h<1$ ⇒ not Lipschitz ⇒ non-uniqueness

- **Onsager's conjecture:** $h<1/3$ in order to dissipate energy
  (Duchon & Robert, *Nonlinearity* 2000)

"Local 4/5 law": $\varepsilon(\boldsymbol{x},t)=-\dfrac{3}{4}\lim\limits_{r\to 0}\dfrac{\langle\delta_r u^{\|}\,|\delta_r\boldsymbol{u}|^2\rangle_{\mathrm{ang}}}{r}$

⇒ close relation between energy dissipation in the limit $Re\to\infty$ and singular behaviors in particle separation

- Recently understood in the case of inviscid Burgers equation
  (Eyink & Drivas, *arXiv* 2014; Frishman & Falkovich, *arXiv* 2014)



Backward-in-time trajectories of entropy solutions are Markovian; velocity is a martingale

Pair dispersion

- Statistics of the two-point motion $\boldsymbol{R}(t) = \boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)$
  $\langle \cdot \rangle_{r_0}$ conditioned on a fixed initial distance $|\boldsymbol{R}(0)| = r_0$

- **Batchelor's ballistic regime:**

$$\left\langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^2 \right\rangle_{r_0} \propto (\varepsilon\, r_0)^{2/3}\ t^2$$

for $t \ll \tau_{r_0} \sim \varepsilon^{-1/3} r_0^{2/3}$ turnover time

Batchelor, *Proc. Camb. Phil. Soc.* 1952

- **Richardson–Obukhov *explosive* law:**

$$\left\langle |\boldsymbol{R}(t)|^2 \right\rangle_{r_0} \sim g\, \varepsilon\, t^3$$ for $\tau_{r_0} \ll t \ll T_{\mathrm{L}}$

Richardson, *Proc. Roy. Soc. Lond.* 1926
Obukhov, *Izv. Akad. Nauk SSSR* 1941

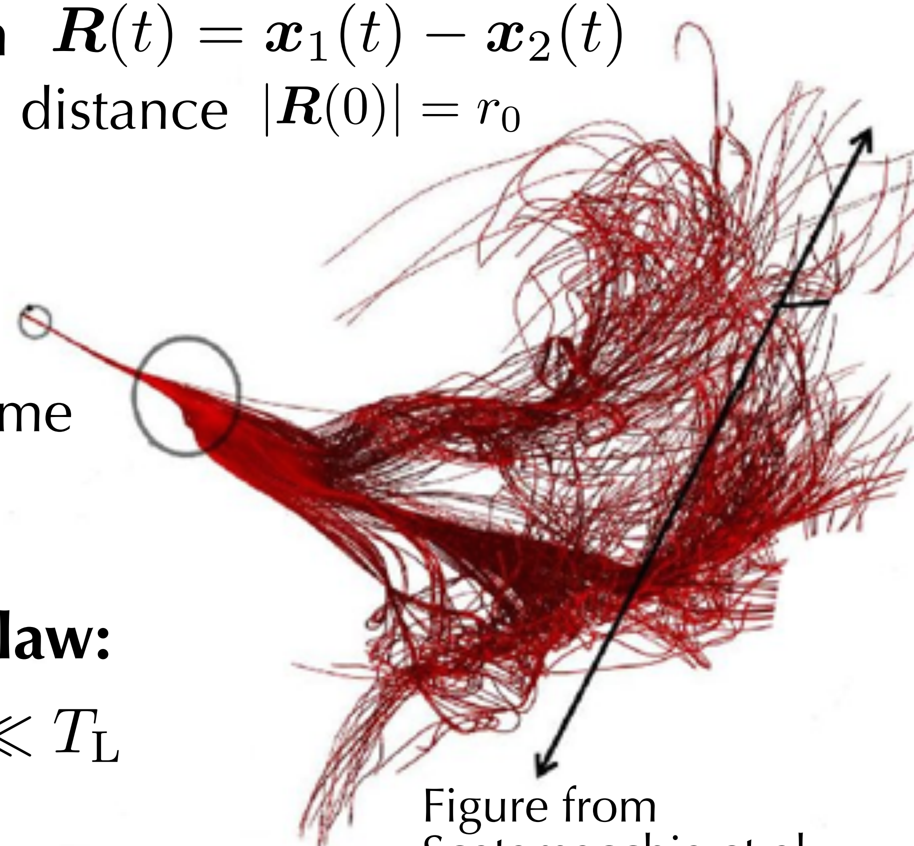


Figure from Scatamacchia et al., *PRL* 2013

Difficult to observe numerically and experimentally because of the large temporal scale separation that is required: $\tau_\eta \ll \tau_{r_0} \ll t \ll T_{\mathrm{L}}$

