Inertial effects on tumbling in a simple shear flow

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Outline

- Jeffery's equation (w/ special detour)
- Inertial corrections
- Sketch of the calculation
- Results

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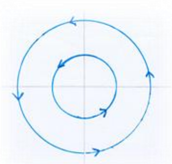
Jeffery eq:

$$\dot{\boldsymbol{n}} = \mathbb{O}\boldsymbol{n} + \Lambda\left(\mathbb{S}\boldsymbol{n} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}\mathbb{S}\boldsymbol{n}\right).$$

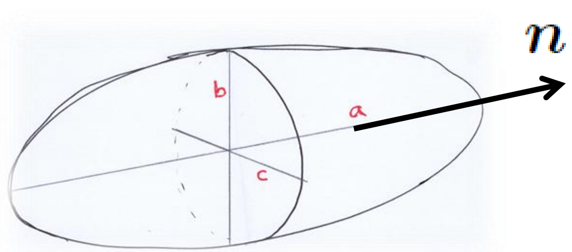


Vorticity $\mathbb{O}$ Strain $\mathbb{S}$

$$\mathbb{O} = \frac{1}{2}(\mathbb{A} - \mathbb{A}^{\mathrm{T}}), \quad \mathbb{S} = \frac{1}{2}(\mathbb{A} + \mathbb{A}^{\mathrm{T}}), \quad \mathbb{A} = \mathbb{O} + \mathbb{S}.$$

$$A_{ij} = \frac{\partial u_i}{\partial x_j}. \qquad \Lambda = \frac{\lambda^2 - 1}{\lambda^2 + 1}.$$

$\lambda = \frac{a}{c}$



Jeffery, G. B. *Proc. R. Soc. Lond. A* **102,** 161–179 (1922).

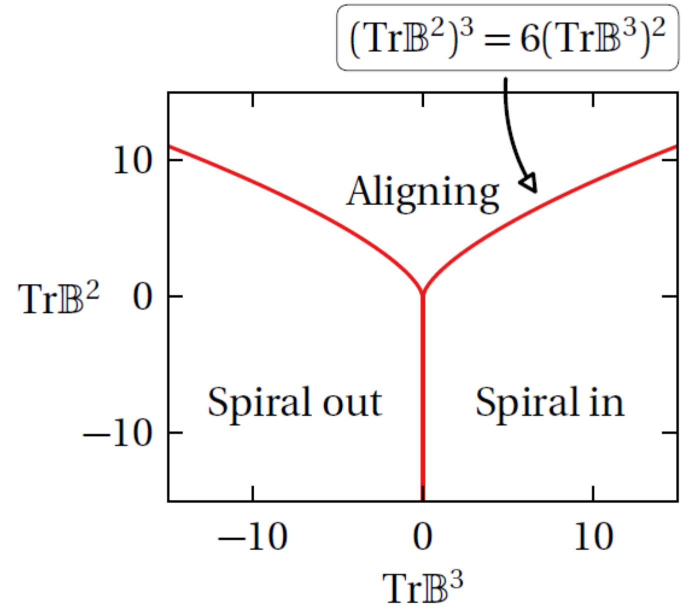
Its solution:

$$\dot{\boldsymbol{q}} = (\mathbb{O} + \Lambda\mathbb{S})\boldsymbol{q}. \qquad \boldsymbol{q}(t) = e^{(\mathbb{O}+\Lambda\mathbb{S})t}\boldsymbol{q}(0).$$

$$\mathbb{B} = \mathbb{O} + \Lambda\mathbb{S} \qquad \boldsymbol{n}(t) = \frac{\boldsymbol{q}(t)}{|\boldsymbol{q}(t)|}.$$

Rod: $\mathbb{B} \longrightarrow \mathbb{A}$

Disk: $\mathbb{B} \longrightarrow -\mathbb{A}^{\mathrm{T}}$

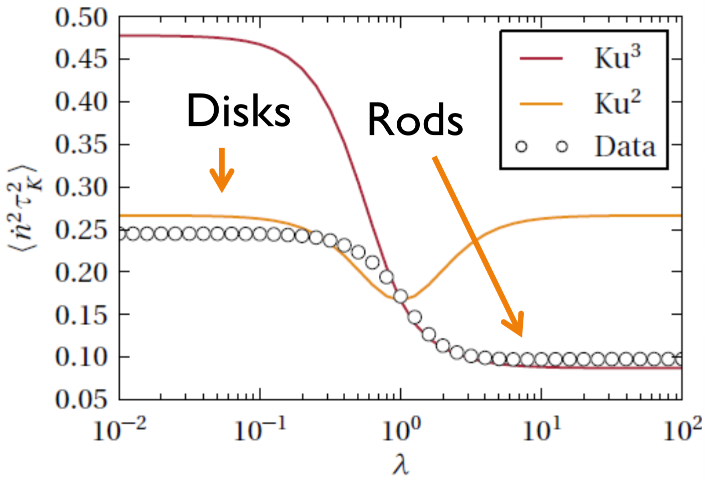


$$\mathrm{Tr}\mathbb{B}^3 = 3\Lambda\mathrm{Tr}\mathbb{O}\mathbb{O}\mathbb{S} + \Lambda^3\mathrm{Tr}\mathbb{S}\mathbb{S}\mathbb{S}.$$

