Is there a C-Theorem in 4D Quantum Gravity?

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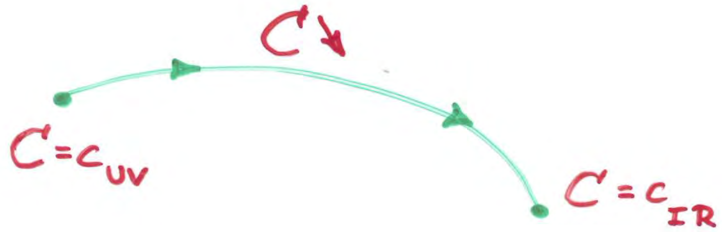
The c-theorem:

Zamolodchikov (1987)

Every 2D euclidean QFT with reflection positivity, rotational invariance, and a conserved $T_{\mu\nu}$ possesses a function C of its couplings which is non-increasing along RG trajectories, and stationary at fixed points where it equals the central charge of the corresponding CFT.



related "intuitive" ideas:

- $-C$ ~ "entropy", increasing along traj.
- ~ "information" loss due to coarse graining
- ~ # "degrees of freedom"
- RG flow irreversible : no limit cycles, etc.

Proposals for analogs in $d > 2$:

- Cardy, Osborn : $c \sim \int_{S^4} d^4x \sqrt{g} \langle T_\mu{}^\mu \rangle$

  trace anomaly

- Cappelli, Friedan, Latorre :

  c from spectral densities of $\langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle$

- Exploit similarity to free energy:

  e.g. "F - theorem" for certain 3D theories on S^3

  ⋮

The a-theorem :

Komargodski, Schwimmer (2011)

c : coefficient of Euler-form action induced on classical 4D background spacetime

1. Background Independence in Quantum Gravity

2. The EAA as a mode-counting device

3. C-function for asymptotically safe quantum gravity: a candidate

Background Independence

The hallmark of classical GR and Quantum Gravity !

- Treat all possible spacetimes on an equal footing
- no "vacuum" singled out a priori
- derive rather than put in "by hand" the arena of all non-gravitational physics (Minkowski space, ...)

Even more fundamental and momentous than the renormalizability issue !

Trying to derive the "arena" from more basic assumptions

Typical problems:

(i) Start, say, from the (strong!!) assumption

"Spacetime ($\equiv$ set of all events) is a differentiable manifold"

$\Rightarrow$ no dynamics, no time, no QM, no causality, no foliation, no ETCR, ...

$$[\hat{\Phi}(\vec{x},t),\ \hat{\pi}(\vec{y},t)] = i\,\delta(\vec{x}-\vec{y})\ ,\ \cdots,\cdots$$

?

(ii) Phase of unbroken diffeomorphism invariance ($g_{\mu\nu} \equiv 0$):

(almost) no interesting local actions!

Ways of implementing Background Independence:

- Literally use no background $\rightarrow$ LQG, CDT
- Use background, but keep it generic: $\rightarrow$ EAA

enforce Background Independence by requiring split-symmetry.

example:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \cdots$$

dynamical $\quad$ backgrd. $\quad$ fluctuation

$$= (\bar{g}_{\mu\nu} + \varepsilon_{\mu\nu}) + (h_{\mu\nu} - \varepsilon_{\mu\nu}) + \cdots$$

split-symmetry transformation:

$$\begin{cases} \delta \bar{g}_{\mu\nu} = \varepsilon_{\mu\nu} & + \cdots \\ \delta h_{\mu\nu} = -\varepsilon_{\mu\nu} & + \cdots \end{cases}$$

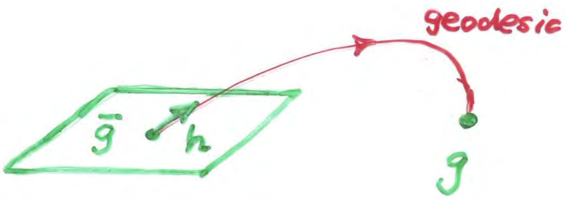
The task:

$$\boxed{\text{Quantize } h_{\mu\nu}\text{-field on a family of spacetimes } (\mathcal{M}, \bar{g}) \text{ such that the physical sector of the resulting QFT respects split-symmetry.}}$$

Beyond the linear split: curved configuration space

$\Rightarrow$ higher orders of the exponential map:

$$\underbrace{g_{\mu\nu} = \bar{g}_{\mu\nu}}_{\text{"coordinates"}} + \underbrace{h_{\mu\nu}}_{\text{"tangent vector"}} + O(h^2) + O(h^3) + \cdots$$

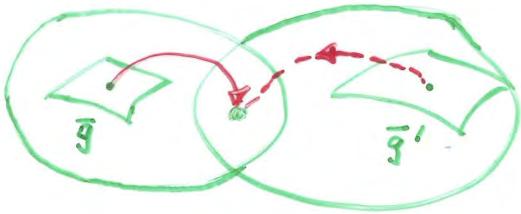


split-sym. transf.:

$$\delta h_{\mu\nu} = -\varepsilon_{\mu\nu} + O(h) + O(h^2) + \cdots$$
$$\delta \bar{g}_{\mu\nu} = \varepsilon_{\mu\nu} + O(h) + O(h^2) + \cdots$$

- Honerkamp (1971): σ-model (χPT)

- Friedan (1985), Becchi, …:



split-sym., consistency of patching $\hat{=}$ zero-curvature condition

connection: $\displaystyle D = \frac{\delta}{\delta \bar{g}_{\mu\nu}} - \frac{\delta}{\delta h_{\mu\nu}} + \cdots$

- MR (1997): analog for symplectic basis manifolds
  (QM as a gauge theory of metaplectic spinor fields)

  $D \sim$ "abelian connection" from Fedosov (~1990) construction of star products on arbitrary symplectic manifolds
  ($\rightarrow$ Kontsevich (1997): arb. Poisson mflds.)

- VilKovisky – DeWitt: "unique off-shell effective action"

Effective Average Action (EAA) approach:

Cutoff $e^{-\Delta S_k}$, $\Delta S_k \sim \int h \, R(-\bar{D}_k) \, h$

and gauge fixing violate split-sym. explicitly

$\Rightarrow$ restoration is a nontrivial issue

Analogy: QED in a reg. scheme violating gauge invariance

photon mass $\neq 0$ in general:

fine-tune m_γ^{bare} such that $m_\gamma^{\text{ren}} = 0$

$\hat{=}$ select a specific class of RG trajectories

> Restoration of split-symmetry $\hat{=}$
>
> selection of an appropriate subset of RG trajectories
>
> is crucial for the correct identification of the desired <u>universality class</u>.

Key Question:

Does the subset contain asym. safe trajectories?

Can Background Independence and Asymptotic Safety be satisfied <u>simultaneously</u>?

EAA for generic field multiplet

$$\Gamma_k[\varphi;\bar{\Phi}] \equiv \Gamma_k[\Phi,\bar{\Phi}]\Big|_{\Phi=\bar{\Phi}+\varphi}$$

$$\Phi \equiv (g_{\mu\nu},\cdots) \equiv \bar{\Phi}+\varphi$$

$$\bar{\Phi} \equiv (\bar{g}_{\mu\nu},\cdots)$$

$$\varphi \equiv (h_{\mu\nu},\cdots)$$

WISS : Ward Identity of Split Symmetry

$$\boxed{\frac{\delta\Gamma_k[\Phi,\bar{\Phi}]}{\delta\bar{\Phi}(x)} = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)}[\Phi,\bar{\Phi}]+\mathcal{R}_k\right)^{-1}\frac{\delta S_{\mathrm{tot}}^{(2)}[\Phi,\bar{\Phi}]}{\delta\bar{\Phi}(x)}\right]}$$

MR, C. Wetterich,

hep-th/9708051

Truncating theory space (metric gravity)

$$\mathcal{T}_{QEG}: \quad \Gamma_k[h, C, \bar{C}; \bar{g}] \equiv \Gamma_k[g, \underline{\bar{g}}, C, \bar{C}]\Big|_{g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}}$$

"extra $\bar{g}$-dependence"

"single metric truncations":

MR, 1996

$$\Gamma_k = \underbrace{\Gamma_k[g]}_{1996 \,\cdots\, 2009} + \int (\mathcal{F}(\bar{g})h)^2 + \int \bar{C} M(\bar{g}) C$$