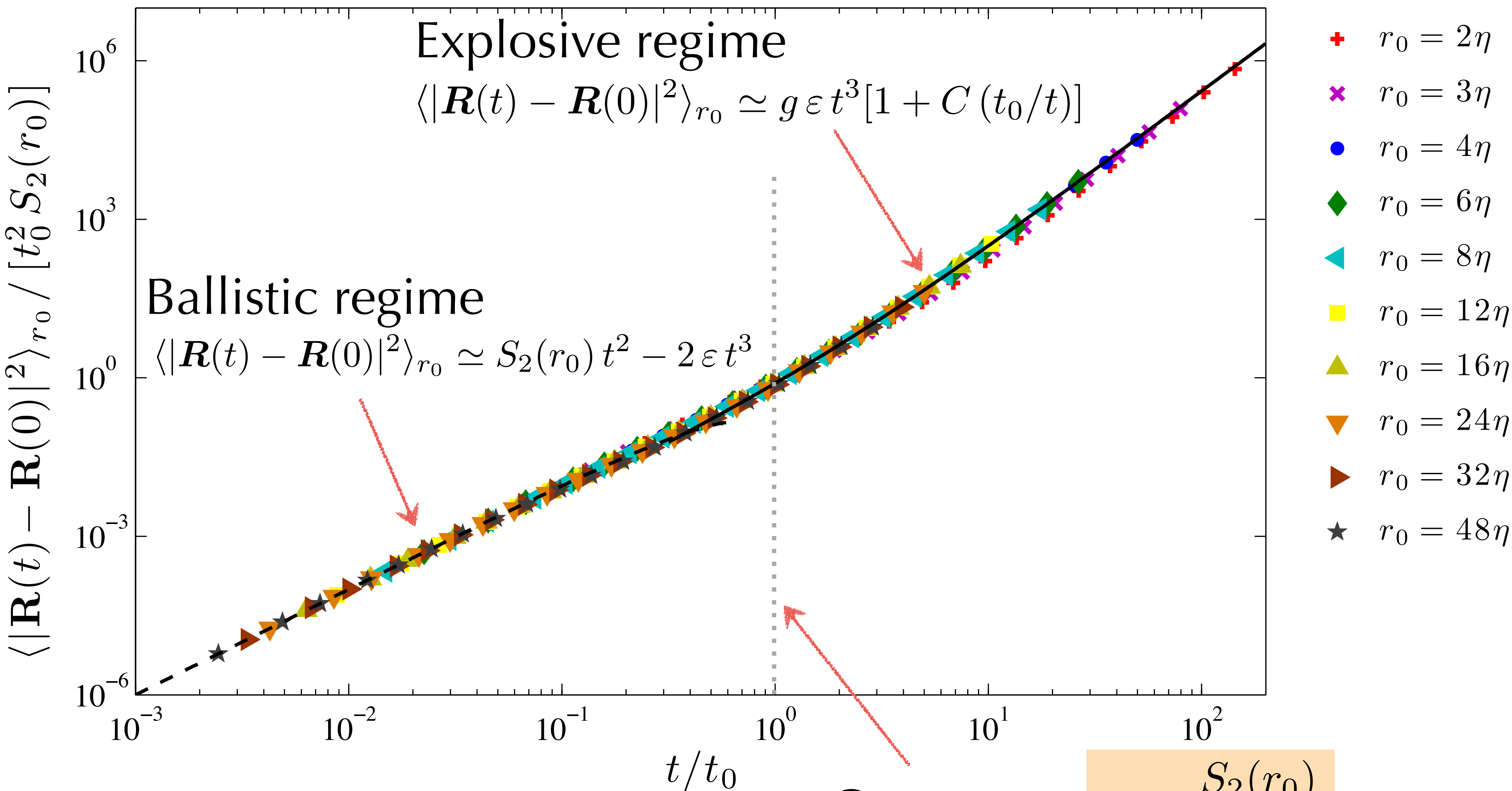
Review by Salazar & Collins
Ann. Rev. Fluid Mech. 2009

$\Rightarrow$ sub-leading terms? Mechanisms?

Transition Ballistic/Explosive

DNS: pseudo-spectral + large-scale forcing — Bitane *et al.* (*PRE* 2012)

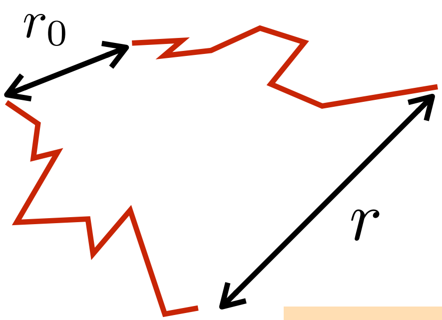
4096^3 collocation points $R_\lambda \approx 730$



