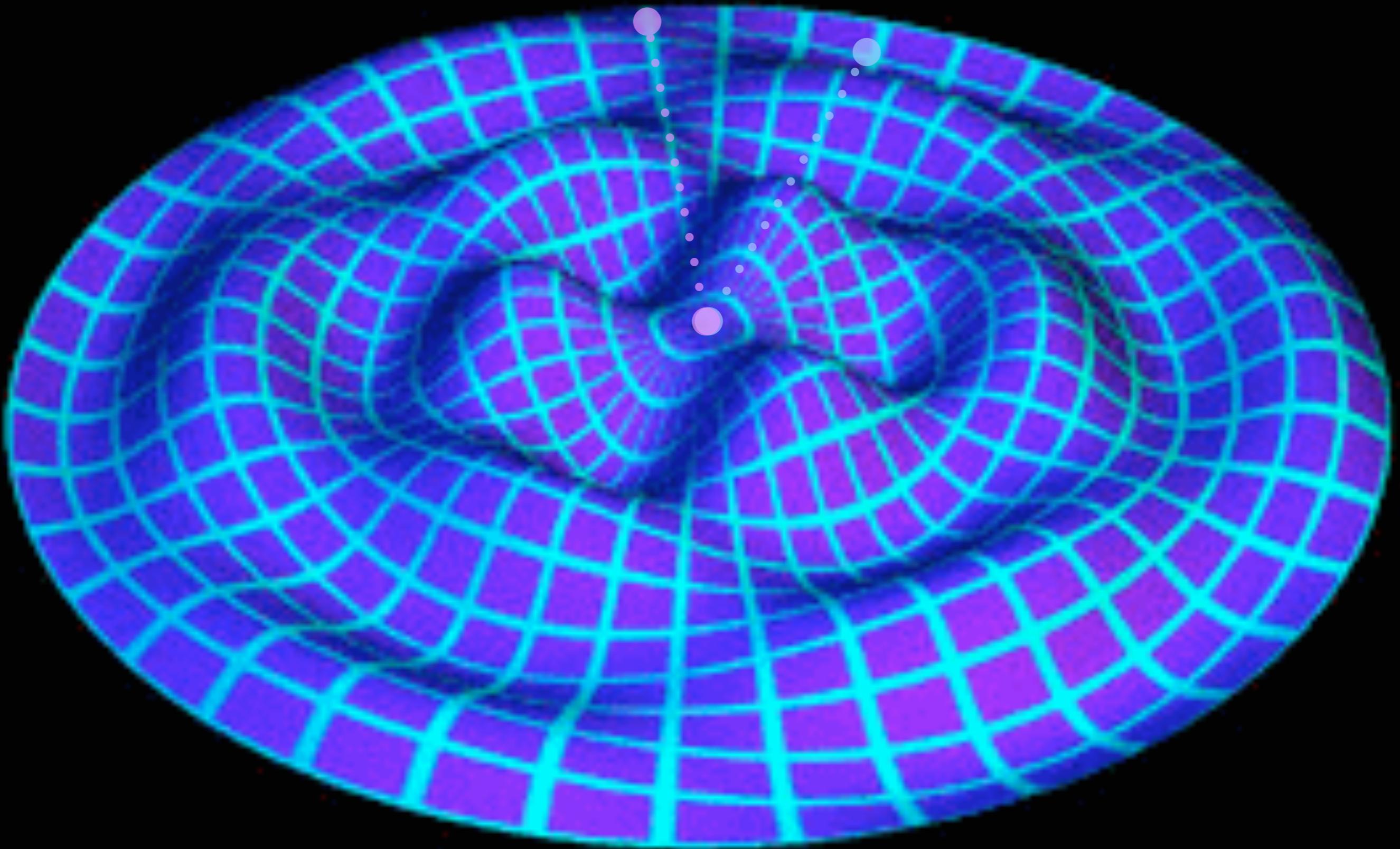


# Background independent ERG for conformally reduced gravity

Probing the fundamental nature of spacetime with the RG  
NORDITA 23/3/15

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# What does scale $1/k$ mean?