Manrique, MR, 2009

Manrique, MR, Saueressig, 2010

D. Becker, MR, 2014

"bi-metric truncations": e.g.

$$\Gamma_k = \Gamma_k[g, \bar{g}] + \int (\mathcal{F}(\bar{g})h)^2 + \int \bar{C} M(\bar{g}) C$$

"*level* expansion":

$$\Gamma_k[g, \bar{g}] \equiv \Gamma_k[h; \bar{g}] = \sum_{p=0}^{\infty} \frac{1}{p!} \int \gamma_k^{(p)}[\bar{g}] \underbrace{h \cdots h}_{p \text{ factors}}$$

p = "level"

"single metric approx.": retain only level $p = 0$,

$g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ not disentangled

Einstein - Hilbert type bimetric truncation

Manrique, MR, Saueressig, 2010
D. Becker, MR 2014

$$\Gamma_k[g,\bar{g}] = -\frac{1}{16\pi G_k^{Dyn}} \int \sqrt{g}\,\{R(g) - 2\Lambda_k^{Dyn}\}$$

$$-\frac{1}{16\pi G_k^{B}} \int \sqrt{\bar{g}}\,\{R(\bar{g}) - 2\Lambda_k^{B}\}$$

expand

$$\equiv \; -\frac{1}{16\pi G_k^{(0)}} \int \sqrt{\bar{g}}\,\{R(\bar{g}) - 2\Lambda_k^{(0)}\}$$

$$+\frac{1}{16\pi G_k^{(1)}} \int \sqrt{\bar{g}}\,\left[\bar{G}^{\mu\nu} + \Lambda_k^{(1)}\,\bar{g}^{\mu\nu}\right] h_{\mu\nu}$$

$$+\frac{1}{16\pi G_k^{(2)}} \int \sqrt{\bar{g}}\; h\,[\cdots \bar{D}^2 \cdots]\,h \;+\; O(h_{\mu\nu}^3)$$

4-dim. theory space $\ni\ (g^{Dyn}, \lambda^{Dyn}, g^{B}, \lambda^{B})$

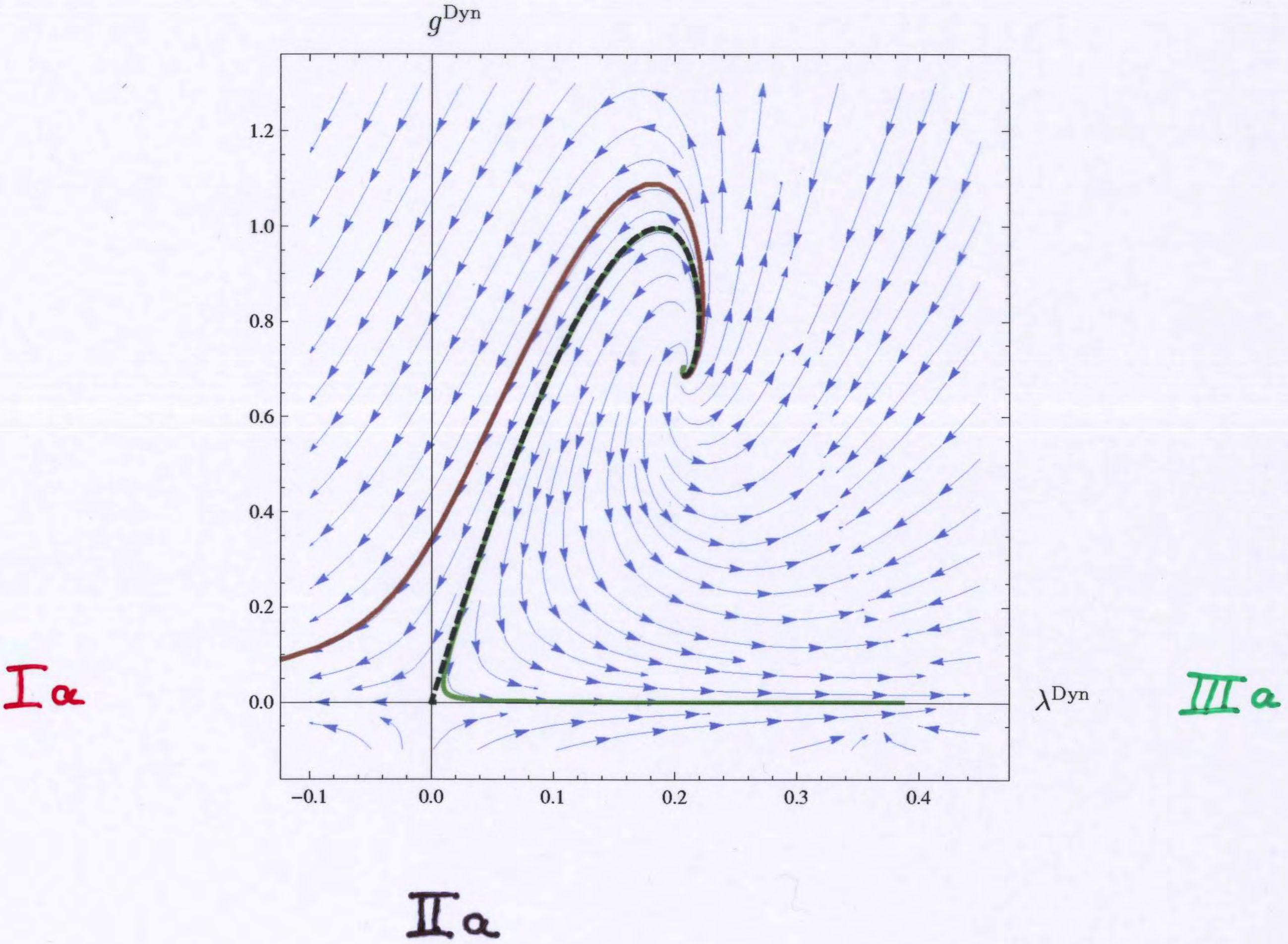
split-symmetry
(WISS, leading order) $\Longleftrightarrow \left(\frac{1}{G_k^B} = 0\ ,\ \frac{\Lambda_k^B}{G_k^B} = 0\right)$

$$\Longleftrightarrow \left(G_k^{(0)} = G_k^{(1)} = G_k^{(2)} = \cdots\ ,\ \Lambda_k^{(0)} = \Lambda_k^{(1)} = \cdots\right)$$

Broken during RG evolution; try to restore in the physical limit ($k \to 0$).

The bi-metric Einstein-Hilbert truncation:

the $g^{\mathrm{Dyn}} - \lambda^{\mathrm{Dyn}}$ - subspace



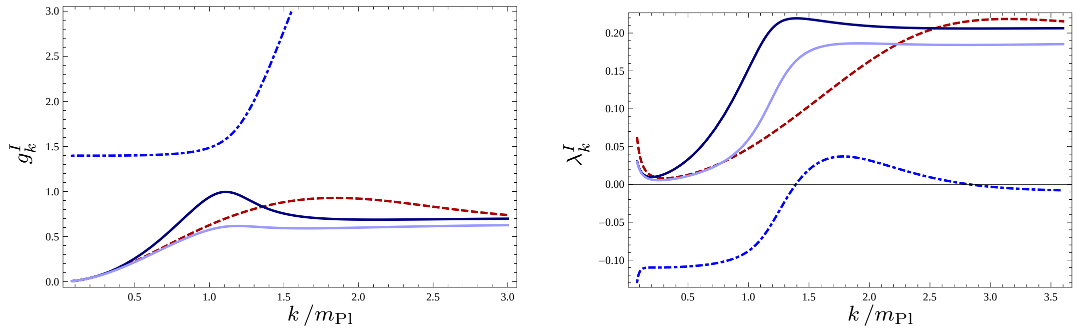


Figure 12. Type (IIIa)$^{\text{Dyn}}$-**(Attr)$^{\mathbf{B}}$**trajectory: dimensionless couplings.