Crossover: $\boxed{t_0 = \dfrac{S_2(r_0)}{2\varepsilon}}$

Richardson's diffusion

Assumption: velocity difference is **uncorrelated** ⇒ separation diffuses



Transition probability density $p_2(r,t|r_0,0)$

$$\partial_t p_2 = \nabla\cdot(K(r)\nabla p_2)$$

+ K41(Obukhov) $\quad K(r)\sim\varepsilon^{1/3}r^{4/3}$

$$\Rightarrow \boxed{p_2(r,t|r_0,0)\propto\frac{r^2}{t^{9/2}}\,\mathrm{e}^{-C\,r^{2/3}/(\varepsilon\,t)}} \quad \text{and} \quad \boxed{\left\langle|\boldsymbol{R}(t)|^2\right\rangle_{r_0}\sim g\,\varepsilon\,t^3}$$

Explosive growth: limiting distribution independent of initial separation r_0

Formalized for the Kraichnan model (Gaussian, δ-correlated velocities)
see Falkovich, Gawedzki, Vergassola, *Rev. Mod. Phys.* 2001

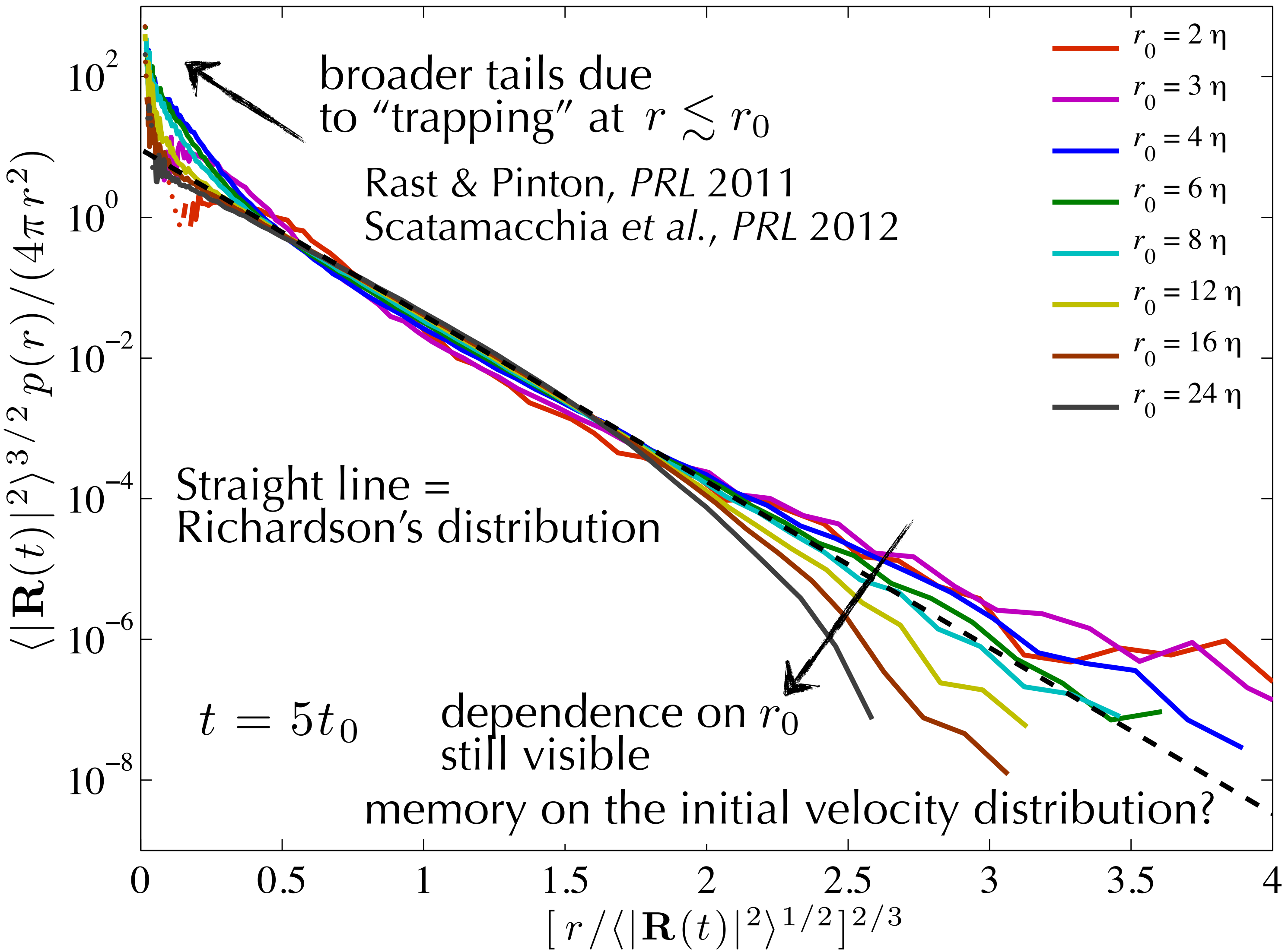
Physical shortcoming: velocity difference get uncorrelated on times O(t)

phenomenology ⇒ correlation time $\quad\tau_r\sim r^{2/3}$

$$+r^2\sim t^3 \quad\Rightarrow\quad \boxed{\tau_r\sim t}$$

Distribution of distances

Comparison to Richardson's distribution $\quad p_2(r,t|r_0,0) \propto \dfrac{r^2}{t^{9/2}}\,\mathrm{e}^{-C\,r^{2/3}/(\varepsilon\,t)}$



