What is the large-scale effective action of $(2+1)$-dimensional causal dynamical triangulations?

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Probing the Fundamental Nature of Spacetime with the Renormalization Group

Nordita, Stockholm

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Acknowledgments

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$(2+1)$-dimensional causal dynamical triangulations

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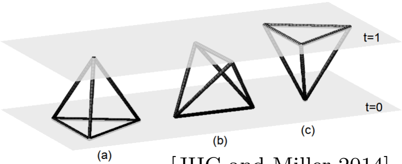
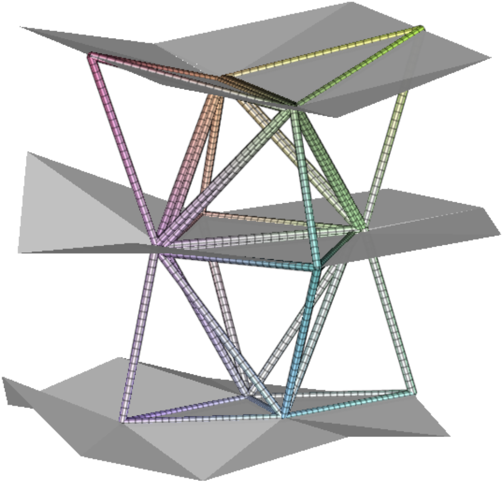
Partition function (after Wick rotation)

$$Z = \sum_{\mathcal{T}_c \cong \mathrm{S}^2 \times \mathrm{S}^1} \frac{1}{C(\mathcal{T}_c)} e^{-\mathcal{S}_{\mathrm{EH}}^{(\mathrm{E})}[\mathcal{T}_c]/\hbar} \quad \text{for} \quad \mathcal{S}_{\mathrm{EH}}^{(\mathrm{E})}[\mathcal{T}_c] = -k_0 N_0(\mathcal{T}_c) + k_3 N_3(\mathcal{T}_c)$$



[JHC and Miller 2014]



(2 + 1)-dimensional causal dynamical triangulations

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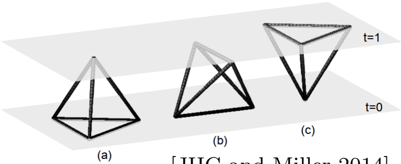

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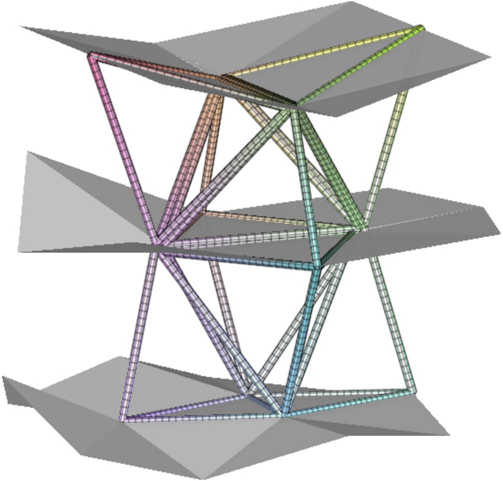
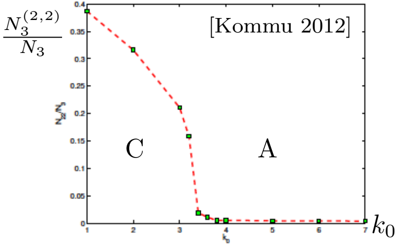