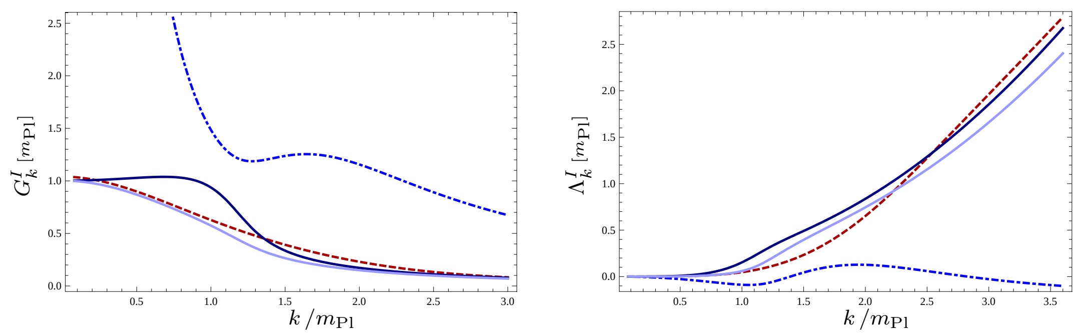


Figure 13. Type (IIIa)$^{\text{Dyn}}$-**(Attr)$^{\mathbf{B}}$**trajectory: dimensionful couplings.

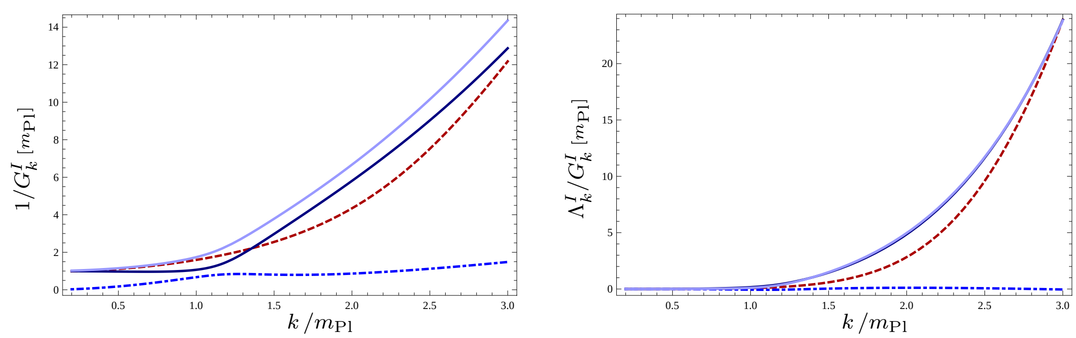


Figure 14. Type (IIIa)$^{\text{Dyn}}$-**(Attr)$^{\mathbf{B}}$**trajectory: the coefficients as they appear in the EAA. Note again the vanishing $1/G_k^{\mathrm{B}}$ and $\Lambda_k^{\mathrm{B}}/G_k^{\mathrm{B}}$, indicative of split-symmetry restoration in the limit $k \to 0$.

dashed (red):	$I = \mathrm{sm}$ (single-metric)
solid (dark-blue):	$I = \mathrm{Dyn} \equiv (p)$ for $p \geq 1$
solid (light-blue):	$I = (0)$
dot-dashed (blue):	$I = \mathrm{B}$

B-Sector: the running UV attractor

hierarchical structure of the coupled diff. eqs.:

$$\underbrace{(g^{Dyn}, \lambda^{Dyn})}_{\text{solve}} \longrightarrow \underbrace{g^{B} \longrightarrow \lambda^{B}}$$

insert solution $\Longrightarrow$

Non-autonomous ODEs:

$(t \equiv \log k)$

$$\partial_t g_k^B = \beta_g^B (g_k^B \,; k)$$

$$\partial_t \lambda_k^B = \beta_\lambda^B (g_k^B, \lambda_k^B \,; k)$$

- Vector field $\beta^B \equiv (\beta_g^B, \beta_\lambda^B)$ on g^B-λ^B-subspace is explicitly RG-time dependent.

- β^B has a (k-dependent) zero which is UV-attractive in both directions:

$$\beta^B (g_\bullet^B(k), \lambda_\bullet^B(k); k) = 0 \qquad \forall k$$

- 4D embedding: "running UV attractor"

$$\mathrm{Attr}^B(k) = (g_k^{Dyn}, \lambda_k^{Dyn}, g_\bullet^B(k), \lambda_\bullet^B(k))$$

The Main Result:

The 2-parameter family of RG trajectories restoring split symmetry in the IR consists of precisely those which approach

$\mathrm{Attr}^{B}(k \to 0)$ in the IR.

Summary:

(Potentially) acceptable QFTs can be based on RG trajectories with

initial point $(k \to \infty)$: $NG^{B} \oplus NG_{+}^{Dyn}$-FP

final point $(k \to 0)$: $\mathrm{Attr}^{B}(k \to 0)$

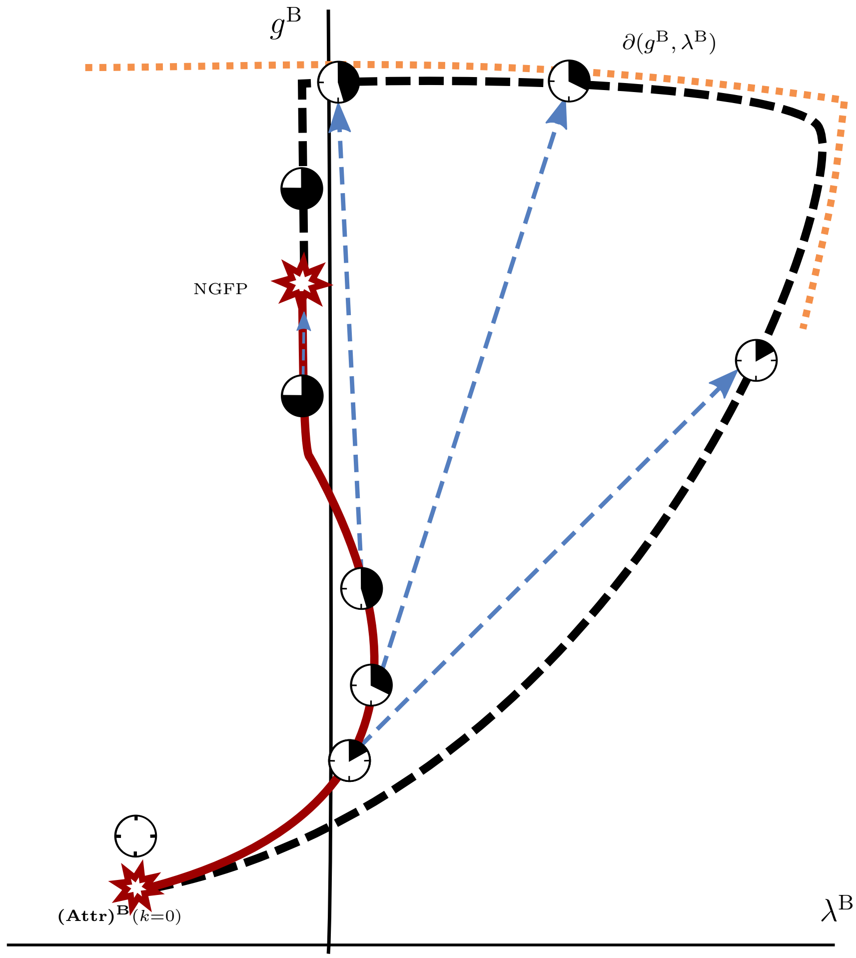


Figure 5. The dashed (black) and solid (red) curve show the k-dependent positions of $\mathbf{(Attr)^B}(k)$ and $\mathrm{Sol}^{\mathrm{B}}_{\bullet}(k)$ on the g^{B}-λ^{B}-plane. Recall that $\mathrm{Sol}^{\mathrm{B}}_{\bullet}(k)$ is an RG trajectory while $\mathbf{(Attr)^B}(k)$ is not. The (orange) dotted curve indicates the boundary '$\partial(g^{\mathrm{B}},\, \lambda^{\mathrm{B}})$' where the B-couplings diverge. The clocks mark equal-time positions on the curves; the black filling indicates the elapsed RG time for upward evolution. The dashed (blue) arrows indicate the direction in which $\mathrm{Sol}^{\mathrm{B}}_{\bullet}(k')$ is pulled at 'time' k', namely the position of the attractor at this instant of time, $\mathbf{(Attr)^B}(k')$. While initially, at $k=0$, the trajectory $\mathrm{Sol}^{\mathrm{B}}_{\bullet}(0)$ coincides with $\mathbf{(Attr)^B}(0)$ the curves depart due to their different velocities. They meet again at $\mathbf{NG^B_+}{\oplus}\mathbf{NG^{Dyn}_+}$**-FP** for $k \to \infty$. However, the motion of $\mathbf{(Attr)^B}(k)$ at intermediate scales is encoded in the indentation of the $\mathrm{Sol}^{\mathrm{B}}_{\bullet}(k)$ curve before it approaches $\mathbf{NG^B_+}{\oplus}\mathbf{NG^{Dyn}_+}$**-FP**.

