# Real time renormalization

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March 24, Nordita, Stockholm

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# The in-out and the in-in formalisms

Quantum field theory is formulated via the path integral. The generating functional $Z[J]$ (transition amplitude) is

$$Z[J] = \langle 0_+ | 0_- \rangle_J = e^{iW[J]} = \int \mathcal{D}\phi e^{i(S+J\phi)}.$$

The system evolves from the **in** vacuum $|0_-\rangle$ to the **out** vacuum $|0_+\rangle$ in the presence of the external source $J$. We can calculate matrix elements of the operator $T$ according to

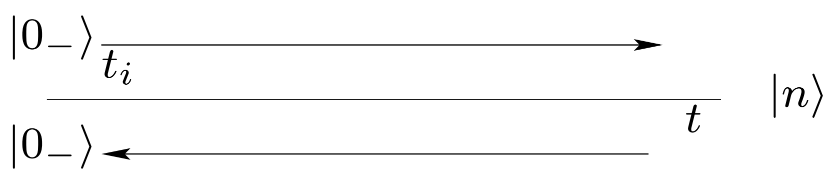
$$\langle 0_+ | T | 0_- \rangle.$$

This in-out matrix element is in general a **complex** quantity. However the expectation value between the **in-in** states is a real quantity

$$\langle 0_- | T | 0_- \rangle.$$

The in-in formalism can provide **expectation values**. The generating functional is

$$e^{iW[J^+,J^-]} = \sum_n \langle 0_- | n \rangle_{J^-} \langle n | 0_- \rangle_{J^+}$$



# The in-in formalism

The in-in generating functional is for a scalar field is

$$
\begin{aligned}
Z[J^+, J^-] &= e^{iW[J^+,J^-]} \\
&= \sum_n \langle 0_- | \bar{T} e^{i\int (H-\phi^- J^-)} | n \rangle \langle n | T e^{-i\int (H-\phi^+ J^+)} | 0_- \rangle \\
&= \mathrm{Tr}\left[ T e^{-i\int (H-\phi^+ J^+)} \underbrace{|0_-\rangle\langle 0_-|}_{\rho(0_-)} \bar{T} e^{i\int (H-\phi^- J^-)} \right] \\
&= \mathrm{Tr}\left[\rho(t)\right] \\
&= \int \mathcal{D}\phi^+ \mathcal{D}\phi^- e^{i\int (S[\phi^+]+J^+\phi^+ - S^*[\phi^-]-J^-\phi^-)},
\end{aligned}
$$

where in $S^*$ we have $+i\epsilon$. We integrate over all field configuration, the $\phi^+$ and $\phi^-$ field configurations coincide at $t = t^f$ (in practice $t^f = +\infty$) (**C**losed **T**ime **P**ath).

- we split the field variable as $\phi = \phi_< + \phi_> = \phi_S + \phi_E$
- the RG method can systematically eliminate the environmental degrees of freedom
- the $\rho_r(t)$ resulting reduced density matrix can account for the decoherence

# The in-in (or CTP) scalar propagator

We introduce the notation $\hat{\mathcal{O}} = \begin{pmatrix} \mathcal{O}_+ \\ \mathcal{O}_- \end{pmatrix}$.

The scalar CTP propagator is

$$\begin{aligned}
i\frac{\delta^2 W[\hat{j}]}{\delta i J_a^+ \delta i J_b^+} &= \langle 0|T[\phi_a\phi_b]|0\rangle = iD_{ab}^{++} = iD_{ab}^{++tr} \\
i\frac{\delta^2 W[\hat{j}]}{\delta i J_a^- \delta i J_b^-} &= \langle 0|T[\phi_b\phi_a]|0\rangle^* = iD_{ab}^{--} = -iD_{ba}^{++*} = -iD_{ab}^{++\dagger} \\
i\frac{\delta^2 W[\hat{j}]}{\delta i J_a^- \delta i J_b^+} &= \langle 0|\phi_a\phi_b|0\rangle = iD_{ab}^{-+} = iD_{ab}^{>} = -iD_{ab}^{-+\dagger} \\
i\frac{\delta^2 W[\hat{j}]}{\delta i J_a^+ \delta i J_b^-} &= \langle 0|\phi_b\phi_a|0\rangle = iD_{ab}^{+-} = iD_{ab}^{<} = -iD_{ab}^{-+*} = iD_{ab}^{-+tr}
\end{aligned}$$

$$\begin{pmatrix} \langle T[\phi_x^i\phi_y^j]\rangle & \langle \phi_y^j\phi_x^i\rangle \\ \langle \phi_x^i\phi_y^j\rangle & \langle T[\phi_y^j\phi_x^i]\rangle^* \end{pmatrix} = i\begin{pmatrix} D & D^{+-} \\ D^{-+} & D^{--} \end{pmatrix}_{x,y}$$