Phase diagram



Large-scale observable 1 in phase C

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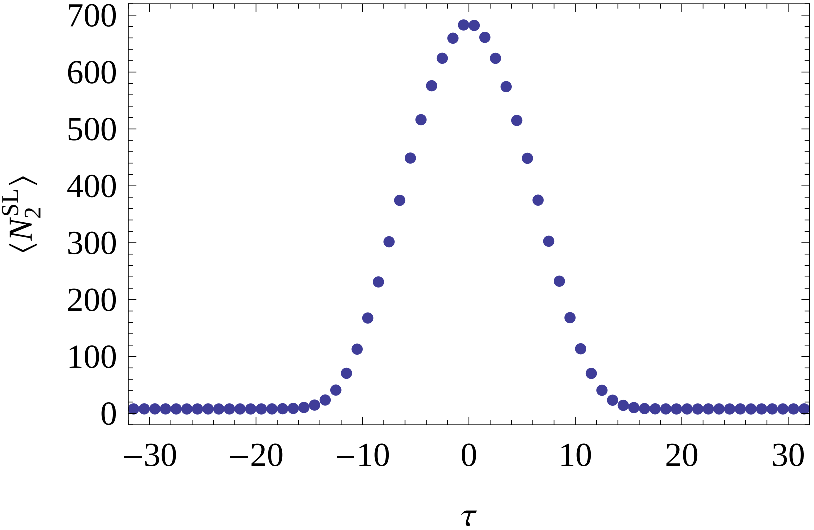
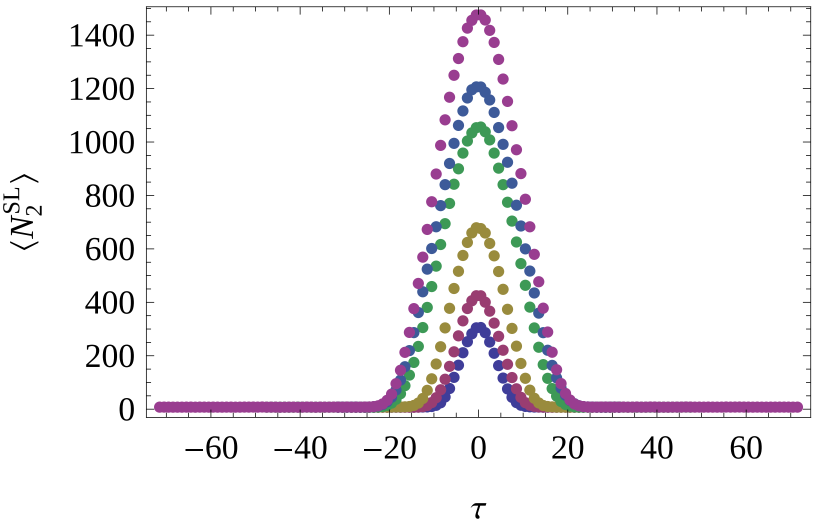
Ensemble average number $\langle N_2^{\mathrm{SL}} \rangle$ of spacelike 2-simplices as a function of the discrete time coordinate τ



Finite size scaling

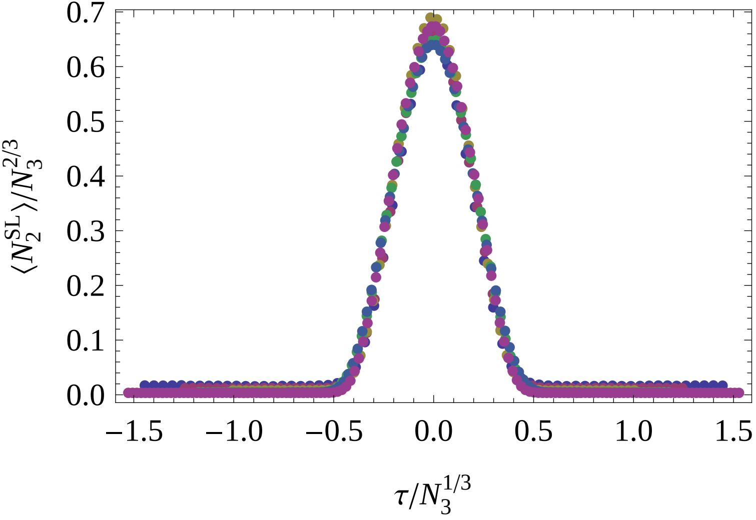
Finite size scaling

$\langle N_2^{\mathrm{SL}}(\tau)\rangle$ for varying N_3 at fixed k_0



Finite size scaling

Finite size scaled $\langle N_2^{\mathrm{SL}}(\tau) \rangle$ for varying N_3 at fixed k_0



Canonical finite size scaling *Ansatz* $V_3 = \lim_{\substack{N_3 \to \infty \\ a \to 0}} C_3 N_3 a^3$