Such a representation emphasizes the collapse of the core of the distribution…

Markovian approaches

Assumption: acceleration differences are **short correlated**

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{A} = \delta \mathrm{D}_t \boldsymbol{u} \longleftarrow$$ components correlated over a time $\mathrm{O}(\tau_\eta)$

Central-Limit Theorem: $\boxed{\boldsymbol{A} \overset{\text{law}}{\equiv} \sqrt{\tau_\eta}\,\mathbb{A}(\boldsymbol{R},\boldsymbol{V}) \circ \boldsymbol{\eta}(t)}$ when $t \gg \tau_\eta$

with $\mathbb{A}^\mathsf{T}\mathbb{A} = \langle \delta \mathrm{D}_t \boldsymbol{u} \otimes \delta \mathrm{D}_t \boldsymbol{u} \,|\, \delta \boldsymbol{u} \rangle$ correlations of acceleration differences conditioned on $\delta\boldsymbol{u}$

General form: $$\begin{cases} \mathrm{d}\boldsymbol{R} = \boldsymbol{V}\,\mathrm{d}t \\ \mathrm{d}\boldsymbol{V} = \boldsymbol{a}(\boldsymbol{R},\boldsymbol{V},t)\mathrm{d}t + \mathbb{B}(\boldsymbol{R},\boldsymbol{V},t)\,\mathrm{d}\boldsymbol{W} \end{cases}$$

Kurbanmuradov & Sabelfeld (1995); Sawford (2001)

$\Rightarrow$ Fokker–Planck equation for $p(\boldsymbol{r},\boldsymbol{v},t|\boldsymbol{r}_0,\boldsymbol{v}_0,0)$

$$\partial_t p + \partial_{r_i}(v_i\,p) + \partial_{v_i}(a_i\,p) = \frac{1}{2}\partial_{v_i}\partial_{v_j}[B_{ik}B_{jk}\,p]$$

Admissibility condition: "well-mixing"

Consistency with Eulerian statistics: $p_E(\boldsymbol{r},\boldsymbol{v})$ is a stationary solution associated to an initial uniform distribution in space (Thomson 1991)

Limits of Markov modeling

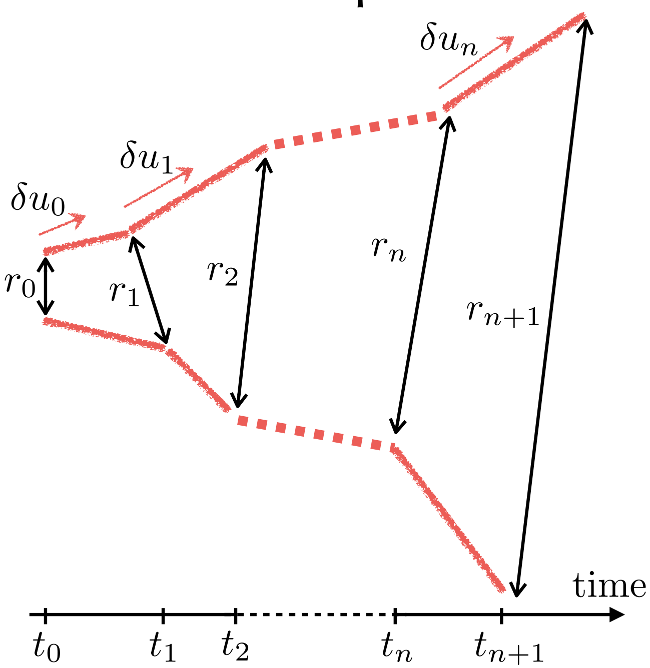
- Is acceleration really short-time correlated?
  - ⇒ OK for components but not amplitude (Mordant *et al.*, *PRL* 2004)
  - ⇒ Stretched exponential correlations (non-mixing process)

- Most models lead to an asymptotic diffusion of velocities.
  Is this the mechanism explaining Richardson's scaling $R \sim t^{3/2}$?
  - ⇒ Is it compatible with the observed intermittent behaviors?
    e.g. for exit times (Boffetta & Sokolov, *PRL* 2002)
  - ⇒ Are finite-*Re* effects solely responsible for lack of scaling?
    (Scatamacchia *et al.*, *PRL* 2012)