Conformal field theory? How to separate scales?  
What does scale  $1/k$  mean? What is long-range scaling?  
How to form a continuum limit?

Use background field method:

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}$$

cutoff:

$$R_k \sim r \left( -\frac{\bar{\nabla}^2}{k^2} \right)$$

- Now the scale  $k$  is defined through  $\bar{g}_{\mu\nu}$  so the notion depends on the choice of background
- The RG itself depends on both  $g_{\mu\nu}$  and  $\bar{g}_{\mu\nu}$  so also is inherently background dependent, and necessarily bi-metric
- Is there a background independent notion of scale  $k$ ?

Why not single metric approximation?

# Scalar field theory @ LPA

I.H. Bridle, J. Dietz & T.R. Morris, JHEP 03 (2014) 093

$$\Gamma[\phi] = \int d^d x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right\}$$

$$\phi = \varphi + \chi$$

$$\mathcal{R}(-\partial^2, \chi(x)) = (k^2 + \partial^2 - h(\chi)) \theta(k^2 + \partial^2 - h(\chi))$$

Single field approximation □ pathologies

Keep both fields & impose msWI (broken split Ward Identity).

Transform to background independent variables.

# Conformally reduced gravity

$$\phi = \varphi + \chi$$

$$g_{\mu\nu} = f(\phi) \delta_{\mu\nu} \quad \text{and} \quad \bar{g}_{\mu\nu} = f(\chi) \delta_{\mu\nu}$$

- Remnant diffeomorphism invariance ...

$$x^\mu \mapsto x^\mu / \lambda, \quad f(\chi) \mapsto \lambda^2 f(\chi)$$

$$\text{F.g. } f(\phi) = \int d^d x \sqrt{2\phi} \left( \frac{1}{2} K \left( \frac{\varphi}{\phi}, \chi \right) \partial^\mu \varphi \partial_\mu \phi - \frac{1}{2} K \left( \frac{\chi}{\phi}, \chi \right) \partial^\mu \chi \partial_\mu \phi \right)$$

Maintain background covariance:

- $\varphi$  at  $\mathcal{O}(\partial^2)$  but slowly varying  $\chi$

- Conformal factor problem  $\phi \mapsto \phi + \ln \lambda$ : wrong sign kinetic term

- $K$  has absorbed the factor  $1/8\pi G$  and  $\sqrt{\bar{g}}(a)$  so (r.b.)  $R_k[\bar{g}]$   $\mapsto$   $1/(6\pi G)$

Break quantum covariance factors of  $f(\chi)$

$$\chi \mapsto \chi, \quad \phi \mapsto \phi + \ln \lambda, \quad \varphi \mapsto \varphi + \ln \lambda$$

# Flow equation

$$\partial_t \Gamma_k[\varphi, \chi] = \frac{1}{2} \int_x \int_y \Delta(x, y) \partial_t R_k(y, x)$$

$$\int_x \equiv \int d^d x f(x)^{\frac{d}{2}}$$

$$f(x) \equiv f(\chi(x))$$

$$\Delta(x, y) = \left[ f(x)^{-\frac{d}{2}} f(y)^{-\frac{d}{2}} \frac{\delta^2 \Gamma_k}{\delta \varphi(x) \delta \varphi(y)} + R_k(x, y) \right]^{-1}$$

$$t = \ln(k/\mu)$$

# Broken split Ward Identity (msWI)

$$\varphi(x) \mapsto \varphi(x) + \varepsilon(x) \quad \chi(x) \mapsto \chi(x) - \varepsilon(x) \quad \phi = \varphi + \chi$$

$$\int_w f^{-\frac{d}{2}} \left( \frac{\delta \Gamma_k}{\delta \chi} - \frac{\delta \Gamma_k}{\delta \varphi} \right) \varepsilon = \frac{1}{2} \int_x \int_y \Delta(x, y) \left[ \frac{d}{2} \partial_\chi \ln f(y) R_k(y, x) + \partial_\chi R_k(y, x) \right] \varepsilon(y)$$

slow  $\varepsilon(x)$

# Sketch of derivative expansion...

Write trace in terms of differential operators...