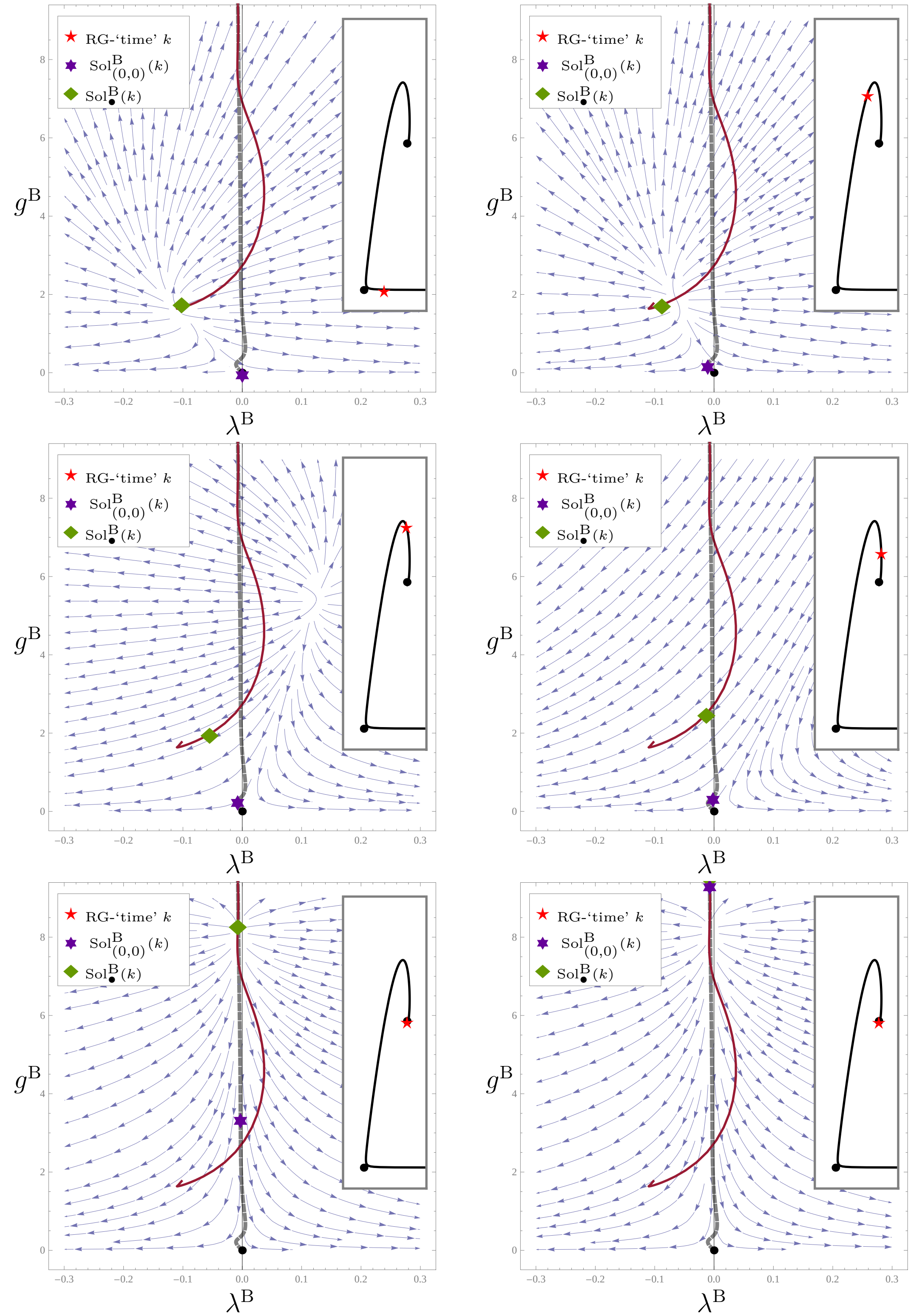


Figure 4. The B-phase portraits at increasing scales k. The underlying type $(\mathrm{IIIa})^{\mathrm{Dyn}}$ trajectory in the Dyn-sector is shown in the inset on the right, and the current RG time is marked with a star therein. The arrows point towards the IR and picture the instantaneous vector field in the B-sector. The (red) solid and the (gray) dashed curve highlight two important solutions in the B-sector, namely $\mathrm{Sol}^{\mathrm{B}}_{\bullet}(k)$ and $\mathrm{Sol}^{\mathrm{B}}_{(0,0)}(k)$, respectively. Their current position is indicated by the (green) diamond and the (violet) six-pointed star, respectively.

A bi-metric truncation: CREH

E. Manrique, M.R., 2009

$$\Gamma_k[g,\bar{g}] \equiv \Gamma_k[\phi, \chi_B] =$$

$$= -\frac{1}{16\pi G_k^{Dyn}} \int d^4x \sqrt{g}\left(R(g) - 2\Lambda_k^{Dyn}\right)$$

$$- \frac{1}{16\pi G_k^{B}} \int d^4x \sqrt{\bar{g}}\left(R(\bar{g}) - 2\Lambda_k^{B}\right)$$

$$- \frac{3}{8\pi} \frac{M_k}{G_k} \int d^4x \left(\sqrt{g}\sqrt{\bar{g}}\right)^{1/2}$$

$$g_{\mu\nu} = \phi^2 \delta_{\mu\nu}$$
$$\bar{g}_{\mu\nu} = \chi_B^2 \delta_{\mu\nu}$$

$$= \frac{3}{8\pi} \int d^4x \left\{ \frac{1}{G_k^{Dyn}} \phi \hat{\Box} \phi + \frac{1}{3} \frac{\Lambda_k^{Dyn}}{G_k^{Dyn}} \phi^4 \right.$$

$$+ \frac{1}{G_k^{B}} \chi_B \hat{\Box} \chi_B + \frac{1}{3} \frac{\Lambda_k^{B}}{G_k^{B}} \chi_B^4$$

$$\left. - \frac{1}{G_k} M_k \chi_B^2 \phi^2 \right\}$$

Generalize non-derivative terms to

bi-metric LPA

$$F_k(\phi, \chi_B) \rightsquigarrow Y_k(\varphi, b)$$

WISS :

$$b\,\partial_b Y_k = -\frac{g_k^2}{24\pi}\,\frac{b^6}{b^2+\partial_\varphi^2 Y_k}$$

FRGE:

$$\left[\partial_t + \varphi\,\partial_\varphi + b\,\partial_b - \eta_N - 2\right] Y_k$$

$$= -\frac{g_k^2}{24\pi}\left(1-\tfrac{1}{6}\eta_N\right)\frac{b^6}{b^2+\partial_\varphi^2 Y_k}$$

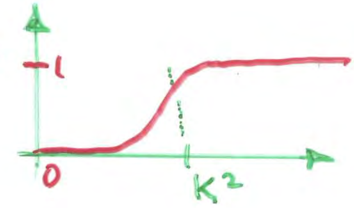
consistent ?

see arXiv: 0907.2617 , appendix B

"Counting" field modes: first hints

(for fields with even Grassmann parity)

(i) $$Z_k = \int \mathcal{D}\hat{\phi}\; e^{-S[\hat{\phi}]}\; e^{-\frac{1}{2}\int \hat{\phi} R_k \hat{\phi}}$$



$\Rightarrow$ $$\boxed{Z_k \searrow k \quad \text{and} \quad -\ln Z_k \nearrow \quad \text{monotonically}}$$

(ii) $$\partial_t \Gamma_k[\varphi;\bar{\phi}] = \frac{1}{2}\mathrm{Tr}\Big[\underbrace{\big(\Gamma_k^{(2)} + R_k\big)^{-1}}_{>0}\;\underbrace{\partial_t R_k}_{>0}\Big]$$

$\forall k > 0$ along well defined and complete RG trajectory

$\Rightarrow$ $$\boxed{\partial_k \Gamma_k[\varphi;\bar{\phi}] \geqslant 0 \qquad \forall \varphi, \bar{\phi}}$$