- Is turbulent relative motion really a Markov process?
  - ⇒ Relation to Lévy walks / waiting times approaches
    (Shlesinger *et al.*, *PRL* 1987; Faller, *JFM* 1996; Rast & Pinton, *PRL* 2011)
  - ⇒ Some deviations might be due to memory effects
    (Ilyin *et al.*, *PRE* 2010; Eyink & Benveniste, *PRE* 2013)

A piecewise-ballistic scenario

- Ballistic regime is key in the convergence to the explosive behavior
- Build a simple model that reproduces some essential mechanisms



Continuous-time random walk

$$\vec{r}_n \mapsto \vec{r}_{n+1} = \vec{r}_n + \Delta t_n \delta \vec{u}_n$$
$$t_n \mapsto t_{n+1} = t_n + \Delta t_n$$

$$\begin{cases} \delta u_n \text{ and } \Delta t_n \text{ depend on } r_n \\ \text{the } \delta u_n\text{'s are independent from each other} \end{cases}$$

non-Markovian w.r.t. to the continuous time

K41 version: $\begin{cases} \delta u_n \sim (\varepsilon\, r_n)^{1/3} \\ \Delta t_n \sim r_n/\delta u_n \sim \varepsilon^{-1/3} r_n^{2/3} \end{cases}$

$$\left.\begin{array}{l} r_{n+1} \simeq r_n + X_n\, r_n \quad \Rightarrow \ln(r_n/r_0) \simeq \displaystyle\sum_k \ln(1+X_k) \propto n \\ t_{n+1} \simeq t_n + Y_n\, r_n^{2/3} \quad \Rightarrow t_n \simeq \displaystyle\sum_k Y_n\, r_n^{2/3} \propto \mathrm{e}^{\frac{2}{3}n} \end{array}\right\} \Rightarrow \ln(r_t/r_0) \simeq \frac{3}{2} \ln(t/t_0)$$

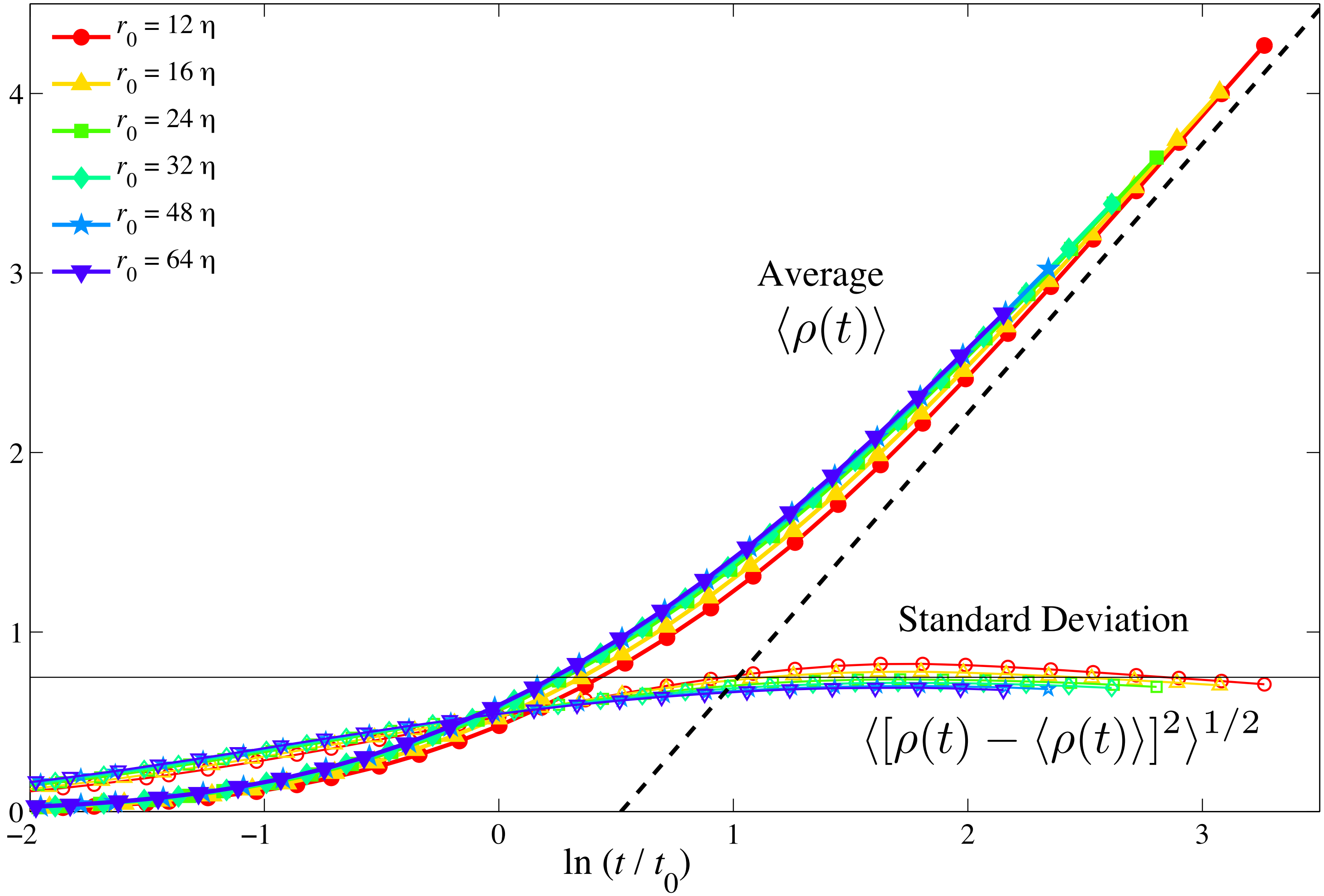
- Is $\ln(|\boldsymbol{R}(t)|/r_0)$ a **self-averaging quantity**?
  Law of large numbers? Central-limit theorem? Large deviations?

