Phase transitions in Tensor Models

Răzvan Gurău

Nordita 2015

Phase transitions in field theory

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Phase transition $\Leftrightarrow$ symmetry breaking

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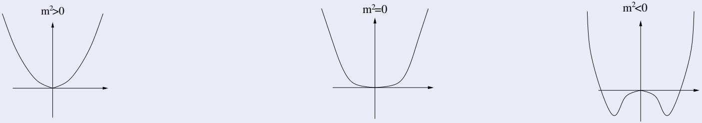

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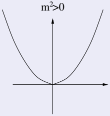

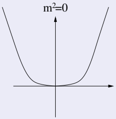

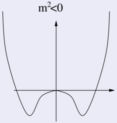

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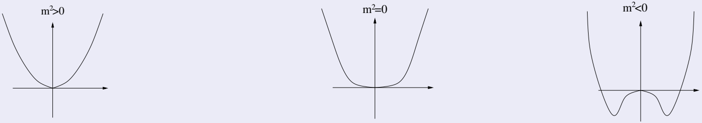

Broken phase → VEV: $\langle \bar{\phi}\phi \rangle = \frac{-m^2}{\lambda} \equiv v^2$. Expand around the VEV $\phi = (v+\rho)e^{\imath\frac{\theta}{v}}$

$$S_{broken} \sim \left(1 + \frac{\rho}{v}\right)^2 \partial\theta\partial\theta + \partial\rho\partial\rho + 2|m^2|\rho^2 + 2|m^2|\rho^3 + \frac{\lambda}{2}\rho^4$$

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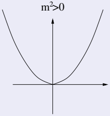

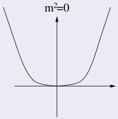

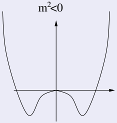

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Phase transition: zero eigenvalue of the "mass matrix"

$$\frac{\delta^2 S^{\text{notkinetic}}}{\delta\bar{\phi}\delta\phi}\Big|_{\bar{\phi}=\phi=0} = m^2 = 0$$

Continuum limit of Dynamical Triangulations

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Continuum limit: send $\lambda \to \lambda_{critical}$, $\sigma \to 0$ keeping the physical volume fixed.

DT continuum limit v.s. Field Theory phase transition

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Dynamical Triangulations are generated by matrix and tensor models:

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with $V_{\rm int}(\bar{T}, T)$ invariant under conjugation by the unitary group

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Tensor models: the continuum limit of the DT = phase transition (breaking of the unitary invariance)

Introduction

Tensor models

Phase transition in the quartic model

Tensor invariants as Edge Colored Graphs

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Building blocks: tensors with no symmetry transforming as

$$T'_{b^1 \dots b^D} = \sum U^{(1)}_{b^1 a^1} \dots U^{(D)}_{b^D a^D} \, T_{a^1 \dots a^D} \ , \quad \bar{T}'_{p^1 \dots p^D} = \sum \bar{U}^{(1)}_{p^1 q^1} \dots \bar{U}^{(D)}_{p^D q^D} \, \bar{T}_{q^1 \dots q^D}$$

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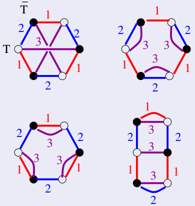
Invariants: colored graphs

$$\mathrm{Tr}_{\mathcal{B}}(T,\bar{T}) = \sum \prod_{v} T_{a^1_v\dots a^D_v} \prod_{\bar{v}} \bar{T}_{q^1_{\bar{v}}\dots q^D_{\bar{v}}} \prod_{c=1}^{D} \prod_{l^c=(w,\bar{w})} \delta_{a^c_w q^c_{\bar{w}}}$$



- White (black) vertices for T ($\bar{T}$).
- Edges for $\delta_{a^c q^c}$ colored by c, the position of the index.

Invariant Actions for Tensor Models

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The most general single trace tensor model

$$S(T,\bar{T}) = \sum T_{a^1\dots a^D} \bar{T}_{q^1\dots q^D} \prod_{c=1}^{D} \delta_{a^c q^c} + \sum_{\mathcal{B}} t_{\mathcal{B}} \mathrm{Tr}_{\mathcal{B}}(\bar{T},T)$$

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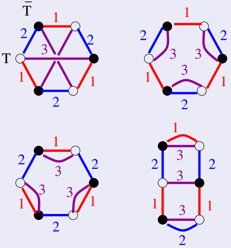

$$\int_{\bar T, T} \quad e^{-N^{D-1}\left(\sum T_{a^1\dots a^D}\bar T_{q^1\dots q^D}\prod_{c=1}^{D}\delta_{a^c q^c}\right)}$$

$$\mathrm{Tr}_{\mathcal{B}_1}(\bar T, T)\mathrm{Tr}_{\mathcal{B}_2}(\bar T, T)\dots$$