The CTP symmetry is $T[\phi_a\phi_b] + \bar{T}[\phi_a\phi_b] = \phi_a\phi_b + \phi_b\phi_a,\;\; D - D^\dagger = D^{+-} - D^{+-*}$.

# The CTP scalar propagator

The scalar CTP propagator is

$$\hat{D}^{ij}_k = -i\int_x e^{iq(x-y)}\begin{pmatrix}\langle T[\phi^i_x\phi^j_y]\rangle & \langle\phi^j_y\phi^i_x\rangle\\ \langle\phi^i_x\phi^j_y\rangle & \langle T[\phi^j_y\phi^i_x]\rangle^*\end{pmatrix} = \begin{pmatrix}D_k & -D^{-+*}_{-k}\\ D^{-+}_k & -D^\dagger_k\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{k^2-m^2+i\epsilon} & -2\pi i\delta(k^2-m^2)\Theta(-k^0)\\ -2\pi i\delta(k^2-m^2)\Theta(k^0) & -\frac{1}{k^2-m^2-i\epsilon}\end{pmatrix}$$

$$= P\frac{1}{k^2-m^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - 2\pi i\delta(k^2-m^2)\begin{pmatrix}\frac{1}{2} & \Theta(-k^0)\\ \Theta(k^0) & \frac{1}{2}\end{pmatrix}.$$

The inverse CTP propagator is

$$\hat{D}^{-1ij}_k = (k^2-m^2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + i\epsilon\begin{pmatrix}1 & -2\Theta(-k^0)\\ -2\Theta(k^0) & 1\end{pmatrix}$$

$$= \begin{pmatrix}k^2-m^2 & i\mathrm{sign}(k^0)\epsilon\\ -i\mathrm{sign}(k^0)\epsilon & -k^2+m^2\end{pmatrix} + i\epsilon\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

# Single time renormalization, blocked action

In Minkowski spacetime the blocking step for the Wilsonian action reads as

$$e^{iS_{k-\Delta k}[\phi]} = e^{i(S_k[\phi+\chi]+\frac{i}{2}\mathrm{Tr}\ln\frac{\delta^2 S_k[\phi+\chi]}{\delta\phi\delta\phi})}$$

The trace takes into account the modes within the momentum shell $p \in [k, k-\Delta k]$. The time axis should be treated separately (the frequency is integrated out) i.e.

$$\mathrm{Tr}\ln\frac{\delta^2 S_k[\phi+\chi]}{\delta\phi\delta\phi} = \int_\omega \int_{k-\Delta k<|\mathbf{p}|<k} \ln\frac{\delta^2 S_k[\phi+\chi]}{\delta\phi_{-\omega-\mathbf{p}}\delta\phi_{\omega\mathbf{p}}}.$$

It results in the Wegner-Houghton equation

$$V_{k-\Delta k} = V_k - \frac{i}{2}\mathrm{Tr}\ln D^{-1}, \text{ with } D^{-1} = \omega^2 - \mathbf{k}^2 - V_k'' + i\epsilon \text{ the inverse propagator,}$$

giving

$$\dot{V}_k = i\alpha k^{d-1}\int_{\omega=-\infty}^{\infty}\ln D^{-1}.$$

After integrating out the frequency component of the scale $k$ and get

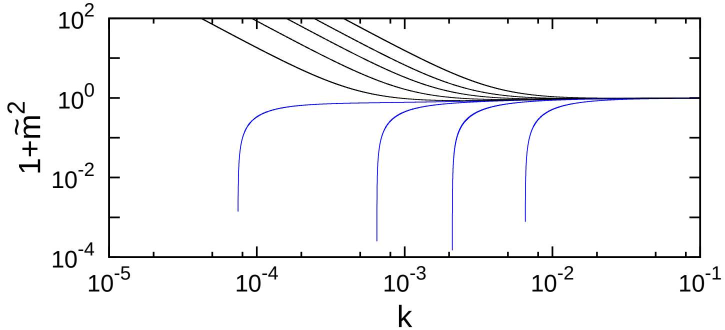
$$\dot{V}_k = -\alpha k^{d-1}\sqrt{k^2+V_k''}.$$