Are distances a multiplicative process?

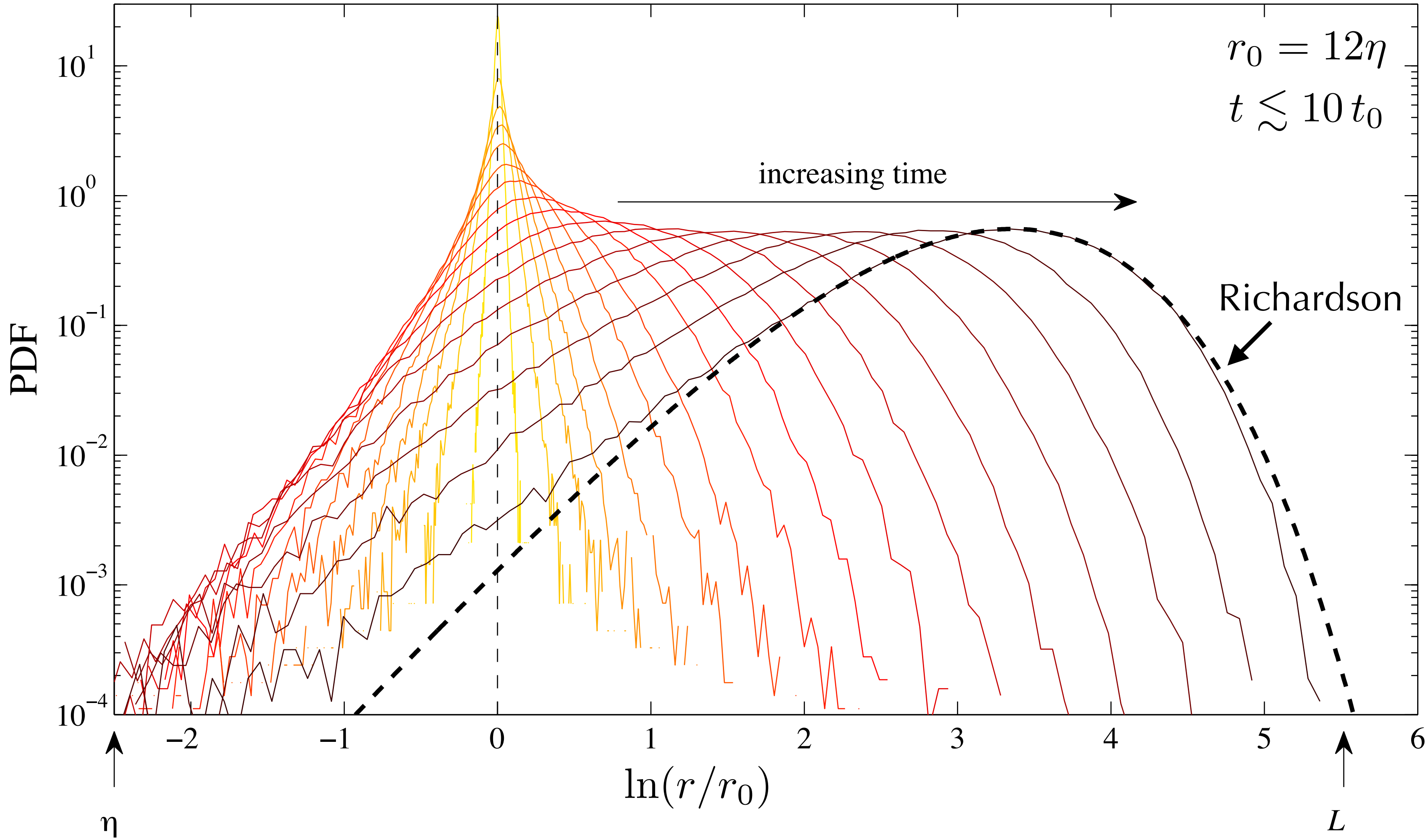
- The ballistic scenario suggests $\rho = \ln(|\boldsymbol{R}(t)|/r_0)$ as a relevant quantity