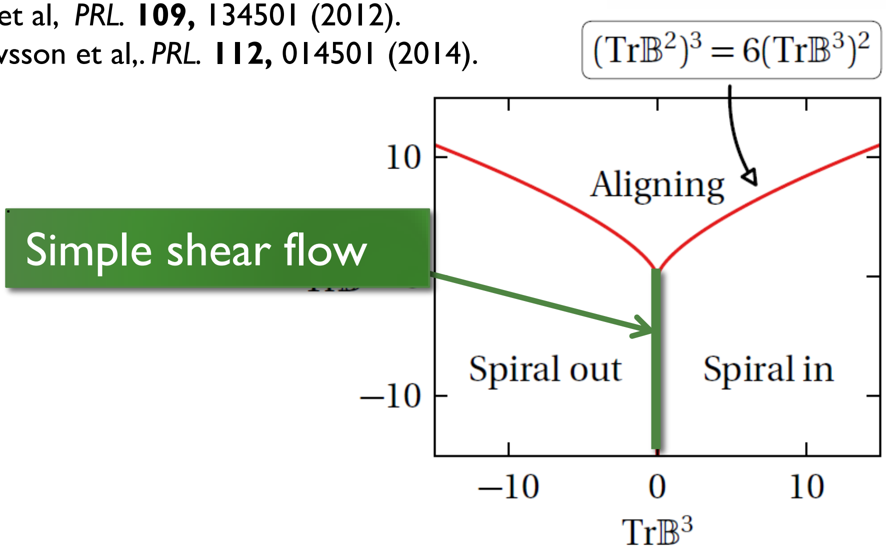
Bretherton, F. P. *JFM* **14,** 284–304 (1962).

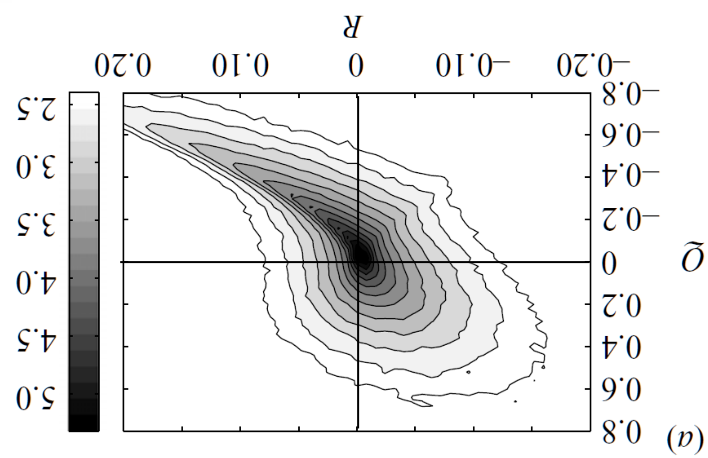
Jeffery detour: Tumbling in turbulence

Tumbling rate in isotropic turbulence (DNS)



Parsa et al, *PRL.* **109,** 134501 (2012).
Gustavsson et al,. *PRL.* **112,** 014501 (2014).





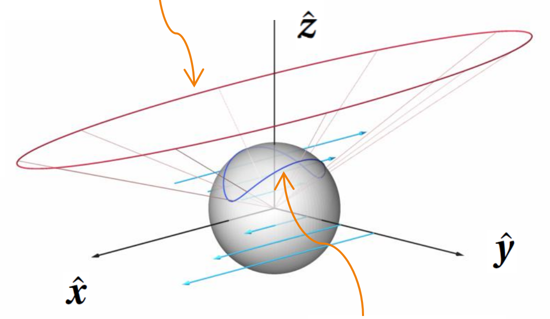
Lüthi et al (2009) JFM 641 (497)

Simple shear – pathological case?

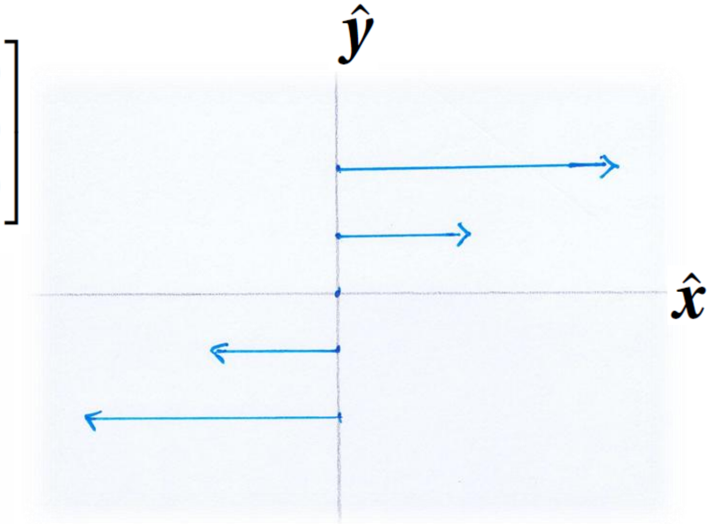
$$\boldsymbol{u}(y) = s\, y\, \hat{\boldsymbol{x}}$$

$$\mathbb{A} = \begin{bmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{q}(t) = e^{(\mathbb{O}+\Lambda\mathbb{S})t}\, \boldsymbol{q}(0)$$



$$\boldsymbol{n}(t) = \frac{\boldsymbol{q}(t)}{|\boldsymbol{q}(t)|}$$



Although this limit may seem to be of limited interest, ...