"pointwise monotonicity"

The dimensionless world of Theory Space ($\mathcal{T}$)

conventions: $\boxed{[x^\mu]=0}$ $\quad [ds^2]=[g_{\mu\nu}dx^\mu dx^\nu] = -2$

$$\Rightarrow \quad [\hat{g}_{\mu\nu}] = [\bar{g}_{\mu\nu}] = [g_{\mu\nu}] = -2$$
$$[\hat{h}_{\mu\nu}] = [h_{\mu\nu}] = -2$$

dimensionless fields: $\tilde{\varphi} := k^{-[\varphi]}\varphi \; , \; \tilde{\bar{\Phi}} = k^{-[\varphi]}\bar{\Phi}$

e.g.: $\tilde{h}_{\mu\nu} := k^2 h_{\mu\nu} \; , \; \tilde{\bar{g}}_{\mu\nu} = k^2 \bar{g}_{\mu\nu}$

dimensionless couplings: $u_\alpha := k^{-d_\alpha}\bar{u}_\alpha \; , \; d_\alpha \equiv [\bar{u}_\alpha]$

Then

$$\Gamma_k[\varphi;\bar{\Phi}] = \sum_\alpha \underbrace{\bar{u}_\alpha(k)}_{d_\alpha} \underbrace{I_\alpha[\varphi;\bar{\Phi}]}_{-d_\alpha}$$

and

$$\mathcal{A}_k[\tilde{\varphi};\tilde{\bar{\Phi}}] := \sum_\alpha \underbrace{u_\alpha(k)}_{0} \underbrace{I_\alpha[\tilde{\varphi};\tilde{\bar{\Phi}}]}_{0}$$

are <u>numerically</u> equal, i.e.

$$\Gamma_k[\varphi;\bar{\Phi}] = \mathcal{A}_k[\tilde{\varphi};\tilde{\bar{\Phi}}]\Big|_{\tilde{\varphi}=k^{-[\varphi]}\varphi \, , \, \tilde{\bar{\Phi}}=k^{-[\varphi]}\bar{\Phi}}$$

.... but the functionals are different :

$$\partial_t \mathcal{A}_k[\tilde{\varphi};\tilde{\bar{\Phi}}] = \sum_\alpha \partial_t u_\alpha \quad I_\alpha[\tilde{\varphi};\tilde{\bar{\Phi}}]$$

$$\partial_t \Gamma_k[\varphi;\bar{\Phi}] = \sum_\alpha \{\partial_t u_\alpha + d_\alpha u_\alpha\} k^{d_\alpha} I_\alpha[\varphi;\bar{\Phi}]$$

$$\overset{\text{FRGE}}{=} \frac{1}{2}\mathrm{Tr}[\cdots]$$

RG flow $(\mathcal{T}, \vec{\beta})$: the components of $\vec{\beta}$ are

$$\beta_\alpha \equiv \partial_t u_\alpha = -d_\alpha u_\alpha + \text{contrib. from } \underbrace{\frac{1}{2}\mathrm{Tr}[\cdots]}$$

pointwise monotonicity applies here only

Search for functions

$$\boxed{\begin{aligned} \mathcal{C} : \mathcal{T} &\longrightarrow \mathbb{R} \\ u &\mapsto \mathcal{C}(u) \end{aligned}}$$

with

- monotonicity along (some) RG trajectories,

$$\partial_t \mathcal{C}(u(k)) > 0$$

- stationarity at fixed points, $\vec{\beta}(u_*) = 0$,

$$\partial_t \mathcal{C}(u_*) = 0 \qquad \text{when } \mathcal{A}_k[\cdot] \longrightarrow \mathcal{A}_*[\cdot] .$$

Strategy:

- Exploit pointwise monotonicity
- compensate canonical terms by appropriate (scale dependent!) arguments

Self-consistent background configuration

- The general source-field relation:

$$\frac{1}{\sqrt{g}} \frac{\delta}{\delta\varphi(x)} \Gamma_k[\varphi;\bar{\Phi}] + \mathcal{R}_k \varphi(x) = J(x)$$

- Is it satisfied by $\varphi \equiv 0$ when $J = 0$?

- Background $\bar{\Phi} \equiv \bar{\Phi}_k^{sc}$ is self-consistent iff

$$\boxed{\frac{\delta}{\delta\varphi(x)} \Gamma_k[\varphi;\bar{\Phi}] \Big|_{\varphi=0,\, \bar{\Phi}=\bar{\Phi}_k^{sc}} = 0}$$

- Resulting on-shell EAA: FIDO $\Rightarrow$

$$e^{-\Gamma_k[0;\bar{\Phi}_k^{sc}]} = \int \mathcal{D}\hat{\varphi}\, e^{-(S+\Delta S_k)[\hat{\varphi};\bar{\Phi}_k^{sc}]}$$

Cf. "counting" property of Z_k !

Candidate for a generalized C-function :

$$\boxed{\mathscr{C}_k = \Gamma_k[\varphi=0;\bar{\Phi}_k^{sc}] \equiv \mathscr{A}_k[\tilde{\varphi}=0;\tilde{\bar{\Phi}}_k^{sc}]}$$

Properties :

(i) Stationarity at fixed points:

$$\mathscr{C}_k \xrightarrow{FP} \mathscr{C}_* = \mathscr{A}_*[0;\tilde{\bar{\Phi}}_k^{sc}] \quad , \quad (\partial_t \mathscr{C}_k)_* = 0$$

(ii) Stationarity in classical regimes:

$$\mathscr{C}_k \xrightarrow{CR} \mathscr{C}_{CR} = \Gamma_{CR}[0;\bar{\Phi}_k^{sc}] \quad , \quad (\partial_t \mathscr{C}_k)_{CR} = 0$$

(iii) Monotonicity for weak split-symmetry violation:

$$\partial_t \mathscr{C}_k = (\partial_t \Gamma_k)[0;\bar{\Phi}_k^{sc}] + \int d^dx \left(\partial_t \bar{\Phi}_k^{sc}(x)\right) \left.\frac{\delta \Gamma_k[0;\bar{\Phi}]}{\delta \bar{\Phi}(x)}\right|_{\bar{\Phi}=\bar{\Phi}_k^{sc}}$$

$$\equiv \underbrace{(\partial_t \Gamma_k)[\bar{\Phi}_k^{sc},\bar{\Phi}_k^{sc}]}_{>0 \text{ by pointwise monotonicity}} + \int d^dx \left(\partial_t \bar{\Phi}_k^{sc}\right) \underbrace{\frac{\delta \Gamma_k[\Phi,\bar{\Phi}]}{\delta \bar{\Phi}(x)}}_{\sim \text{"extra" } \bar{\Phi}\text{-dependence; } = 0 \text{ at exact split-sym.}}\Bigg|_{\Phi=\bar{\Phi}=\bar{\Phi}_k^{sc}}$$

FRGE

WISS

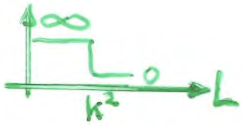
Interpretation of $\mathcal{C}_k$:

evaluates / defines "number of modes" integrated out

$$\mathcal{N}_{k_1 k_2} \equiv \mathcal{C}_{k_2} - \mathcal{C}_{k_1} \quad = \# \text{ eigenvalues of } \Gamma_k^{(2)}[0;\bar{\Phi}_k^{sc}] \text{ in } [k_1^2, k_2^2]$$

Simplest situation:

- violation of split-sym. negligible
- cutoff operator : $\Gamma_k^{(2)}[0;\bar{\Phi}_k^{sc}] \equiv L$
- sharp cutoff :



$$R_k = \hat{R}\,\Theta(k^2 - L) \quad , \quad \hat{R} \to \infty$$

$$\Rightarrow \quad \boxed{\frac{d}{dk^2}\mathcal{C}_k = \mathrm{Tr}\left[\delta\left(k^2 - \Gamma_k^{(2)}[0;\bar{\Phi}_k^{sc}]\right)\right]}$$

if k-dep. is weak :

$$\mathcal{C}_k = \mathrm{Tr}\left[\Theta\left(k^2 - \Gamma_k^{(2)}[0;\bar{\Phi}_k^{sc}]\right)\right] + \text{const}$$

exactly a spectral density !