Large-scale observable 2 in phase C

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Ensemble average covariance $\langle n_2^{\mathrm{SL}}(\tau)\, n_2^{\mathrm{SL}}(\tau')\rangle$ of the fluctuations
$n_2^{\mathrm{SL}}(\tau) = N_2^{\mathrm{SL}}(\tau) - \langle N_2^{\mathrm{SL}}(\tau)\rangle$

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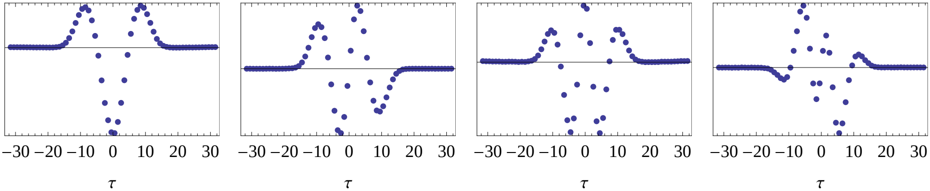
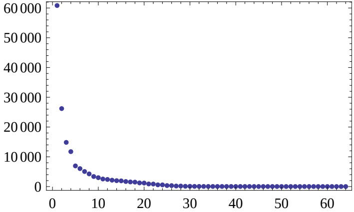
Eigenvectors



Eigenvalues



Large-scale effective action for $N_2^{\mathrm{SL}}(\tau)$

Large-scale effective action for $N_2^{\mathrm{SL}}(\tau)$

Construction from numerical measurements of $N_3^{\mathrm{SL}}(\tau)$ [Ambjørn *et al* 2008b]

$$\mathcal{S}_{\mathrm{eff}}^{(\mathrm{E})}[N_3^{\mathrm{SL}}] = c_1 \sum_{\tau=1}^{T} \left\{ \frac{\left[N_3^{\mathrm{SL}}(\tau+1) - N_3^{\mathrm{SL}}(\tau-1)\right]^2}{N_3^{\mathrm{SL}}(\tau)} + c_1 \left[N_3^{\mathrm{SL}}(\tau)\right]^{1/3} + c_3 N_3^{\mathrm{SL}}(\tau) \right\}$$

See also [Ambjørn *et al* 2004, 2005a, 2005b, 2008a, 2011, 2012, 2014]

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Naive continuum limit

$$S_{\mathrm{EH}}^{(\mathrm{E})}[V_3] = \frac{1}{24\pi G} \int \mathrm{d}t\, N \left[\frac{\dot{V}_3^2(t)}{N^2 V_3(t)} + 2^{2/3} 3^2 \pi^{4/3} V_3^{1/3}(t) - 3\Lambda V_3(t) \right]$$

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Corresponding naive continuum limit in $2+1$ dimensions

$$S_{\mathrm{EH}}^{(\mathrm{E})}[V_2] = \frac{1}{32\pi G} \int \mathrm{d}t\, N \left[\frac{\dot{V}_2^2(t)}{N^2 V_2(t)} - 4\Lambda V_2(t) \right]$$

See [Ambjørn *et al* 2001], [JHC, Anderson *et al* 2012], [Jordan and Loll 2013], [JHC and Miller 2014], [Benedetti and Henson 2014]

Bogacz, Burda, and Waclaw 2012

Bogacz, Burda, and Waclaw 2012

Perform a numerical analysis of the statistical mechanical model defined by the partition function

$$Z = \sum_{\{N_3^{\rm SL}(\tau)\}} e^{-\mathcal{S}_{\rm eff}^{(\rm E)}[N_3^{\rm SL}]/\hbar}$$

$$\mathcal{S}_{\rm eff}^{(\rm E)}[N_3^{\rm SL}] = c_1 \sum_{\tau=1}^{T} \left\{ \frac{\left[N_3^{\rm SL}(\tau+1) - N_3^{\rm SL}(\tau-1)\right]^2}{N_3^{\rm SL}(\tau)} + c_1 \left[N_3^{\rm SL}(\tau)\right]^{1/3} + c_3 N_3^{\rm SL}(\tau) \right\}$$