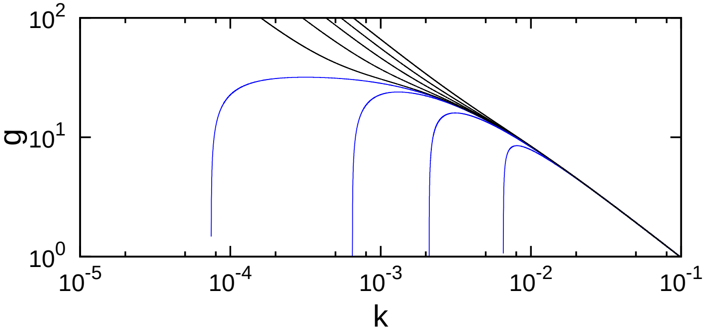
# Minkowski evolution of the $\phi^4$ model

The evolution equations are

$$\dot{\tilde{m}}^2 = -2\tilde{m}^2 - \alpha \frac{\tilde{g}}{2(1+\tilde{m}^2)^{\frac{1}{2}}}$$

$$\dot{\tilde{g}} = (d-4)\tilde{g} + \alpha \frac{3\tilde{g}^2}{4(1+\tilde{m}^2)^{\frac{3}{2}}}.$$



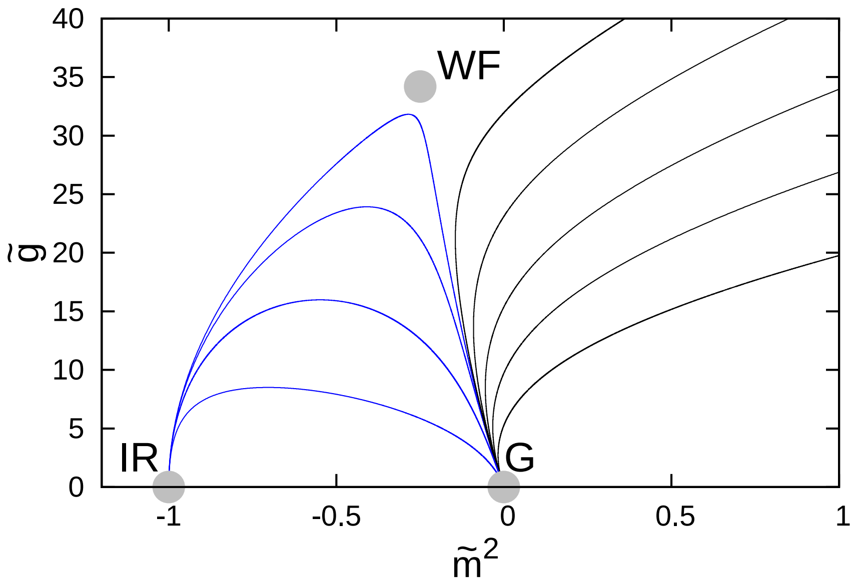




The evolution equations have 3 fixed points:

**G:** UV attractive fixed point: $g_{2G}^* = 0$, $g_{4G}^* = 0$ (asymptotic freedom),

**CO:** hyperbolic fixed point: $g_{2UV}^* = -1/3$, $g_{4UV}^* \approx 35$,

**IR:** IR attractive fixed point: $g_{2IR}^* = -1$, $g_{4IR}^* = 0$.



# Renormalization in the CTP formalism

The single time Wegner-Houghton equation is

$$\dot{V}_k = i\alpha k^{d-1} \int_\omega \ln D^{-1},$$

which can be generalized into a CTP RG equation

$$\dot{V}_k = i\alpha k^{d-1} \int_\omega \mathrm{Tr} \ln \hat{D}^{-1},$$

where we replaced the single time propagator to its CTP version

$$D^{-1} = \omega^2 - k^2 - V_k'' + i\epsilon \;\Rightarrow\; \hat{D}^{-1} = \begin{pmatrix} \omega^2 - k^2 - V_k^{++} + i\epsilon & -2i\Theta(-\omega)\epsilon - V_k^{+-} \\ -2i\Theta(\omega)\epsilon - V_k^{-+} & -\omega^2 + k^2 - V_k^{--} + i\epsilon \end{pmatrix}.$$