  Richardson's distribution: $\langle \rho(t) \rangle = (3/2)\ln(t/t_0) + (1/2)\ln g - 0.46$

  $$\langle [\rho(t) - \langle \rho(t) \rangle]^2 \rangle^{1/2} = 0.748$$



Probability distribution of log-distances



Further modeling

Time increment: dissipation time $\Delta t_n = |\delta\vec{u}_n|^2/\varepsilon$

$\alpha_n = \delta u_n^{\parallel}/|\delta\vec{u}_n|$

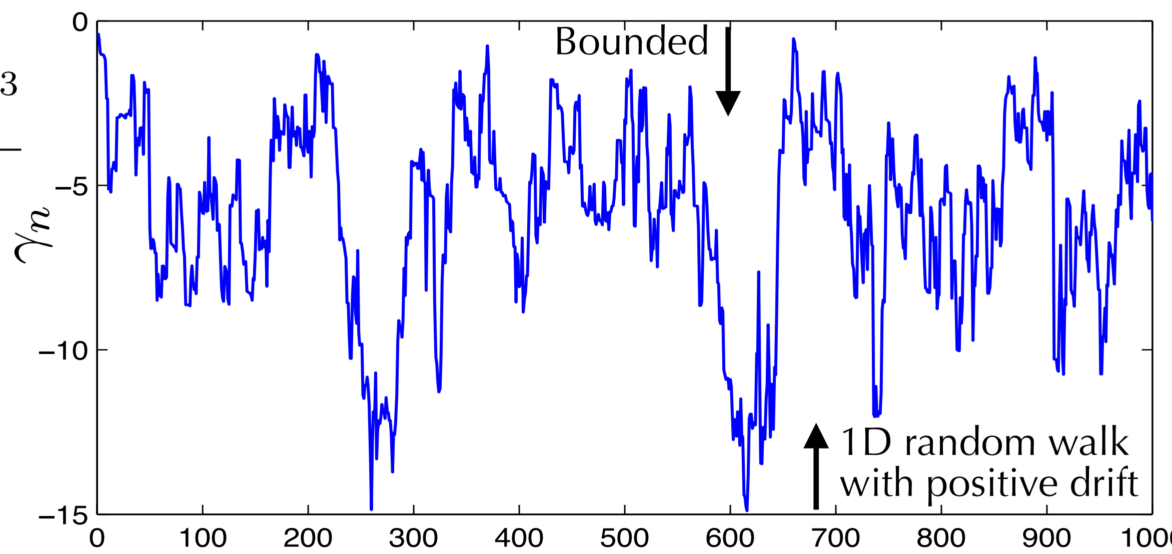
$\beta_n = |\delta\vec{u}_n|^3/(\varepsilon r_n)$

with statistics independent of r_n (K41)

$$\begin{cases} r_{n+1} = r_n\sqrt{1+2\alpha_n\beta_n+\beta_n^2} \\ t_{n+1} = t_n + \varepsilon^{-1/3}\beta_n^{2/3}r_n^{2/3} \end{cases}$$

Change of variables: $\gamma_n = \ln(r_n/r_0) - (3/2)\ln(t/t_0)$ $\quad t_0 = \varepsilon^{-1/3}r_0^{2/3}$

$$\gamma_{n+1} = \frac{3}{2}\ln\frac{\left(1+2\alpha_n\beta_n+\beta_n^2\right)^{1/3}}{\beta_n^{2/3}+\mathrm{e}^{-\frac{2}{3}\gamma_n}}$$