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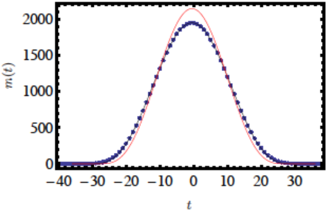
Find a phase diagram qualitatively comparable to that of $(3+1)$-dimensional causal dynamical triangulations

Condensate (droplet) phase



Quantum-mechanical 'widening'

Mean field configuration

- For $\tau \in \left[-\frac{\pi s_0 N_4^{1/4}}{2}, +\frac{\pi s_0 N_4^{1/4}}{2}\right]$,

  $N_3^{\mathrm{SL}}(\tau) = \frac{3}{4} \frac{N_4}{s_0 N_4^{1/4}} \cos^3\left(\frac{\tau}{s_0 N_4^{1/4}}\right)$

- For $\tau \in \left[-\frac{T}{2}, -\frac{\pi s_0 N_4^{1/4}}{2}\right) \cup \left(+\frac{\pi s_0 N_4^{1/4}}{2}, +\frac{T}{2}\right]$,

  $N_3^{\mathrm{SL}}(\tau) = 0$

Benedetti and Henson 2014

Benedetti and Henson 2014

Observe that the minisuperspace truncation of $S^{(\mathrm{E})}_{\mathrm{EH}}[\mathbf{g}]$ in $2+1$ dimensions does not admit a condensate (droplet) solution

Generalize to $(2+1)$-dimensional projectable Hořava-Lifshitz gravity

$$S^{(\mathrm{E})}_{\mathrm{HL}}[\mathbf{g}] = \frac{1}{16\pi G} \int_{\Sigma\times\mathrm{I}} \mathrm{d}^2x\,\mathrm{d}t\,N\sqrt{h}\left[K_{ij}K^{ij} - \lambda K^2 - 2\Lambda + \beta R_\Sigma + \gamma R_\Sigma^2\right]$$

- Condensate is an absolute minimum of $S^{(\mathrm{E})}_{\mathrm{HL}}[\mathbf{g}]$ but not strictly a solution

$$V_2(t) = 4\pi A^2 \begin{cases} (1-a)\cos^2(\omega t) + a & \text{for} \quad t \in \left[-\frac{\pi}{2\omega}, +\frac{\pi}{2\omega}\right] \\ a & \text{for} \quad t \in \left[-\frac{T}{2}, -\frac{\pi}{2\omega}\right) \cup \left(+\frac{\pi}{2\omega}, +\frac{T}{2}\right] \end{cases}$$

$$\text{with} \quad a = \frac{2\gamma}{A^4\Lambda} \quad \text{and} \quad \omega^2 = \frac{N^2\Lambda}{2\lambda - 1}$$

- Discretize $V_2(t)$ and perform 3-parameter fits to $\langle N_2^{\mathrm{SL}}(\tau)\rangle$

- Find evidence for deviation from round 3-sphere in the thermodynamic limit by extrapolating a finite size scaling analysis to infinite N_3

Comprehensive comparative statistical analysis

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Strategy for determining the large-scale effective action of $(2+1)$-dimensional causal dynamical triangulations

1. Formulate several competing hypotheses for the effective action (and its mean field configuration)
2. Perform analyses of $\langle N_2^{\mathrm{SL}}(\tau)\rangle$ and $\langle n_2^{\mathrm{SL}}(\tau)\, n_2^{\mathrm{SL}}(\tau')\rangle$ in terms of these hypotheses
3. Compare statistically the results of these analyses

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$$\text{Hypothesis 1} = \begin{cases} \text{Einstein}-\text{Hilbert action} \\ \text{Round 3}-\text{sphere} \\ \text{Proper time} \end{cases}$$

$$\text{Hypothesis 2} = \begin{cases} \text{Einstein}-\text{Hilbert action} \\ \text{Round 3}-\text{sphere} \\ \text{Coordinate time} \end{cases} = \begin{cases} \text{Horava}-\text{Lifshitz action } (\gamma = 0) \\ \text{Stretched 3}-\text{sphere} \\ \text{Proper time} \end{cases}$$

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Tests of hypotheses by fitting to $\langle N_2^{\mathrm{SL}}(\tau)\rangle$