... these small changes can have a strong (or even dominant) cumulative effect on the particle's position or orientation. This occurs for the class of so-called "indeterminate" particle motions, in which *no* position or orientation is intrinsically favored under "standard conditions."

L. G. Leal

Ann. Rev. Fluid Mech. 1980.

Ancestry – some inertial corrections

Phys. Fluids 26, 883 (1983); doi: 10.1063/1.864230

Equation of motion for a small rigid sphere in a nonuniform flow

Martin R. Maxey[a]
Department of Chemical Engineering, The Johns Hopkins University, Baltimore, Maryland 21218
James J. Riley
Flow Research Company, Kent, Washington 98031

Int. J. Multiphase Flow Vol. 16, No. 1, pp. 153–166, 1990

THE ACCELERATED MOTION OF RIGID BODIES IN NON-STEADY STOKES FLOW

E. GAVZE

J. Fluid Mech. (1965), *vol.* 22, *part* 2, *pp.* 385–400

The lift on a small sphere in a slow shear flow

By P. G. SAFFMAN

J. Fluid Mech. (1991), *vol.* 224, *pp.* 261–274

Inertial migration of a small sphere in linear shear flows

By JOHN B. McLAUGHLIN

J. Fluid Mech. (2005), *vol.* 535, *pp.* 383–414.

Inertial effects on fibre motion in simple shear flow

By G. SUBRAMANIAN AND DONALD L. KOCH

J. Fluid Mech. (2006), *vol.* 557, *pp.* 257–296.

Inertial effects on the orientation of nearly spherical particles in simple shear flow

By G. SUBRAMANIAN AND D. L. KOCH

$$\rho_f\left(\frac{\partial}{\partial t}\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=-\nabla p+\mu\nabla^2\boldsymbol{u}$$

Governing equations and numbers

Particle Eq. of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} n_i = \varepsilon_{ijk}\omega_j n_k$$

$$\mathrm{St}\frac{\mathrm{d}}{\mathrm{d}t}\left[I_{ij}\omega_j\right] = T_i$$

$$I_{ij} = A^I n_i n_j + B^I(\delta_{ij} - n_i n_j)$$

Dimensionless numbers

$$\mathrm{Re}_s = \frac{s a^2 \rho_f}{\mu} \qquad \mathrm{St} = \frac{\rho_p}{\rho_f}\mathrm{Re}_s$$

$$\text{particle aspect ratio } \lambda$$

Strongly coupled system of ODEs and PDE. To proceed:

- Reciprocal theorem for T_i
- Perturbation theory in Re_s , St

Fluid Eq. of motion

$$\mathrm{Re}_s\left(\partial_t u_i + u_j \partial_j u_i\right) = -\partial_i p + \partial_j \partial_j u_i$$

$$u_i = \varepsilon_{ijk}\omega_j r_k \qquad \boldsymbol{r} \in S\,, \qquad \partial_i u_i = 0$$

$$u_i = u_i^\infty, \qquad |\boldsymbol{r}| \to \infty\,.$$

Aim: Effective vector field, correction to Jeffery's Equation

$$\dot{n}_i = F_i^{(0)}(\boldsymbol{n}) + \mathrm{St}F_i^{(\mathrm{St})}(\boldsymbol{n}) + \mathrm{Re}_s F_i^{(\mathrm{Re}_s)}(\boldsymbol{n})$$

Particle inertia (Re = 0)

Particle Eq. of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} n_i = \varepsilon_{ijk}\omega_j n_k$$

$$\mathrm{St}\frac{\mathrm{d}}{\mathrm{d}t}\left[I_{ij}\omega_j\right] = T_i$$

$$I_{ij} = A^I n_i n_j + B^I(\delta_{ij} - n_i n_j)$$

Step 1: Perturbation ansatz in angular velocity

$$\omega_i = \omega_i^{(0)} + \mathrm{St}\,\omega_i^{(\mathrm{St})}$$

Step 2: Write down the Jeffery torque

$$T_j^{(0)} = c_\xi\left(A^R n_j n_k + B^R(\delta_{jk} - n_j n_k)\right)\left(\Omega_k^\infty - \omega_k\right) + c_\xi C^R \varepsilon_{jkm} n_k n_l S_{ml}^\infty$$

Step 3: Insert all of this into Eq. of motion and solve order by order:

$$\dot{n}_i = \varepsilon_{ipq}\Omega_p^\infty n_q + \frac{C^R}{B^R}\left(S_{ip}^\infty n_p - n_i n_p n_q S_{pq}^\infty\right) +$$

$$\mathrm{St}\frac{B^I C^R}{c_\xi (B^R)^2}\left[-(\delta_{ij} - n_i n_j) S_{jk}^\infty \varepsilon_{klm}\omega_l^{(0)} n_m + \varepsilon_{ijk}\omega_j^{(0)} n_k n_l S_{lm}^\infty n_m\right]$$

$$+\,\mathrm{St}\frac{A^I - B^I}{c_\xi B^R}(\delta_{ij} - n_i n_j)\omega_j^{(0)} n_k \omega_k^{(0)}$$

$$\boxed{\dot{n}_i = F_i^{(0)}(\boldsymbol{n}) + \mathrm{St}\,F_i^{(\mathrm{St})}(\boldsymbol{n})}$$