Here

$$V'' \quad \Rightarrow \quad V_k^{ij} = \begin{pmatrix} \frac{\delta^2 V_k}{\delta\phi_+ \delta\phi_+} & \frac{\delta^2 V_k}{\delta\phi_+ \delta\phi_-} \\ \frac{\delta^2 V_k}{\delta\phi_- \delta\phi_+} & \frac{\delta^2 V_k}{\delta\phi_- \delta\phi_-} \end{pmatrix} \equiv \begin{pmatrix} V_k^{++} & V_k^{+-} \\ V_k^{-+} & V_k^{--} \end{pmatrix}$$

# The CTP potential

The general form of the potential is

$$V = \frac{m^2 - i\mu^2}{2}\phi^{+2} - \frac{m^2 - i\mu^2}{2}\phi^{-2} + i\mu^2\phi^+\phi^- \\ +i\frac{h}{4}\phi^{+2}\phi^{-2} + \frac{\lambda}{6}\phi^{+3}\phi^- - \frac{\lambda^*}{6}\phi^+\phi^{-3} + \frac{G}{4!}\phi^{+4} - \frac{G^*}{4!}\phi^{-4},$$

with $G = g - 3ih + 3\lambda - \lambda^*$, where $m^2$ and $g$ are real. The evolution equations for the dimensionless couplings are

$$\begin{aligned}
\dot{\tilde{m}}^2 &= -2\tilde{m}^2 + \Re\, \dot{\tilde{V}}^{++}\Big|_{\phi_+=0,\phi_-=0} \\
\dot{\tilde{\mu}}^2 &= -2\tilde{\mu}^2 + \Im\, \dot{\tilde{V}}^{+-}\Big|_{\phi_+=0,\phi_-=0} \\
\dot{\tilde{\lambda}}_r &= (d-4)\tilde{\lambda}_r + \Re\, \dot{\tilde{V}}^{+++-}\Big|_{\phi_+=0,\phi_-=0} \\
\dot{\tilde{\lambda}}_i &= (d-4)\tilde{\lambda}_i + \Im\, \dot{\tilde{V}}^{+++-}\Big|_{\phi_+=0,\phi_-=0} \\
\dot{\tilde{h}} &= (d-4)\tilde{h} + \Im\, \dot{\tilde{V}}^{++--}\Big|_{\phi_+=0,\phi_-=0} \\
\dot{\tilde{g}} &= (d-4)\tilde{g} + \Re\, \dot{\tilde{V}}^{++++}\Big|_{\phi_+=0,\phi_-=0} - 2\dot{\tilde{\lambda}}_r
\end{aligned}$$

# CTP evolution, $T = 0$

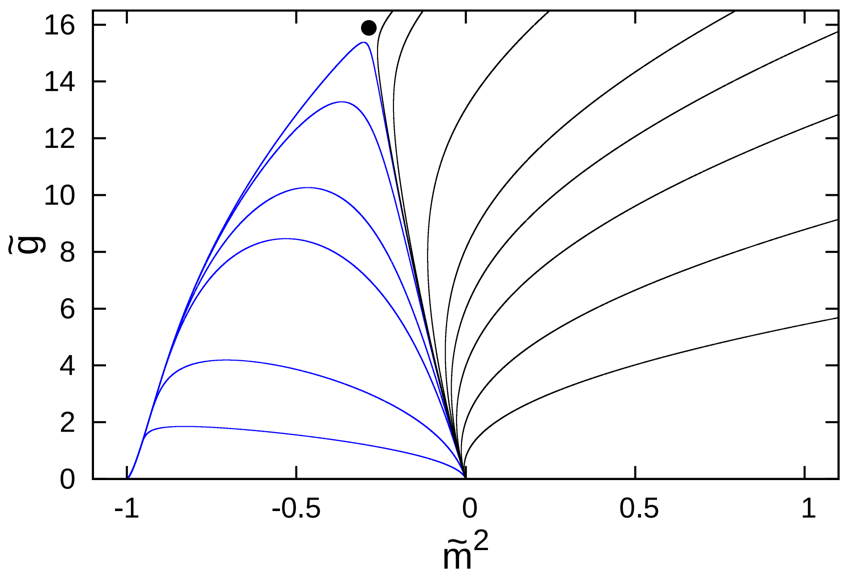
The initial value of the mixing couplings are zero, and they do not evolve.
The evolution of couplings are

$$\dot{\tilde{m}}^2 = -2\tilde{m}^2 - \alpha\frac{\tilde{g}}{(1+\tilde{m}^2)^{1/2}} + \frac{1}{2}\alpha\frac{\tilde{g}\tilde{m}^2}{(1+\tilde{m}^2)^{3/2}}$$

$$\dot{\tilde{g}} = (d-4)\tilde{g} + \frac{3}{2}\alpha\frac{\tilde{g}^2}{(1+\tilde{m}^2)^{3/2}}.$$