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Hypothesis 1

$$\left\{ \begin{array}{l} \text{EH action} \\ \text{Round 3}-\text{sphere} \\ \text{Proper time} \end{array} \right.$$

Hypothesis 2

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or

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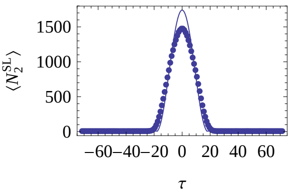
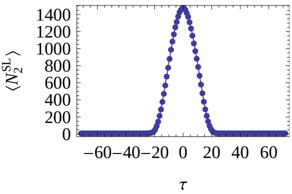
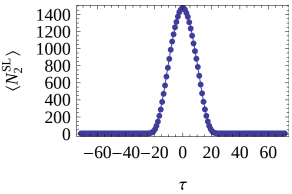


$\chi^2_{\mathrm{pdf}} = 423.94$



$\chi^2_{\mathrm{pdf}} = 9.37$



$\chi^2_{\mathrm{pdf}} = 7.12$

Tests of hypotheses by fitting to $\langle N_2^{\mathrm{SL}}(\tau)\rangle$

Hypothesis 1	Hypothesis 2	Hypothesis 3
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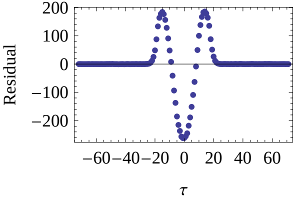

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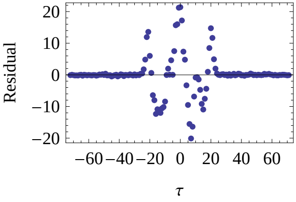

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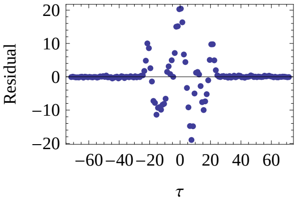

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Variation of residuals with N_3 at fixed k_0

Variation of residuals with N_3 at fixed k_0

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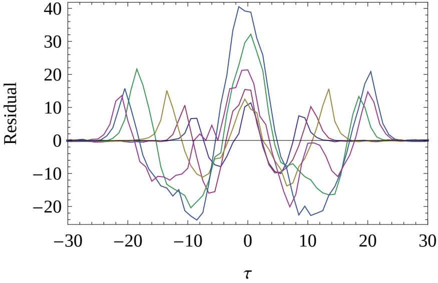

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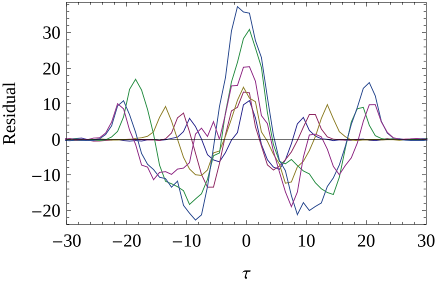

Hořava-Lifshitz parameter $a = 2\gamma/A^4\Lambda$

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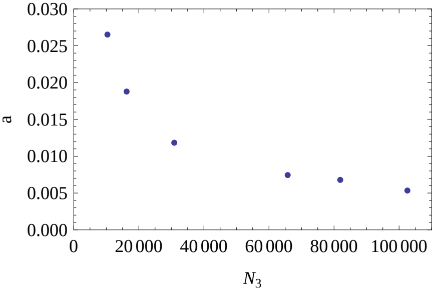
Varying N_3 at fixed k_0



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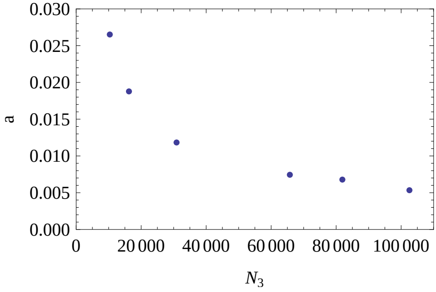
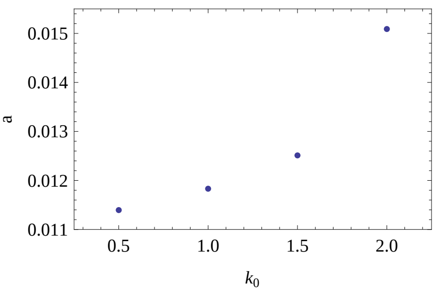