$$E.g. \quad R_{k,x} = R_k (-\bar{\nabla}_x^2) = R_k (-f(\chi)^{-1} \partial_x^2)$$

$$\int d^d x \Delta_x \mathcal{K}_x \delta(x - x') \Big|_{x' = x} = \int d^d x \frac{d^d p}{(2\pi)^d} \underbrace{\left\{ e^{-ip \cdot x} \Delta_x e^{ip \cdot x} \right\}}_{\text{---}} \mathcal{K}(f^{-1} p^2)$$

$$Q(p, x) \equiv e^{-ip \cdot x} \left[ \Gamma^{(2)} + R \right]^{-1} e^{ip \cdot x}$$

satisfies differential equation:

$$\underbrace{e^{-ip \cdot x} R \left\{ e^{ip \cdot x} Q \right\}}_{\text{---}} = 1 - e^{-ip \cdot x} \Gamma^{(2)} \left\{ e^{ip \cdot x} Q \right\}$$

$$\begin{aligned} R(p^2/f)Q - 2i\partial_{p^2}R(p^2/f)p^\mu\partial_\mu Q - \partial_{p^2}R(p^2/f)\partial^2 Q \\ - 2\partial_{p^2}^2R(p^2/f)p^\mu p^\nu\partial_\mu\partial_\nu Q + \mathcal{O}(\partial^3) \end{aligned}$$

iterate

# Flow equations, msWIs

$$\partial_t V(\varphi, \chi) = f(\chi)^{-\frac{d}{2}} \int_0^\infty dp p^{d-1} \frac{\dot{R}(p^2/f)}{\partial_\varphi^2 V - K p^2/f + R(p^2/f)}$$

$$\partial_\chi V - \partial_\varphi V + \frac{d}{2} \partial_\chi \ln f V = f(\chi)^{-\frac{d}{2}} \int_0^\infty dp p^{d-1} \frac{\partial_\chi R + \frac{d}{2} \partial_\chi \ln f R}{\partial_\varphi^2 V - K p^2/f + R(p^2/f)}$$

$$f^{-1} \partial_t K(\varphi, \chi) = 2 f^{-\frac{d}{2}} \int_0^\infty dp p^{d-1} P \dot{R}$$

$$f^{-1} \left\{ \partial_\chi K - \partial_\varphi K + \frac{d-2}{2} \partial_\chi \ln f K \right\} = 2 f^{-\frac{d}{2}} \int_0^\infty dp p^{d-1} P \left[ \partial_\chi R + \frac{d}{2} \partial_\chi \ln f R \right]$$

$$\begin{aligned} P &= -\frac{1}{2} \frac{\partial_\varphi^2 K}{f} Q_0^2 + \frac{\partial_\varphi K}{f} \left( 2 \partial_\varphi^3 V - \frac{2d+1}{d} \frac{\partial_\varphi K}{f} p^2 \right) Q_0^3 \\ &\quad - \left[ \left\{ \frac{4+d}{d} \frac{\partial_\varphi K}{f} p^2 - \partial_\varphi^3 V \right\} \left( \partial_{p^2} R - \frac{K}{f} \right) + \frac{2}{d} p^2 \partial_{p^2}^2 R \left( \frac{\partial_\varphi K}{f} p^2 - \partial_\varphi^3 V \right) \right] \left( \partial_\varphi^3 V - \frac{\partial_\varphi K}{f} p^2 \right) Q_0^4 \\ &\quad - \frac{4}{d} p^2 \left( \partial_{p^2} R - \frac{K}{f} \right)^2 \left( \partial_\varphi^3 V - \frac{\partial_\varphi K}{f} p^2 \right)^2 Q_0^5 \end{aligned}$$