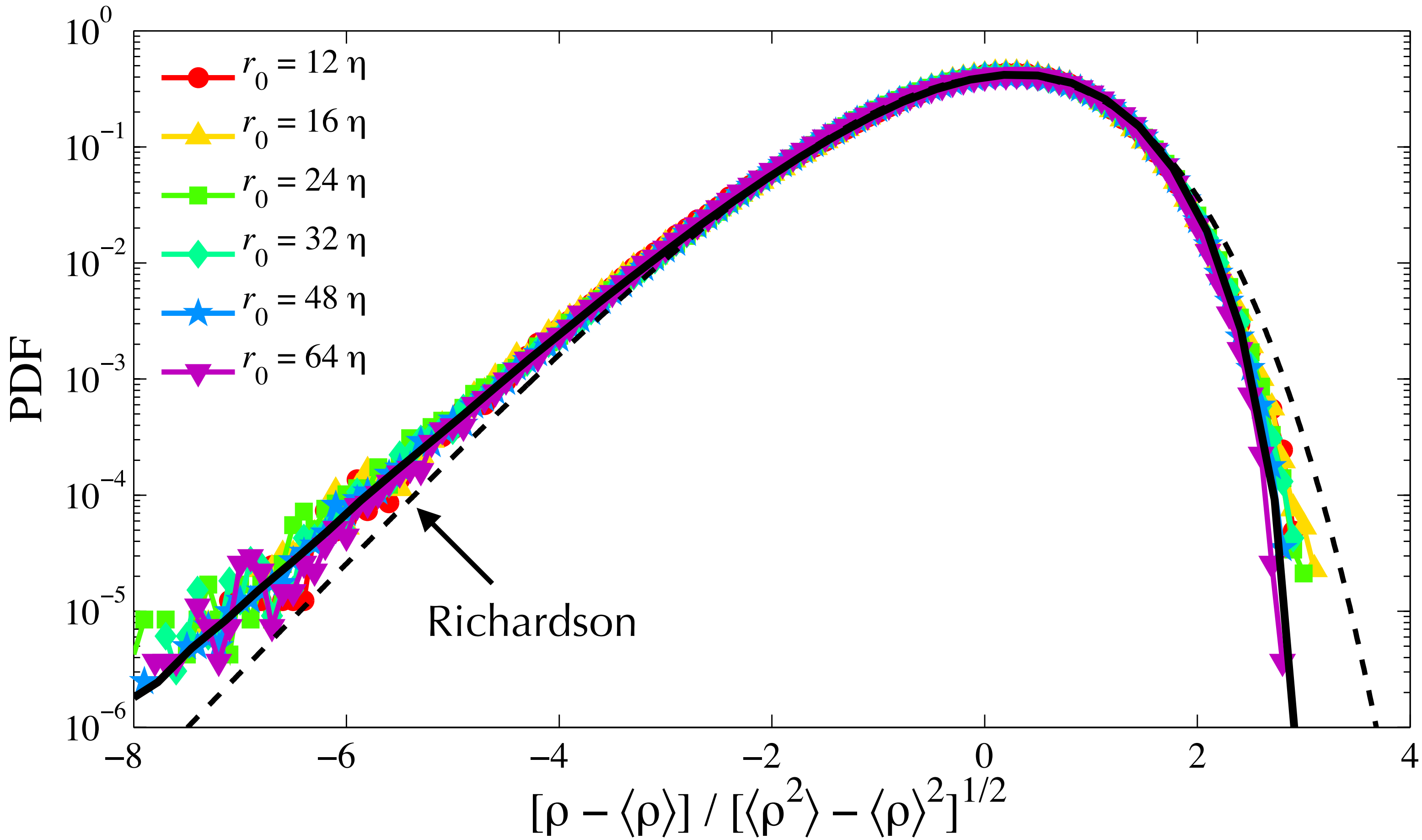
The γ_n's are becoming **stationary**

This suggests for $\rho = \ln(|\boldsymbol{R}(t)|/r_0)$

$\langle\rho\rangle \simeq (3/2)\ln(t/t_0) + \langle\gamma\rangle$ $\quad \mathrm{Var}\,[\rho] \simeq \mathrm{Var}\,[\gamma] = \mathrm{const}$ $\quad \mathrm{PDF}\,(\rho) \simeq \Psi(\rho - \langle\rho\rangle)$

Distribution of the log-separation

Scale invariance for the distribution of $\rho = \ln(|\boldsymbol{R}(t)|/r_0)$



The collapsing distribution can be reproduced by properly choosing the distribution of $\alpha_n = \delta u_n^{\parallel}/|\delta\vec{u}_n|$ and $\beta_n = |\delta\vec{u}_n|^3/(\varepsilon r_n)$

Open questions / Extensions

$$\begin{cases} r_{n+1} = r_n\sqrt{1+2\alpha_n\beta_n+\beta_n^2} \\ t_{n+1} = t_n + \varepsilon^{-1/3}\beta_n^{2/3}r_n^{2/3} \end{cases}$$

$$\alpha_n = \delta u_n^{\parallel}/|\delta\vec{u}_n|$$
$$\beta_n = |\delta\vec{u}_n|^3/(\varepsilon r_n)$$

- **Effect of the fluid velocity intermittency**

How is the scaling behavior affected when K41 is not fulfilled?
⇒ Studying extensions of the model assuming multifractal statistics
e.g. $\beta_n \propto r_n^{3h_n-1}$ with $p(h_n) \propto r_n^{3-D(h_n)}$
How is scale invariance broken?

- **Time irreversibility**

Relative dispersion is faster backward in time than forward
What are the underlying mechanisms? How to quantify?
⇒ In the model, the only symmetry-breaking quantity is α_n
How is the "Richardson constant" altered when $\alpha_n \mapsto -\alpha_n$?