III a

Self-consistent background:

Gravitational instantons, Einstein spaces $\mathcal{M}$

tadpole eq.:
$$\boxed{G_{\mu\nu}\left(\bar{g}^{\,sc}_k\right) = -\Lambda^{(1)}_k\, \bar{g}^{\,sc}_{k\,\mu\nu}}$$

$$\Rightarrow \qquad \bar{g}^{\,sc}_{k\,\mu\nu} = \frac{\mathring{\Lambda}}{\Lambda^{(1)}_k}\,\mathring{g}_{\mu\nu} \qquad \text{with} \qquad G_{\mu\nu}(\mathring{g}) = -\mathring{\lambda}\,\mathring{g}_{\mu\nu}$$

$\mathscr{C}$-function:
$$\boxed{\begin{aligned}\mathscr{C}_k &= \Gamma^{\text{grav}}_k\left[0;\bar{g}^{\,sc}_k\right] \\ &= -\frac{1}{8\pi G^{(0)}_k}\left[\left(\frac{d}{d-2}\right)\Lambda^{(1)}_k - \lambda^{(0)}_k\right]\cdot \mathrm{Vol}\left(\mathcal{M},\bar{g}^{\,sc}_k\right)\end{aligned}}$$

$$\mathrm{Vol}\left(\mathcal{M},\bar{g}^{\,sc}_k\right) = 8\pi\left[\Lambda^{(1)}_k\right]^{-d/2}\,\vartheta$$ must be finite

$$\vartheta \equiv \frac{1}{8\pi}\,\mathring{\lambda}^{\,d/2}\,\mathrm{Vol}\left(\mathcal{M},\mathring{g}\right)$$ independent of $\mathring{\lambda}$, has topological significance

$$\boxed{\begin{aligned}\mathscr{C}_k &\equiv \mathscr{C}\left(g^{(0)}_k, \lambda^{(0)}_k, \lambda^{(1)}_k\right) \\ &\equiv \underbrace{Y\left(g^{(0)}_k, \lambda^{(0)}_k, \lambda^{(1)}_k\right)}_{\text{dep. on trajectory}}\cdot \underbrace{\vartheta}_{\text{dep. on instanton}} \\ Y\left(g^{(0)}, \lambda^{(0)}, \lambda^{(1)}\right) &= \frac{-\left(\frac{d}{d-2}\right)\lambda^{(1)} + \lambda^{(0)}}{g^{(0)}\left[\lambda^{(1)}\right]^{d/2}}\end{aligned}}$$

$$\mathscr{C}, Y : \mathcal{T} \longrightarrow \mathbb{R}$$

Some 4D gravitational instantons:

metric $\mathring{g}$	$\mathcal{M}$	$\mathcal{V}$
eucl. de Sitter	S^4	3π
Page	$P_2 + \overline{P}_2$	$1.8\,\pi$
$S^2 \times S^2$	$S^2 \times S^2$	2π
Fubini-Study	$P_2(\mathbb{C})$	$9\pi/4$

$\boxed{d=4}$

bi-metric :

$$Y(g^{(0)}, \lambda^{(0)}, \lambda^{(1)}) = - \frac{2\lambda^{(1)} - \lambda^{(0)}}{g^{(0)} [\lambda^{(1)}]^2}$$

bi-metric, B-Dyn language :

$$Y(g^{B}, g^{Dyn}, \lambda^{B}, \lambda^{Dyn}) =$$

$$= - \frac{1}{g^{Dyn} \lambda^{Dyn}} - \frac{1}{g^{B} \lambda^{Dyn}} \left[2 - \frac{\lambda^{B}}{\lambda^{Dyn}} \right]$$

single-metric truncation:

$$Y^{sm}(g^{sm}, \lambda^{sm}) = - \frac{1}{g^{sm} \lambda^{sm}}$$

Numerical results for type IIIa trajectories

1. Single-metric EH-truncation:

$\mathscr{C}_k$ is non-monotone along all trajectories.

2. Bi-metric EH-truncation:

- $\mathscr{C}_k$ is monotone along all trajectories which restore split-symmetry.
- Monotonicity can be destroyed by selecting (unphysical!) symmetry-non-restoring trajectories.

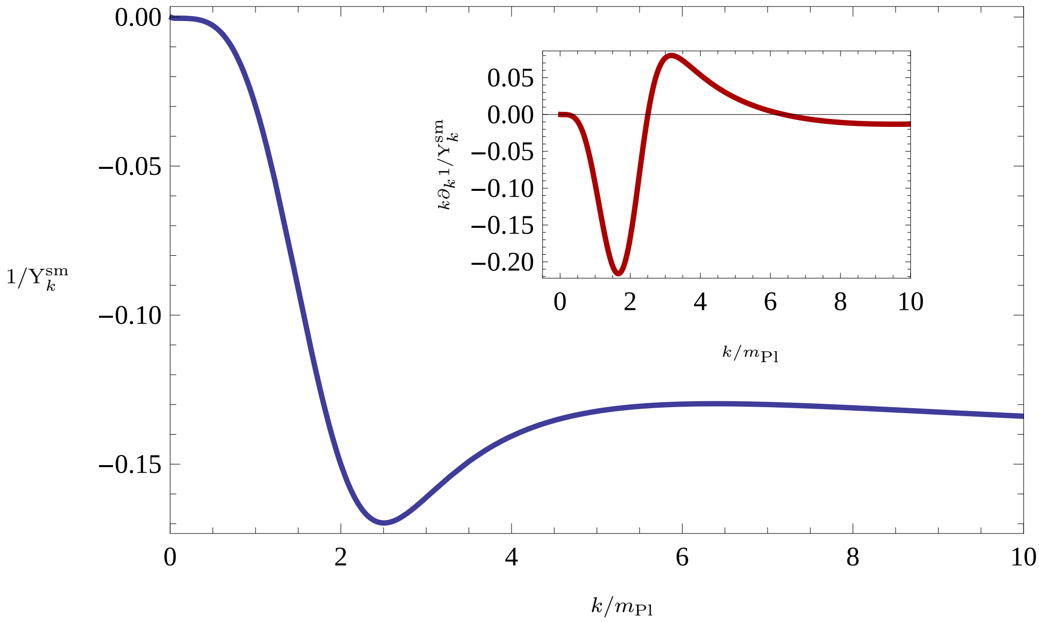


Figure 1. The inverse of $\mathrm{Y}_k^{\mathrm{sm}}$ for a typical single-metric type IIIa trajectory. The inset shows its k-derivative whose positive values indicate a violation of monotonicity.

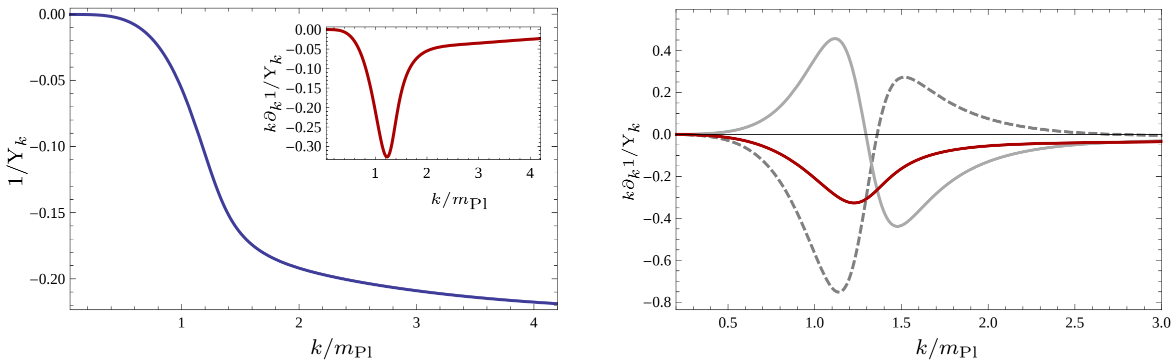


Figure 2. The left plot shows $1/\mathrm{Y}_k$ and its scale derivative for a typical bi-metric type IIIa trajectory that restores split-symmetry in the IR. It is based on the RG equations of [I]. For these trajectories, $\mathscr{C}_k$ is always found to be perfectly monotone. The inset in the left plot shows $k\partial_k 1/\mathrm{Y}_k$, which is decomposed in the right plot into the derivative of the split-symmetric component $1/\mathrm{Y}_k^{\text{split-sym},(1)}$ (dashed, gray curve) and of $\Delta(1/\mathrm{Y}_k)$ (solid, gray curve). Neither of the two contributions is negative definite separately, but their sum is (solid, dark red curve).