Effect of particle inertia: "orbit drift"

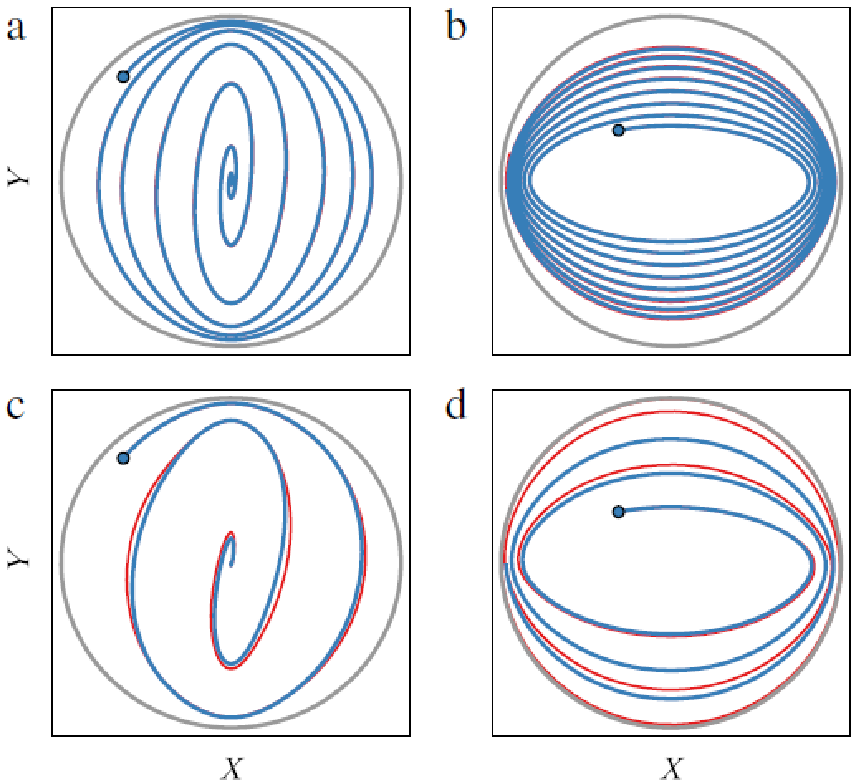
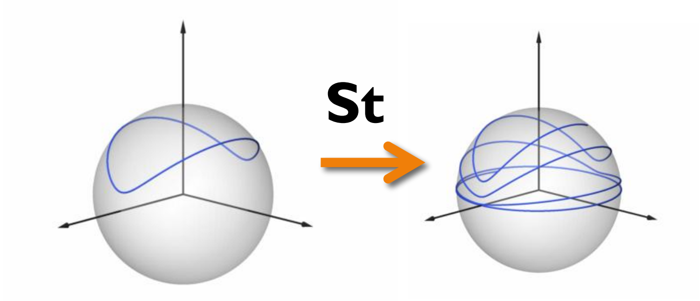
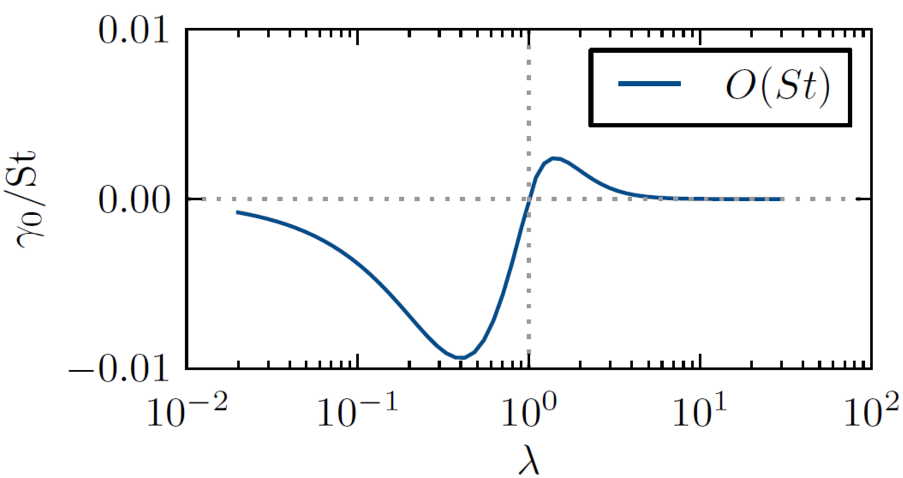


Fig. 1. Numerical solutions of Eq. (1) (red), and Eq. (8) (blue) for four different particles in a simple shear flow. (a) $\lambda = 1/5$, $\widetilde{\mathrm{St}} = 0.1$; (b) $\lambda = 5$, $\widetilde{\mathrm{St}} = 0.25$; (c) $\lambda = 1/5$, $\widetilde{\mathrm{St}} = 0.25$; (d) $\lambda = 5$, $\widetilde{\mathrm{St}} = 0.7$. Trajectories on the sphere are shown



Linear stability of "log rolling" orbit



Higher St, see e.g
Lundell, F. & Carlsson, A. *PRE* **81**, 016323 (2010).