The phase structure remains qualitative unchanged, i.e.

- two phases,
- the fixed points have similar types,
- $\tilde{g} \to 0$ and $1 + \tilde{m}^2 \to 0$ in the deep IR region of the broken phase.



# Finite temperature evolution

The free finite temperature propagator is

$$\hat{D}_{0k}^{-1} = \begin{pmatrix} \omega^2 - k^2 - m^2 + i(\mu_n^2 + \epsilon) & -i\mu_n^2 - 2i\Theta(-\omega)\epsilon \\ -i\mu_n^2 - 2i\Theta(\omega)\epsilon & -\omega^2 + k^2 + m^2 + i(\mu_n^2 + \epsilon) \end{pmatrix},$$

with $\mu_n^2 = \mu^2 + 2n\epsilon$ and $n = \frac{1}{\exp(\beta\epsilon_{\mathbf{k}})-1}$.

- The finite temperature turns on the evolution of the mixing couplings.
- The mixing mass coupling $\tilde{\mu}^2$ is taken into account exactly.
- The quartic couplings are treated perturbatively (Taylor expansion of the Wegner-Houghton equation in $U^{ij}$)

$$\hat{D}_k^{-1} = \hat{D}_{0k}^{-1} - \partial\partial U \equiv \hat{D}_{0k}^{-1} - \begin{pmatrix} U^{++} & U^{+-} \\ U^{-+} & U^{--} \end{pmatrix}.$$

- The expansion around $\phi = 0$ yields

$$\dot{V}_k = 2i\alpha k^{d-1} \int_\omega \left[ \mathrm{Tr} \ln \hat{D}_{0k}^{-1} - \sum_{n=1}^{\infty} \int_\omega \frac{1}{n} \mathrm{Tr}[\partial\partial U \hat{D}_{0k}]^n \right].$$