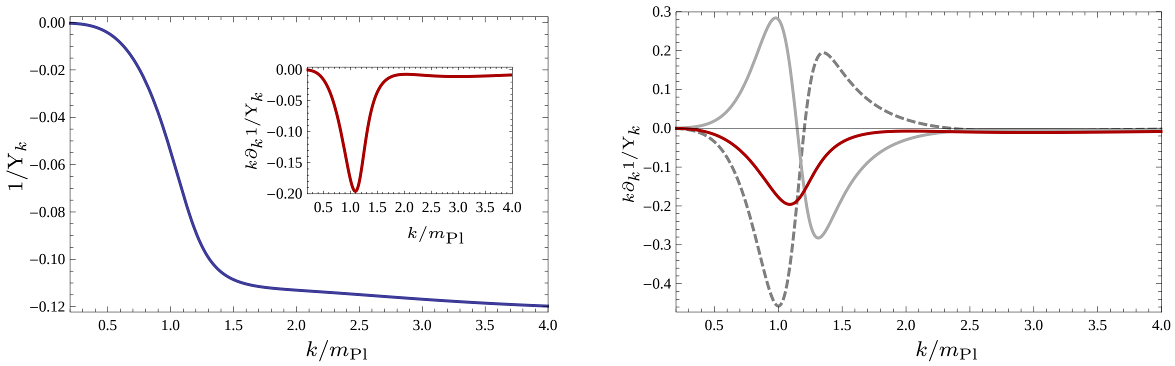


Figure 3. The function $1/\mathrm{Y}_k$ as in Fig. 2, but now based on the RG equations of [II].

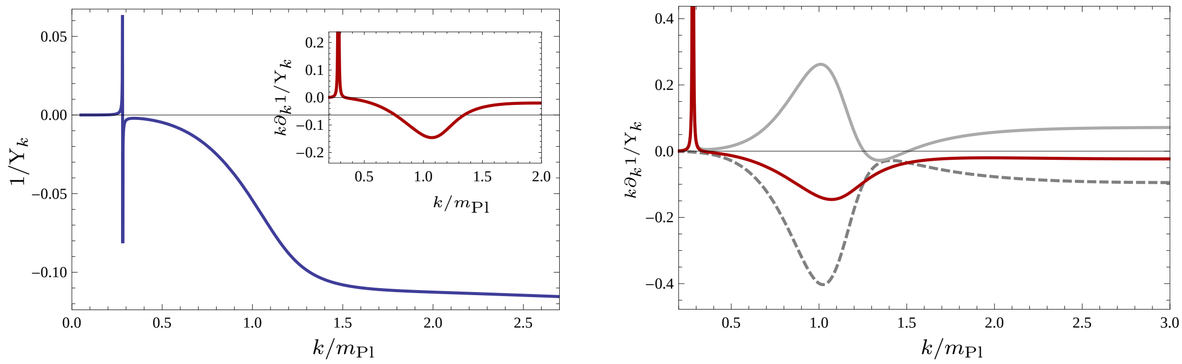


Figure 4. The left plot shows the function $1/\mathrm{Y}_k$ for a bi-metric trajectory of type IIIa that does *not* restore split-symmetry in the IR. It is based on the RG equations of [I]. We observe a sign-change of $k\partial_k(1/\mathrm{Y}_k)$ at moderate values of k, indicating a violation of monotonicity. In the decomposed form of $k\partial_k(1/\mathrm{Y}_k)$, shown in the right plot, we see that the contribution $1/\mathrm{Y}_k^{\text{split-sym},(1)}$ is in fact monotone, but the correction term $\Delta(1/\mathrm{Y}_k)$ is not, and neither is their sum. Not restoring split-symmetry in the IR results in a violation of the monotonicity of $\mathscr{C}_k$ along the trajectory considered.

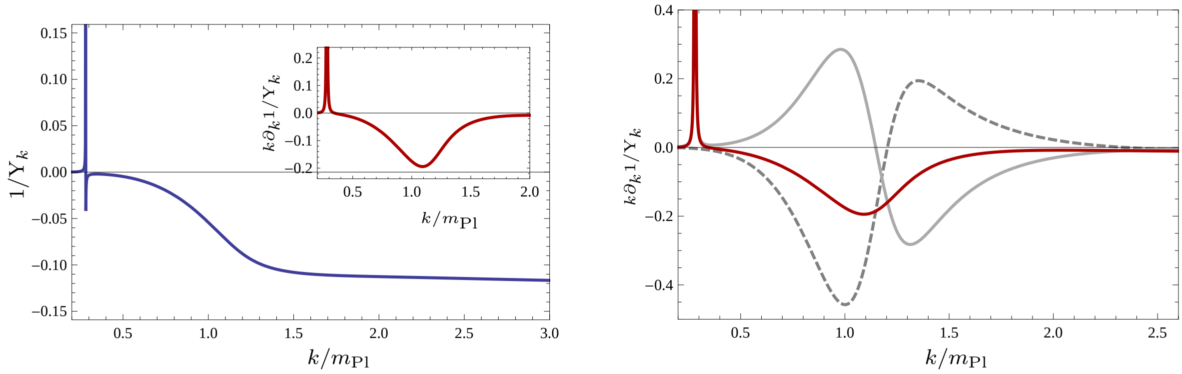


Figure 5. The function $1/\mathrm{Y}_k$ as in Fig. 4, but now based on the RG equations of [II].

- $\partial_t \mathcal{C}_k = \left(\beta_\alpha \dfrac{\partial Y}{\partial u_\alpha}\right)(u(k)) \cdot \vartheta \quad \Rightarrow$

$\partial_t \mathcal{C}_k > 0$ if trajectory is inside

$$\mathcal{T}_+ \equiv \left\{ u \in \mathcal{T} \;\middle|\; \beta_\alpha(u) \frac{\partial Y}{\partial u_\alpha}(u) > 0 \right\} \subset \mathcal{T}$$

- Fix $(g_k^{Dyn}, \lambda_k^{Dyn})$ - trajectory. $\Rightarrow$

$\partial_t \mathcal{C}_k > 0$ if projected trajectory is inside subset of $g^B - \lambda^B$ — plane :

$$\mathcal{T}_+^B(k) \equiv \left\{ (g^B, \lambda^B) \in \mathbb{R}^2 \;\middle|\; (g_k^{Dyn}, \lambda_k^{Dyn}, g^B, \lambda^B) \in \mathcal{T}_+ \right\}$$