$$Q_0 = \frac{1}{\partial_\varphi^2 V - K p^2/f + R}$$

Want momentum dependence so that:

$$\dot{R}(p^2/f) \propto \partial_\chi R + \frac{d}{2} \partial_\chi \ln f R$$

Cutoff dimensionless cutoff profile

$$R(p^2/f) = -k^{d-\eta-\frac{d}{2}d_f} r\left(\frac{p^2}{k^{2-d_f} f}\right)$$

$$\implies \partial_\chi \ln f \dot{R} = (2 - d_f) \left[ \partial_\chi R + \frac{d}{2} \partial_\chi \ln f R \right] - \eta \partial_\chi \ln f R$$

•  $\eta = 0$  ✓

$$\eta \partial_\chi \ln f R \propto \partial_\chi R + \frac{d}{2} \partial_\chi \ln f R \implies f \frac{\partial R}{\partial f} \propto R$$

•  $r(z) = \frac{1}{z^n}$  ✓

# Linear PDEs

$$\alpha = 2 \left( 1 - \frac{\eta}{d+2n} \right) - d_f$$

$$\partial_\chi \ln f \partial_t V = \alpha \left( \frac{d}{2} \partial_\chi \ln f V + \partial_\chi V - \partial_\varphi V \right)$$

$$\partial_\chi \ln f \partial_t K = \alpha \left( \partial_\chi K - \partial_\varphi K + \frac{d-2}{2} \partial_\chi \ln f K \right)$$

## Solutions

$$V(k, \varphi, \chi) = f(\chi)^{-\frac{d}{2}} \tilde{V}(\tilde{k}, \phi), \quad K(k, \varphi, \chi) = f(\chi)^{-\frac{d}{2}+1} \tilde{K}(\tilde{k}, \phi), \quad \tilde{k} = k f(\chi)^{\frac{1}{\alpha}}$$

## Dimensionless variables

$$\hat{k} = \tilde{k}^{\frac{1}{1+d_f/\alpha}} = k^{\frac{1}{1+d_f/\alpha}} f(\chi)^{\frac{1}{\alpha+d_f}} \quad [\hat{k}] = 1$$

$$\tilde{V} = \hat{k}^d \hat{V}_{\hat{k}}(\hat{\phi}), \quad \tilde{K} = \hat{k}^{d-2-\eta} \hat{K}_{\hat{k}}(\hat{\phi}), \quad \phi = \hat{k}^{\frac{\eta}{2}} \hat{\phi}, \quad p = \hat{k} \hat{p}$$

# msWIs, flow equations

$$\partial_t V(\varphi, \chi) = f(\chi)^{-\frac{d}{2}} \int_0^\infty dp p^{d-1} \frac{\dot{R}(p^2/f)}{\partial_\varphi^2 V - K p^2/f + R(p^2/f)}$$

$$\partial_\chi V - \partial_\varphi V + \frac{d}{2} \partial_\chi \ln f V = f(\chi)^{-\frac{d}{2}} \int_0^\infty dp p^{d-1} \frac{\partial_\chi R + \frac{d}{2} \partial_\chi \ln f R}{\partial_\varphi^2 V - K p^2/f + R(p^2/f)}$$

$$f^{-1} \partial_t K(\varphi, \chi) = 2 f^{-\frac{d}{2}} \int_0^\infty dp p^{d-1} P \dot{R}$$

$$f^{-1} \left\{ \partial_\chi K - \partial_\varphi K + \frac{d-2}{2} \partial_\chi \ln f K \right\} = 2 f^{-\frac{d}{2}} \int_0^\infty dp p^{d-1} P \left[ \partial_\chi R + \frac{d}{2} \partial_\chi \ln f R \right]$$