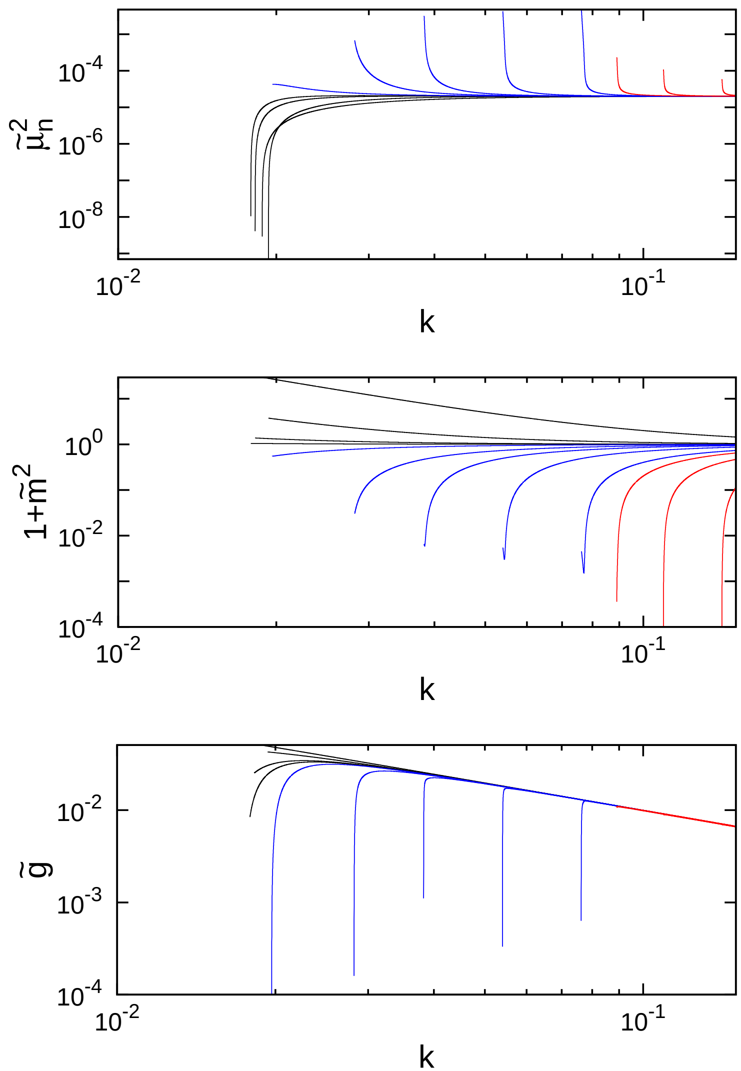
# Finite temperature evolution

According to the scaling of $\tilde{m}^2$ there are two phases in the model:

- when $1 + \tilde{m}^2 > 1$ then the phase is **symmetric** (black curves),
- when $0 < 1 + \tilde{m}^2 < 1$ then we have a **broken** phase (blue and red curves).

The coupling $\tilde{\mu}_n^2$ can signal the decoherence. The off-diagonal elements of the reduced denisty matrix is proportional to $\exp(-\tilde{\mu}_n^2)$, i.e.

- when $\tilde{\mu}_n^2$ is finite, then the state is coherent (quantum),
- when $\tilde{\mu}_n^2 \to \infty$, then the decoherence appears (classical).

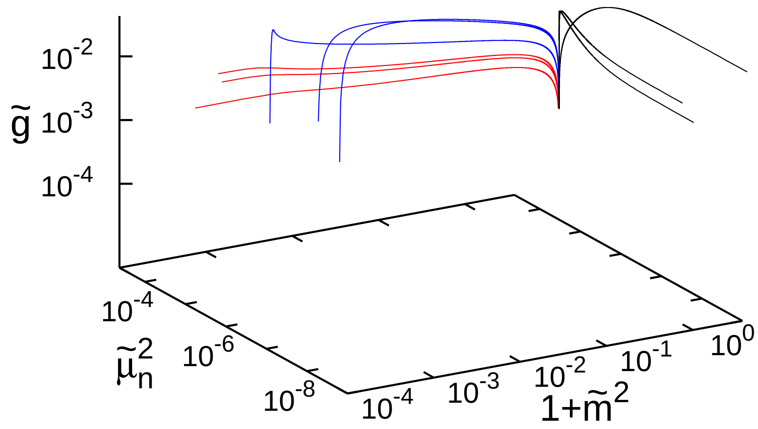



In the broken phase either $\tilde{g}$ or $1 + \tilde{m}^2$ tends to zero.

# Finite temperature evolution

The model has **three** phases

- symmetric phase, $\tilde{\mu}^2 \to 0$
- quantum broken phase, $\tilde{g} \to 0$
- classical broken phase, $1+\tilde{m}^2 \to 0$




| | **CTP phases** | | | **STP phases** | |
|---|---|---|---|---|---|
| **coupling** | **symmetric** | **quantum broken** | **classical broken** | **symmetric** | **broken** |
| $\tilde{m}^2$ | $1+\tilde{m}^2 > 1$ | $1 > 1+\tilde{m}^2 > 0$ | $1+\tilde{m}^2 \to 0$ | $1+\tilde{m}^2 > 1$ | $1+\tilde{m}^2 \to 0$ |
| $\tilde{g}$ | $\tilde{g} > 0$ | $\tilde{g} \to 0$ | $\tilde{g} > 0$ | $\tilde{g} > 0$ | $\tilde{g} \to 0$ |
| $\tilde{\mu}_n^2$ | $\tilde{\mu}_n^2 \to 0$ | $\tilde{\mu}_n^2 > 0$ | $\tilde{\mu}_n^2 \to \infty$ | – | – |

# Outlook

- CTP Wetterich equation
- Complex scalar fields, quantum electrodynamics
- CTP quantum Einstein gravity

# Acknowledgments

This research was supported by the European Union and the State of Hungary, cofinanced by the European Social Fund in the framework of the TÁMOP 4.2.4.A/2-11-1-2012-0001 'National Excellence Program'. Sándor Nagy acknowledges financial support from a János Bolyai Grant of the Hungarian Academy of Sciences and the Hungarian National Research Fund OTKA (K112233).