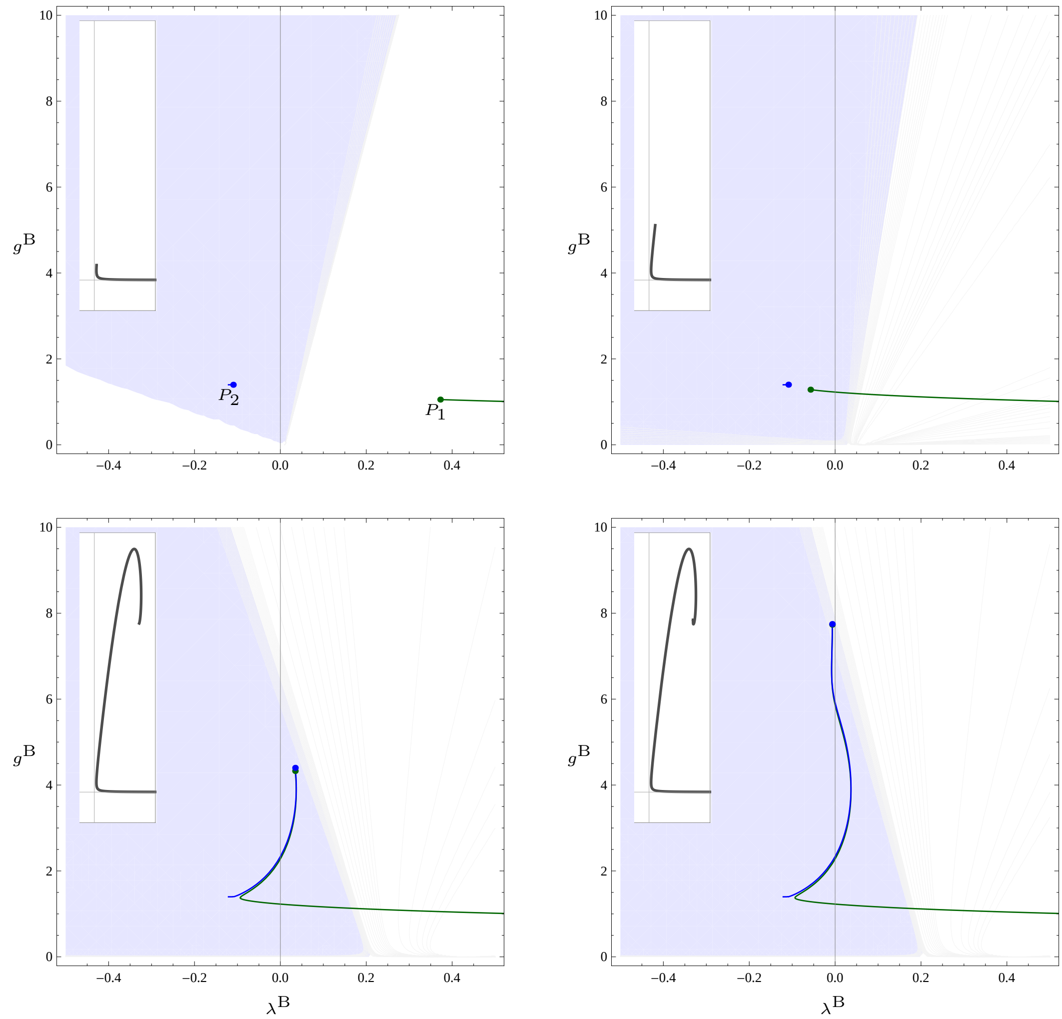


Figure 6. This series of snapshots represents the g^{B}-λ^{B}-plane at four RG times which increase from the upper left to the lower right diagram. They are given by the maximum k-value of the incomplete dynamical trajectory $k \mapsto (g_k^{\mathrm{Dyn}}, \lambda_k^{\mathrm{Dyn}})$ shown in the respective inset. The shaded regions correspond to $\mathcal{T}_+^{\mathrm{B}}(k)$ at that particular time; hence every trajectory in the shaded (white) region will give rise to a positive (negative) value of $k\partial_k \mathscr{C}_k$ at the instant of time k. Furthermore, two different B-trajectories that are evolved upward (towards increasing scales k) are shown at the corresponding moments. The one passing the point P_1 (P_2) is split-symmetry violating (restoring). The symmetry restoring trajectory starts its upward evolution close to P_2, the position of the running UV attractor [**?**]; we see that this trajectory never leaves the shaded area, and thus its $\mathscr{C}_k$-function is strictly monotone. This is different for the trajectory through P_1: Attracted by the running UV-attractor, it is pulled into the shaded region, thus unavoidably crossing the boundary of $\mathcal{T}_+^{\mathrm{B}}(k)$, which causes a sign flip of $\partial_k \mathscr{C}_k$, rendering $\mathscr{C}_k$ non-monotone.

Testing pointwise monotonicity

$$\partial_t \Gamma_k[\varphi;\bar{\Phi}] \geq 0 \qquad \forall \varphi,\bar{\Phi} \quad \forall k$$

could be destroyed by truncation!

Check at special arguments:

$$(\partial_t \Gamma_k)[0;\bar{\Phi}_k^{sc}] \overset{?}{\geq} 0$$

Results:

(1) sm-trunc. : violated!

(2) bm-trunc., no sym. restoration: violated!

(3) bm-trunc., sym. restoration: satisfied!

Implication:

Violated $\mathcal{C}_k$-monotonicity in cases (1) and (2) is not due to structural defect of the $\mathcal{C}_k$-candidate but rather to an insufficient truncation.

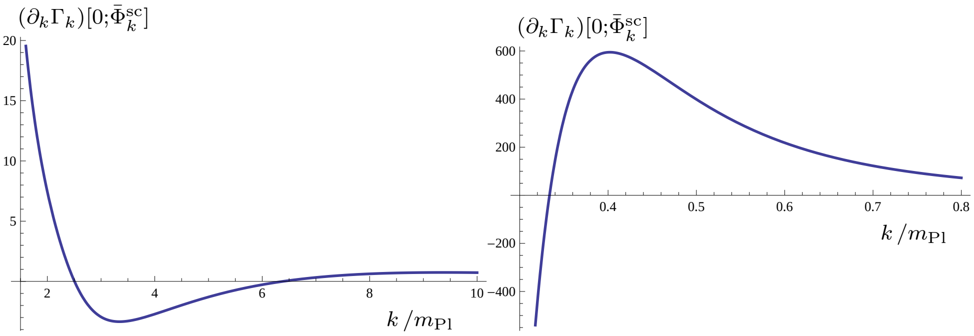


Figure 1. The quantity $(\partial_k\Gamma_k)\,[0;\bar{\Phi}_k^{\rm sc}]$ is evaluated for a typical single-metric (left) and split-symmetry violating bi-metric (right) trajectory. In both cases, it is seen to be negative for certain scales. This indicates a severe failure of the underlying approximation since, at the exact level, $(\partial_k\Gamma_k)$ is known to be positive at all field arguments and for any k.

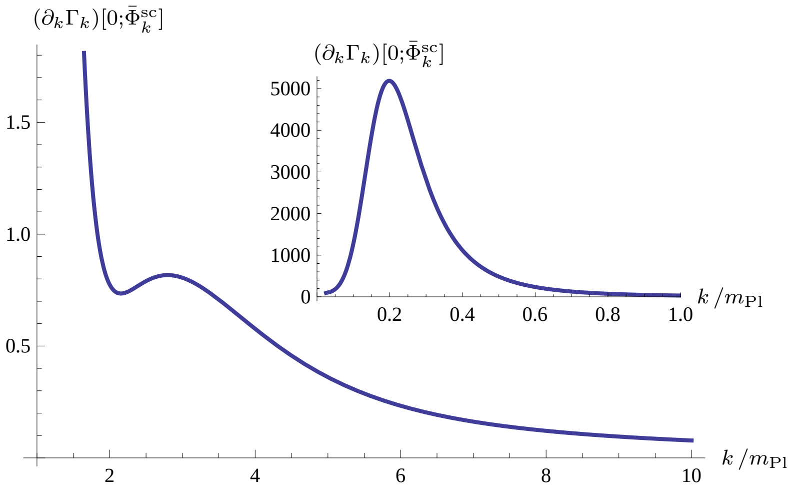


Figure 2. The quantity $(\partial_k\Gamma_k)\,[0;\bar{\Phi}_k^{\rm sc}]$ is now evaluated for a split-symmetry restoring bi-metric trajectory. It always stays non-negative, even in those regimes where the single-metric or the split-symmetry violating bi-metric (see inset) trajectories fail the pointwise monotonicity test.

Generalized cross-over FP $\longrightarrow$ CR

$$\mathcal{C}^{UV} = \mathcal{C}_* = -O(1)$$

$$\mathcal{C}^{IR} = \mathcal{C}_{CR} = -\frac{3\pi}{G_{CR}\Lambda_{CR}}$$

Analog of <u>IR-central charge</u>:

Bekenstein-Hawking entropy of de Sitter space.

<u>"Integrated c-theorem":</u> # modes $\sim$

$$\boxed{\mathcal{N} \equiv \mathcal{C}^{UV} - \mathcal{C}^{IR} \approx \frac{3\pi}{G_{CR}\Lambda_{CR}}}$$

$\mathcal{C}^{UV}$, $\mathcal{C}^{IR}$, $\mathcal{N}$ are finite!

for type IIIa only !!

Thanks to Asymptotic Safety!

[Cf. Banks, Bousso: "Λ-N-connection", N-bound]

Conclusion

- There seems to be significant evidence supporting the existence of a "C-function" in QEG_4 which is stationary in FP- and classical regimes, and monotonically decreasing along all Background Independence-restoring RG trajectories with $\Lambda > 0$ on all scales.

- The candidate $\mathscr{C}_k = \Gamma_k[0; \Phi_k^{sc}]$ achieves an impressive compensation of the symmetry-violation inherent in $\Gamma_k[\cdot]$ by its running argument.