Einarsson, J., Angilella, J. R. & Mehlig, B. *Physica D* **278–279**, 79–85 (2014).

Finite Re

Fluid Eq. of motion

$$\mathrm{Re}_s\left(\partial_t u_i + u_j \partial_j u_i\right) = -\partial_i p + \partial_j \partial_j u_i$$

$$u_i = \varepsilon_{ijk}\omega_j r_k \qquad \boldsymbol{r} \in S\,, \qquad \partial_i u_i = 0$$

$$u_i = u_i^\infty\,, \qquad |\boldsymbol{r}| \to \infty\,.$$

- Problem:
  - Need to calculate the torque on the particle to O(Re)
  - ... but we don't want to solve Navier-Stokes equations.
- Resolution: A reciprocal theorem
  - One solvable auxiliary problem: $(\tilde{u}_i, \tilde{\sigma}_{ij})$
  - The disturbance flow of the "real" problem: (u_i', σ_{ij}')
  - The trick is: choose auxiliary problem to make surface integral proportional to torque:

$$\int_S \mathrm{d}\tilde{F}_i u_i' + \int_V \mathrm{d}V\, u_i' \partial_j \tilde{\sigma}_{ij} = \int_S \mathrm{d}F_i' \tilde{u}_i + \int_V \mathrm{d}V\, \tilde{u}_i \partial_j \sigma_{ij}'$$

Kim, S. *Microhydrodynamics: principles and selected applications*. (1991)
Subramanian, G. & Koch, D. L. *Journal of Fluid Mechanics* **535,** 383–414 (2005).

Finite Re

$$T_j = T_j^{(0)} - \mathrm{Re}_s \int_V \mathrm{d}V \tilde{U}_{ij} f_i(\boldsymbol{u}')$$

Solutions to Stokes Equation

Jeffery's Torque

$$f_i(\boldsymbol{u}') = \partial_t u_i' + u_j^\infty \partial_j u_i' + u_j' \partial_j u_i^\infty + u_j' \partial_j u_i'$$

Repeat Step 3: Insert all of this into Eq. of motion and solve order by order:

$$\begin{aligned}
\dot{n}_i = {} & \varepsilon_{ipq}\Omega_p^\infty n_q + \frac{C^R}{B^R}\left(S_{ip}^\infty n_p - n_i n_p n_q S_{pq}^\infty\right) + \\
& \mathrm{St}\frac{B^I C^R}{c_\xi (B^R)^2}\left[-(\delta_{ij} - n_i n_j) S_{jk}^\infty \varepsilon_{klm}\omega_l^{(0)} n_m + \varepsilon_{ijk}\omega_j^{(0)} n_k n_l S_{lm}^\infty n_m\right] \\
& + \mathrm{St}\frac{A^I - B^I}{c_\xi B^R}(\delta_{ij} - n_i n_j)\omega_j^{(0)} n_k \omega_k^{(0)} \\
& + \frac{\mathrm{Re}_s}{c_\xi B^R}\int_V \mathrm{d}V \varepsilon_{ijk} n_j \tilde{U}_{kl} f_l(\boldsymbol{u}')
\end{aligned}$$

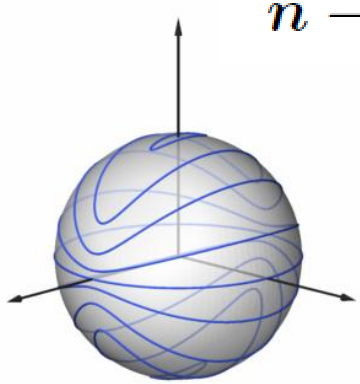
$$\boxed{\dot{n}_i = F_i^{(0)}(\boldsymbol{n}) + \mathrm{St} F_i^{(\mathrm{St})}(\boldsymbol{n}) + \mathrm{Re}_s F_i^{(\mathrm{Re}_s)}(\boldsymbol{n})}$$