$$\begin{aligned} P = & -\frac{1}{2} \frac{\partial_\varphi^2 K}{f} Q_0^2 + \frac{\partial_\varphi K}{f} \left( 2 \partial_\varphi^3 V - \frac{2d+1}{d} \frac{\partial_\varphi K}{f} p^2 \right) Q_0^3 \\ & - \left[ \left\{ \frac{4+d}{d} \frac{\partial_\varphi K}{f} p^2 - \partial_\varphi^3 V \right\} \left( \partial_{p^2} R - \frac{K}{f} \right) + \frac{2}{d} p^2 \partial_{p^2}^2 R \left( \frac{\partial_\varphi K}{f} p^2 - \partial_\varphi^3 V \right) \right] \left( \partial_\varphi^3 V - \frac{\partial_\varphi K}{f} p^2 \right) Q_0^4 \\ & - \frac{4}{d} p^2 \left( \partial_{p^2} R - \frac{K}{f} \right)^2 \left( \partial_\varphi^3 V - \frac{\partial_\varphi K}{f} p^2 \right)^2 Q_0^5 \end{aligned}$$

$$Q_0 = \frac{1}{\partial_\varphi^2 V - K p^2/f + R}$$

# Background Independent flow equations

$$\partial_{\hat{t}} \hat{V} + d\hat{V} - \frac{\eta}{2} \hat{\phi} \hat{V}' = - (d - \eta + 2n) \int_0^\infty d\hat{p} \, \hat{p}^{d-1} \hat{Q}_0 r(\hat{p}^2)$$

- No  $\chi$ :  $\hat{V} \equiv \hat{V}_{\hat{k}}(\hat{\phi})$ ,  $\hat{K} \equiv \hat{K}_{\hat{k}}(\hat{\phi})$ !

$$\partial_{\hat{t}} \hat{K} + (d - 2 - \eta) \hat{K} - \frac{\eta}{2} \hat{\phi} \hat{K}' = -2(d - \eta + 2n) \int_0^\infty d\hat{p} \, \hat{p}^{d-1} \hat{P} r(\hat{p}^2)$$

- No  $f$  !  $\hat{t} = \ln(\hat{k}/\mu)$

$$\begin{aligned} \hat{P} = & -\frac{1}{2} \hat{K}'' \hat{Q}_0^2 + \hat{K}' \left( 2\hat{V}''' - \frac{2d+1}{d} \hat{K}' \hat{p}^2 \right) \hat{Q}_0^3 \\ & + \left[ \left\{ \frac{4+d}{d} \hat{K}' \hat{p}^2 - \hat{V}''' \right\} \left( r'(\hat{p}^2) + \hat{K} \right) + \frac{2}{d} \hat{p}^2 r''(\hat{p}^2) \left( \hat{K}' \hat{p}^2 - \hat{V}''' \right) \right] \left( \hat{V}''' - \hat{K}' \hat{p}^2 \right) \hat{Q}_0^4 \\ & - \frac{4}{d} \hat{p}^2 \left( r'(\hat{p}^2) + \hat{K} \right)^2 \left( \hat{V}''' - \hat{K}' \hat{p}^2 \right)^2 \hat{Q}_0^5 \end{aligned}$$

$$\hat{Q}_0 = \frac{1}{\hat{V}'' - \hat{K} \hat{p}^2 - r(\hat{p}^2)}$$

# Two notions of fixed point

$$V(\varphi, \chi) = k^{d - \frac{d}{2} d_f} \bar{V}_k(\bar{\varphi}, \bar{\chi}), \quad \varphi = k^{\eta/2} \bar{\varphi}, \quad \chi = k^{\eta/2} \bar{\chi}$$