Varying k_0 at fixed N_3



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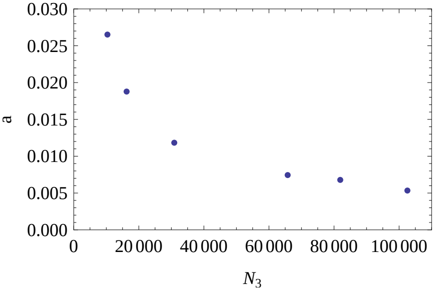

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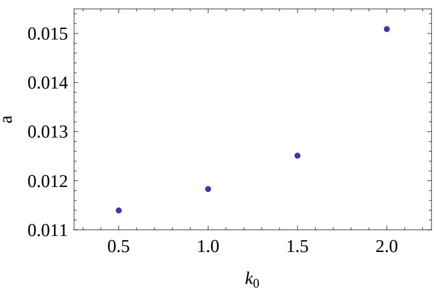

What happens to $\langle N_2^{\mathrm{SL}}(\tau) \rangle$ near the first-order phase transition?

Hořava-Lifshitz parameter $a = 2\gamma/A^4\Lambda$

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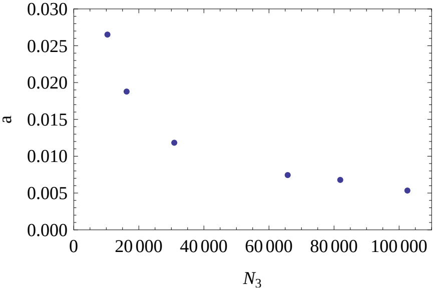

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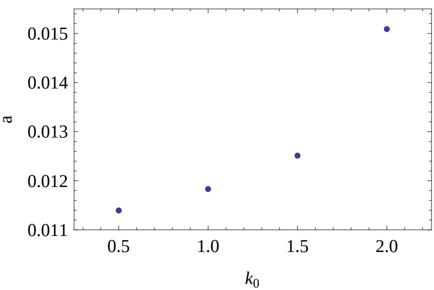

What happens to $\langle N_2^{\mathrm{SL}}(\tau)\rangle$ near the first-order phase transition?

k_0	χ^2_{pdf}(Hypothesis 1)	χ^2_{pdf}(Hypothesis 2)	χ^2_{pdf}(Hypothesis 3)
0.5	566.10	23.91	17.38
1.0	291.79	24.59	15.79
1.5	104.68	23.12	17.65
2.0	43.12	18.96	12.04

Ongoing research

- Alternative hypotheses in the form of other large-scale effective actions
  - Fourth order gravity
  - Topologically massive gravity
  - New massive gravity
  - *et cetera*

- Simultaneous fits to $\langle N_2^{\mathrm{SL}}(\tau)\rangle$ and $\langle n_2^{\mathrm{SL}}(\tau)\, n_2^{\mathrm{SL}}(\tau')\rangle$

- Quantification of the accuracy of the second order expansion of $S_{\mathrm{eff}}[V_2]$

  $S_{\mathrm{eff}}[V_2] = S_{\mathrm{eff}}^{(0)}[V_2^{\mathrm{cl}}] + S_{\mathrm{eff}}^{(1)}[v_2] + S_{\mathrm{eff}}^{(2)}[v_2] + S_{\mathrm{eff}}^{(3)}[v_2] + \cdots$ with $v_2 = V_2 - V_2^{\mathrm{cl}}$

- Parametrization of corrections to canonical finite size scaling

$$V_3 = C_3 N_3 a^3 + O(N_3^{1-\epsilon}, a^{3+\varepsilon}) \quad \text{at finite } N_3 \text{ and finite } a$$

- Transition amplitudes

Controversial statement

The large-scale effective action of $(3+1)$-dimensional causal dynamical triangulations is the minisuperspace truncation of the Einstein-Hilbert action.