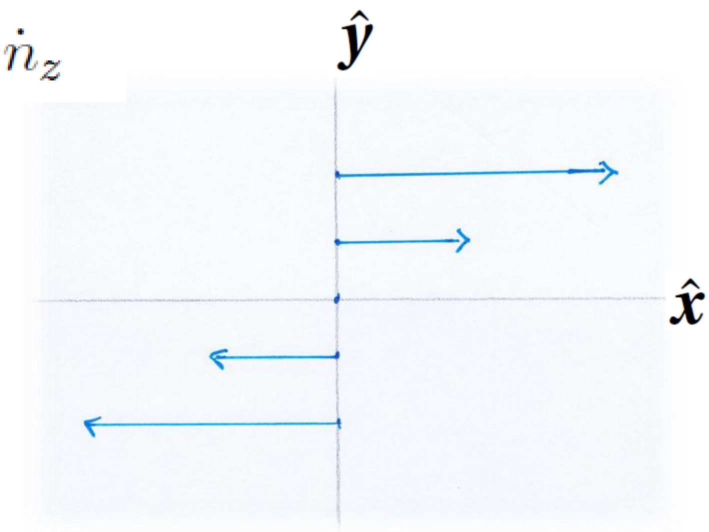
Symmetries of $\dot{n}_i$

- Problem has two distinct discrete symmetries

$$\boldsymbol{n} \to -\boldsymbol{n} \implies \dot{n}_z \to -\dot{n}_z$$

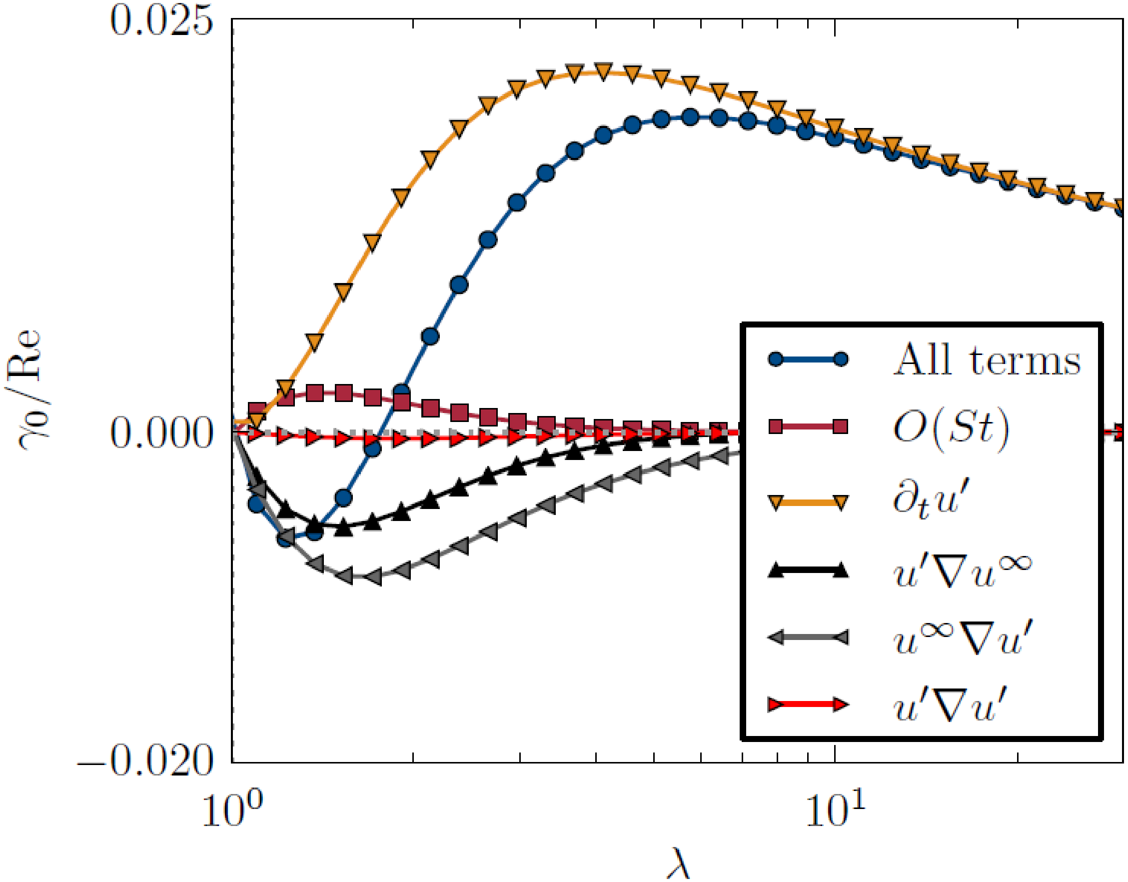
$$\boldsymbol{n} \to \mathbb{R}_z(\pi)\boldsymbol{n} \implies \dot{n}_z \to \dot{n}_z$$





- Conclusion: Equator and poles of the unit sphere must be stationary points.
- We can perform linear stability analysis!

Results (Prolate spheroids)