$$= f(\chi)^{-\frac{d}{2}} \hat{k}^d \hat{V}_{\hat{k}}(\hat{\phi})$$

$$\partial_t \bar{V} = (k^{-d_f} f)^{\frac{d\eta}{2(d-\eta+2n)}} \left\{ \frac{1}{\alpha + d_f} \left( \alpha + \frac{\eta}{2} \chi \partial_\chi \ln f \right) \partial_{\hat{t}} \hat{V}(\hat{\phi}) \right.$$

$$\left. + \left( \frac{\eta}{2} \chi \partial_\chi \ln f - d_f \right) \left( \frac{d\eta}{2(d-\eta+2n)} \hat{V} - \frac{\eta}{2(\alpha + d_f)} \hat{\phi} \hat{V}' \right) \right\}$$

If  $\eta \chi \partial_\chi \ln f$  varies with  $\chi$  then

$$\partial_t \bar{V} = 0 \implies \partial_{\hat{t}} \hat{V} = 0 \quad \text{and}$$

$$\frac{d\eta}{2(d-\eta+2n)} \hat{V} - \frac{\eta}{2(\alpha + d_f)} \hat{\phi} \hat{V}' = 0 \implies \hat{V} \propto \hat{\phi}^{\frac{2d}{d+2n}}$$



# Two notions of fixed point

$$V(\varphi, \chi) = k^{d - \frac{d}{2} d_f} \bar{V}_k(\bar{\varphi}, \bar{\chi}), \quad \varphi = k^{\eta/2} \bar{\varphi}, \quad \chi = k^{\eta/2} \bar{\chi}$$

$$= f(\chi)^{-\frac{d}{2}} \hat{k}^d \hat{V}_{\hat{k}}(\hat{\phi})$$

$$\partial_t \bar{V} = (k^{-d_f} f)^{\frac{d\eta}{2(d-\eta+2n)}} \left\{ \frac{1}{\alpha + d_f} \left( \alpha + \frac{\eta}{2} \chi \partial_\chi \ln f \right) \partial_{\hat{t}} \hat{V}(\hat{\phi}) \right.$$

$$\left. + \left( \frac{\eta}{2} \chi \partial_\chi \ln f - d_f \right) \left( \frac{d\eta}{2(d-\eta+2n)} \hat{V} - \frac{\eta}{2(\alpha + d_f)} \hat{\phi} \hat{V}' \right) \right\}$$

If  $\eta \chi \partial_\chi \ln f$  varies with  $\chi$  then  $\partial_t \bar{V} \neq 0$  (!)

~~k-fixed points exist if:  $\hat{k}$  and fixed points coincide if:~~

$$\frac{d\eta}{2(d-\eta+2n)} \hat{V} - \eta \frac{d\eta}{2(\alpha + d_f)} \hat{\phi} \hat{V}' = 0 \implies \bullet \quad \hat{V} d_f \neq \hat{\phi}^{\frac{2d}{d+2n}}$$

- $f(\chi) \propto \chi^\gamma$

- ~~$d_f = \gamma \frac{\eta}{2}$~~

Background independence is incompatible  
with k-fixed points!

$$f(\phi) = k_0^{d_f} F\left(\frac{\phi}{k_0^{\eta/2}}\right)$$

$\implies$

$$\bar{f}(\bar{\chi}) = F\left(\bar{\chi} e^{\frac{\eta}{2}(t-t_0)}\right) e^{d_f(t_0-t)}$$

k-fixed points exist if: **hat{k}-fixed points coincide if:**

- $\eta = 0$  •  $d_f = 0$
- $f(\chi) \propto \chi^\gamma$  •  $d_f = \gamma \frac{\eta}{2}$

# In fully fledged quantum gravity?....

- Background independent notion of scale  $\hat{k}$  exists
- which is also independent of the form of  $f(\phi)$
- Generally  $k$ -fixed points are incompatible with background independence (i.e. the msWI)

...exceptions depending on form (running) of  $f(\chi)$

$\hat{k}$ -fixed points exist independently of all this...  
....at a simpler deeper level