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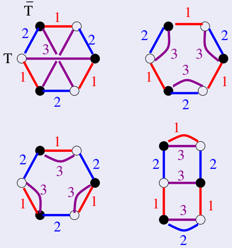

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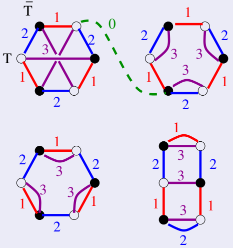

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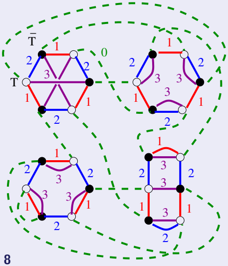

Graphs $\mathcal{G}$ with $D + 1$ colors.

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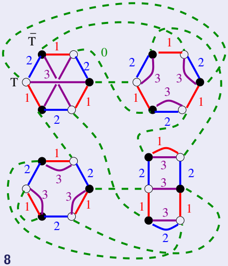

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Represent triangulated D dimensional spaces.

Colored Graphs as gluings of colored simplices

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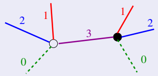
White and black $D+1$ valent vertices connected by edges with colors $0, 1 \dots D$.



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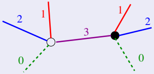
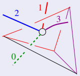
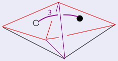


Vertex $\leftrightarrow$ colored D simplex .



Edges $\leftrightarrow$ gluings along $D-1$ simplices respecting all the colorings



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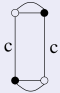
The simplest quartic invariants correspond to "melonic" graphs with four vertices $\mathcal{B}^{(4),c}$

$$\sum\Big(T_{a^1\dots a^D}\bar T_{q^1\dots q^D}\prod_{c'\neq c}\delta_{a^{c'}q^{c'}}\Big)\delta_{a^c p^c}\delta_{b^c q^c}\Big(T_{b^1\dots b^D}\bar T_{p^1\dots p^D}\prod_{c'\neq c}\delta_{b^{c'}p^{c'}}\Big)$$



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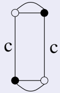

The simplest interacting theory: coupling constants $t_{\mathcal{B}} = \begin{cases} \frac{\lambda}{2}\,, & \mathcal{B} = \mathcal{B}^{(4),c} \\ 0\,, & \text{otherwise} \end{cases}$

Amplitudes and Dynamical Triangulations

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$$A^G(N) = \lambda^{\# Vertices} N^{D-\dots} = e^{\kappa_{D-2}(\lambda,N) Q_{D-2} - \kappa_D(\lambda,N) Q_D}$$

Discretized Einstein Hilbert action on the dual triangulation with Q_D equilateral D-simplices and Q_{D-2} $(D-2)$-simplices.

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DT continuum limit: $\lambda \to \lambda_c$

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Hubbard Stratanovich transformation

$$\int d\bar{\phi} d\phi \;\; e^{-\bar{\phi}\phi - \frac{\lambda}{2}(\bar{\phi}\phi)^2} = \int d\bar{\phi} d\phi \;\; e^{-\bar{\phi}\phi} \int dh \;\; e^{-\frac{1}{2}h^2 + (\imath\sqrt{\lambda})\bar{\phi} h \phi}$$

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integrate out $\bar{\phi}, \phi$ (Gaussian) to get an effective theory for h

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$$Z(\lambda) = \int \left(\prod_c [dH^c] \right) \; e^{-\frac{1}{2}\sum_c N^{D-1} \mathrm{Tr}_c[H^c H^c] + \mathrm{Tr}_{\mathcal{D}}[\ln R(H)]} \;,$$

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The vacuum

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Field theory for the matrix fields H^c, $c = 1, 2 \dots D$ with action:

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The unique unitary invariant solution is $H^c = a\mathbf{1}$ with

$$a = \frac{\imath\sqrt{\lambda}}{1 - \imath\sqrt{\lambda}Da} \Rightarrow a = \imath\sqrt{\lambda}\ \frac{-1 + \sqrt{1 + 4D\lambda}}{2D\lambda}$$

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with $Q(M) \sim M^3$ (hence $M = 0$ is the invariant vacuum)

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- the (diagonalized) mass matrix:

$$\frac{\delta^2 S}{\delta M^c_{\alpha\beta}\delta M^{c'}_{\gamma\delta}}\Big|_{M=0} = N^{D-1}(1-a^2)\left(\delta^{cc'}\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{DN}\delta_{\alpha\beta}\delta_{\gamma\delta}\right) + N^{D-1}(1-Da^2)\left(\frac{1}{DN}\delta_{\alpha\beta}\delta_{\gamma\delta}\right)$$

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Field theory phase transition: zero eigenvalue of the mass matrix $a^2 = \frac{1}{D}$