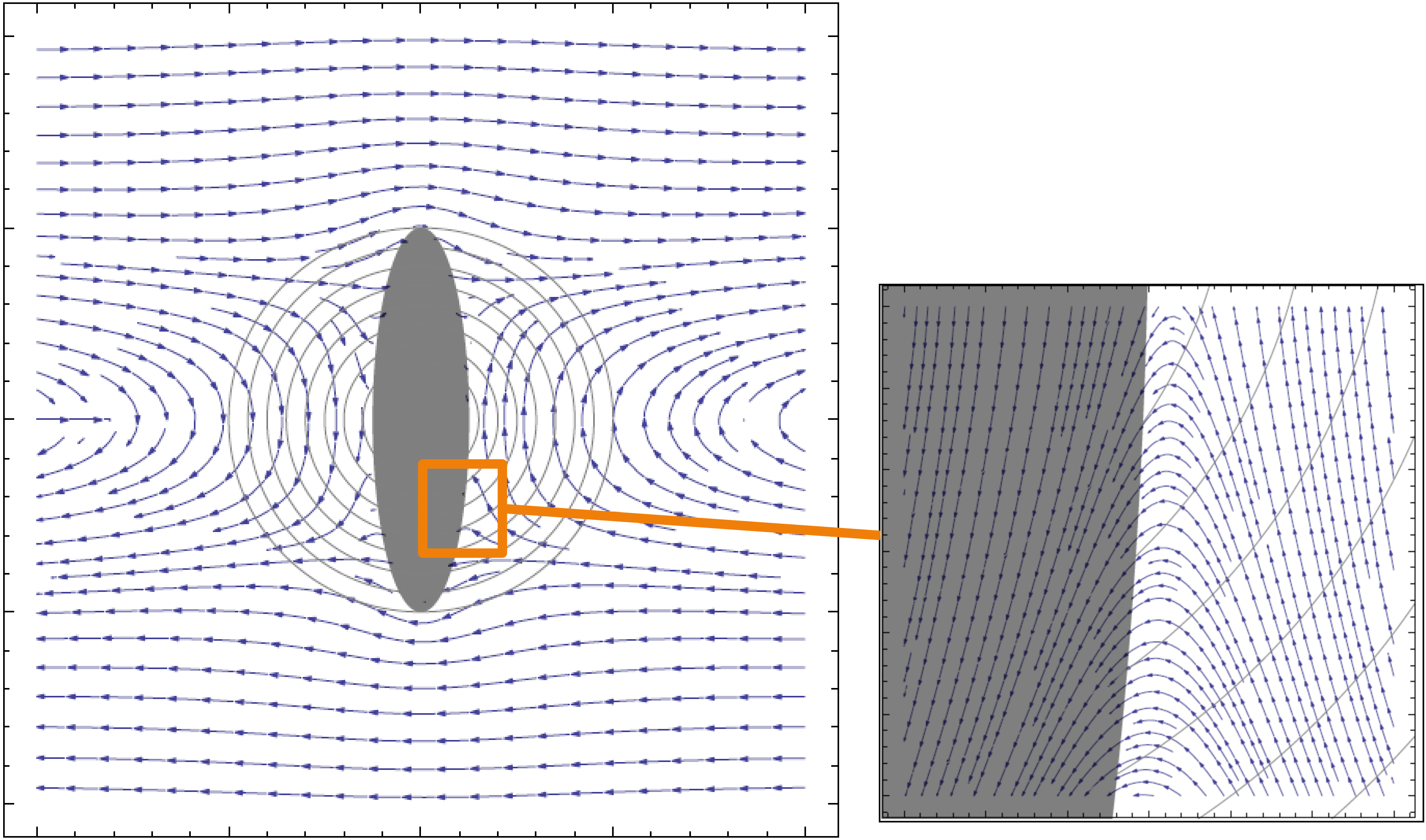
Einarsson, J, Angilella, JR, Candelier, F, Lundell, F, Mehlig, B. *Unpublished*.

Conclusions

- Tumbling of non-spherical particles in shear flow determined by competition between unsteady inertia and convective inertia
- Inertia destroys the Jeffery orbits and determines the resulting attractors.
- Non-trivial dependence on particle shape
- Results presented for shear flow – calculation similar for any steady linear flow, but small inertia likely to have small effect
- Small inertia: small effect for large particles in turbulence
- Large inertia: Relative angles of nearby particles

?!

Stokes flow solutions



Stokes flow solutions

$$\mathcal{G}_{ij} = \frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3},$$

$$\mathcal{G}_{ij,k} = -\frac{\delta_{ij} x_k}{r^3} + \frac{\delta_{ik} x_j}{r^3} + \frac{\delta_{jk} x_i}{r^3} - \frac{3 x_i x_j x_k}{r^5},$$

$$\mathcal{G}_{ij,ll} = \nabla^2 \mathcal{G}_{ij} = \frac{2\delta_{ij}}{r^3} - \frac{6 x_i x_j}{r^5},$$

$$\mathcal{G}_{ij,llk} = \nabla^2 \mathcal{G}_{ij,k} = -\frac{6}{r^5}\left(\delta_{ij} x_k + \delta_{jk} x_i + \delta_{ik} x_j\right) + \frac{30 x_i x_j x_k}{r^7}.$$

"Rotlet" & "Stresslet"

$$\mathcal{G}^R_{ij,k} = \frac{1}{2}\left(\mathcal{G}_{ij,k} - \mathcal{G}_{ik,j}\right) = \frac{1}{r^3}\left(\delta_{ik} x_j - \delta_{ij} x_k\right),$$

$$\mathcal{G}^S_{ij,k} = \frac{1}{2}\left(\mathcal{G}_{ij,k} + \mathcal{G}_{ik,j}\right) = \frac{\delta_{kj} x_i}{r^3} - \frac{3 x_i x_j x_k}{r^5}.$$

$$\mathcal{Q}^R_{ij,k}(\boldsymbol{r}, \boldsymbol{n}) = \int_{-c}^{c} \mathrm{d}\xi (c^2 - \xi^2) \mathcal{G}^R_{ij,k}(\boldsymbol{r} - \xi \boldsymbol{n}),$$

$$\mathcal{Q}^S_{ij,k}(\boldsymbol{r}, \boldsymbol{n}) = \int_{-c}^{c} \mathrm{d}\xi (c^2 - \xi^2) \mathcal{G}^S_{ij,k}(\boldsymbol{r} - \xi \boldsymbol{n}),$$

$$\mathcal{Q}_{ij,llk}(\boldsymbol{r}, \boldsymbol{n}) = \int_{-c}^{c} \mathrm{d}\xi (c^2 - \xi^2)^2 \mathcal{G}_{ij,llk}(\boldsymbol{r} - \xi \boldsymbol{n}).$$

"Spheroidal multipoles"

$$\begin{aligned} u'_i = {} & \mathcal{Q}^R_{ij,k} \varepsilon_{jkl} \left[\left(A^R n_l n_m + B^R (\delta_{lm} - n_l n_m) \right) \Omega_m + C^R \varepsilon_{lmn} n_m S_{no} n_o \right] \\ & + \left(\mathcal{Q}^S_{ij,k} + \alpha \mathcal{Q}_{ij,llk} \right) \left[\left(A^S n^A_{jklm} + B^S n^B_{jklm} + C^S n^C_{jklm} \right) S^\infty_{lm} \right. \\ & \left. - C^R \left(\varepsilon_{jlm} n_k n_m + \varepsilon_{klm} n_j n_m \right) \Omega_l \right] \end{aligned}$$

Chwang, A. T. & Wu, T. Y.-T. *Journal of Fluid Mechanics* **67,** 787–815 (1975).

Effects of noise (Brownian rotations)

$$\partial_t P(\boldsymbol{n}, t) = -\partial_{\boldsymbol{n}}\left[(F_0(\boldsymbol{n}) + \mathrm{St}F_1(\boldsymbol{n}))P\right] + \mathrm{Pe}^{-1}\partial_{\boldsymbol{n}}^2 P \equiv \hat{J}P$$

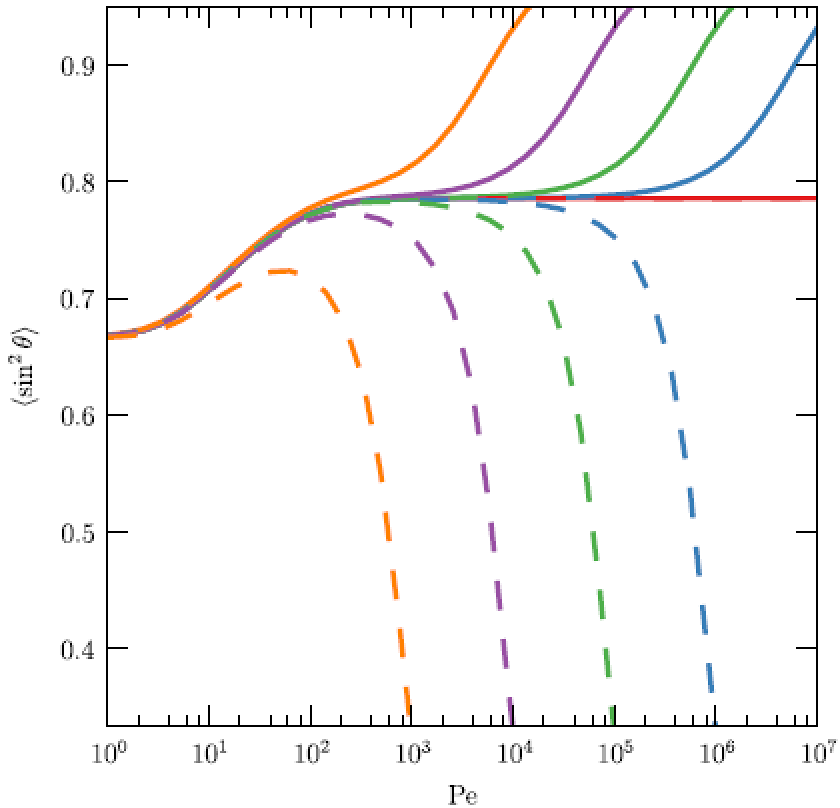


Fig. 4. Stationary orientational average as function of Pe for different $\widetilde{\mathrm{St}}$ and λ. Solid lines are a prolate particle $\lambda = 5$, dashed lines oblate $\lambda = 1/5$. The red solid line is $\widetilde{\mathrm{St}} = 0$ result, then blue $\widetilde{\mathrm{St}} = 10^{-4}$, green $\widetilde{\mathrm{St}} = 10^{-3}$, purple $\widetilde{\mathrm{St}} = 10^{-2}$ and orange $\widetilde{\mathrm{St}} = 10^{-1}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$
\begin{aligned}
\mathcal{Q}^{S}_{ij,k}(\boldsymbol{r},\boldsymbol{n}) = {} & \delta_{kj} x_i J^0_3(u) - \delta_{kj} n_i J^1_3(u,v) - 3 r_i r_j r_k J^0_5(u,v) - \delta_{jk} n_i J^1_3(u,v) \\
& + 3(n_i r_j r_k + n_j r_i r_k + n_k r_i r_j) J^1_5(u,v) - 3(r_i n_j n_k + r_j n_i n_k + r_k n_i n_j) J^2_5(u,v) \\
& + 3 n_i n_j n_k J^3_5(u,v)
\end{aligned}
$$

$$
\begin{aligned}
I^n_m &= \int_{-c}^{c} \mathrm{d}\xi \frac{\xi^n}{|\boldsymbol{r} - \xi \boldsymbol{n}|^m}, \\
J^n_m &= c^2 I^n_m - I^{n+2}_m, \\
K^n_m &= c^2 J^n_m - J^{n+2}_m = c^4 I^n_m - 2c^2 I^{n+2}_m + I^{n+4}_m.
\end{aligned}
$$