Renormalization Group Flow in Causal Dynamical Triangulations

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Stockholm, March 24th, 2015

Work done in collaboration with:
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Outline

Gravitational path integral

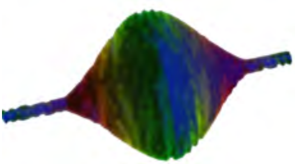
What is Causal Dynamical Triangulations?

Causal Dynamical Triangulations (CDT) is a background independent approach to quantum gravity.

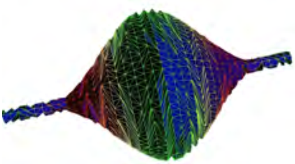
CDT provides a lattice regularization of the formal gravitational path integral via a sum over causal triangulations.

$$\int \mathrm{D}[g]\, e^{iS^{EH}[g]} \longrightarrow \sum_{\mathcal{T}} e^{-S^{R}[\mathcal{T}]}$$

continuous



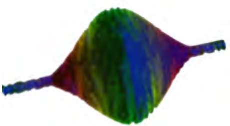
discrete



Regge action

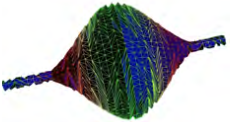
The Einstein-Hilbert action has a natural realization on piecewise linear geometries called Regge action

$$S^E[g] = -\frac{1}{G}\int \mathrm{d}t \int \mathrm{d}^D x \sqrt{g}(R - 2\Lambda)$$



$$\Downarrow$$

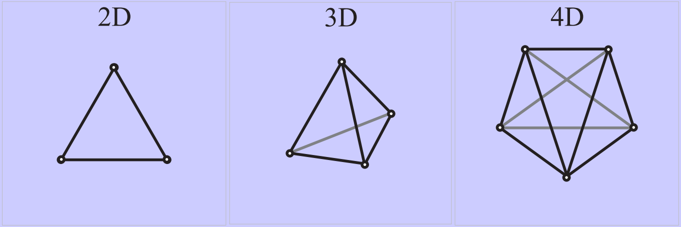
$$S^R[\mathcal{T}] = -K_0 N_0 + K_4 N_4 + \Delta(N_{14} - 6N_0)$$



- N_0 number of vertices
- N_4 number of simplices
- N_{14} number of simplices of type $\{1,4\}$
- K_0 K_4 Δ bare coupling constants $(G, \Lambda, \alpha = a_t/a_s)$

CDT in a nutshell

- To make sense of the gravitational path integral one uses the standard method of regularization - discretization.
- The partition function of quantum gravity, defined as a formal integral over all geometries, is written as a nonperturbative sum over all causal triangulations.
- Due to global proper-time foliation, the Wick rotation is well defined $(a_t \to i a_t)$.
- Using **Monte Carlo** techniques we can approximate expectation values of observables.



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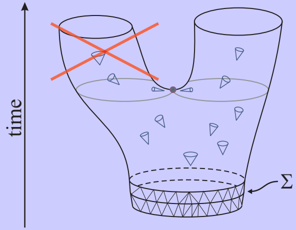

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$$\int \mathrm{D}[g] e^{i S^{EH}[g]} \rightarrow \sum_{\mathcal{T}} e^{i S^{R}[\mathcal{T}]} \rightarrow \sum_{\mathcal{T}} e^{-S^{R}[\mathcal{T}]}$$

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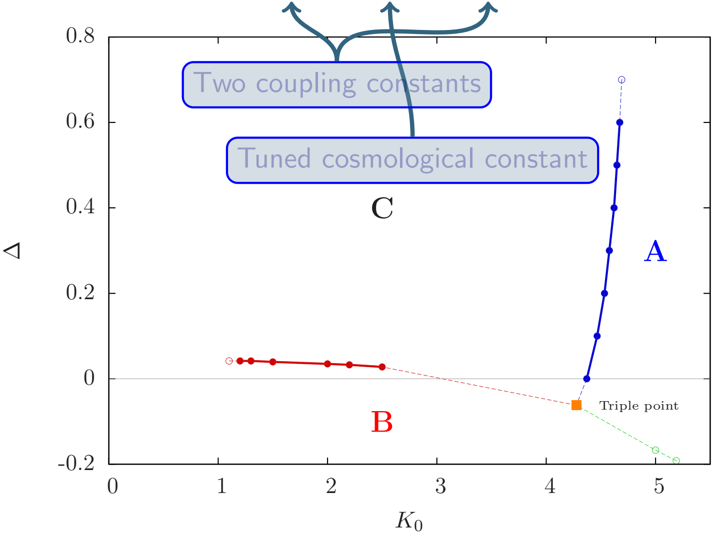
$$\langle \mathcal{O}[\mathcal{T}] \rangle \approx K^{-1} \sum_{i=1}^{K} \mathcal{O}[\mathcal{T}^{(i)}]$$

Example observable is $N(i)$, the number of tetrahedra building slice i, $i = 1, \ldots, T$.

A glimpse at the phase diagram

$$S[\mathcal{T}] = -K_0 N_0 + K_4 N_4 + \Delta(N_{14} - 6N_0)$$



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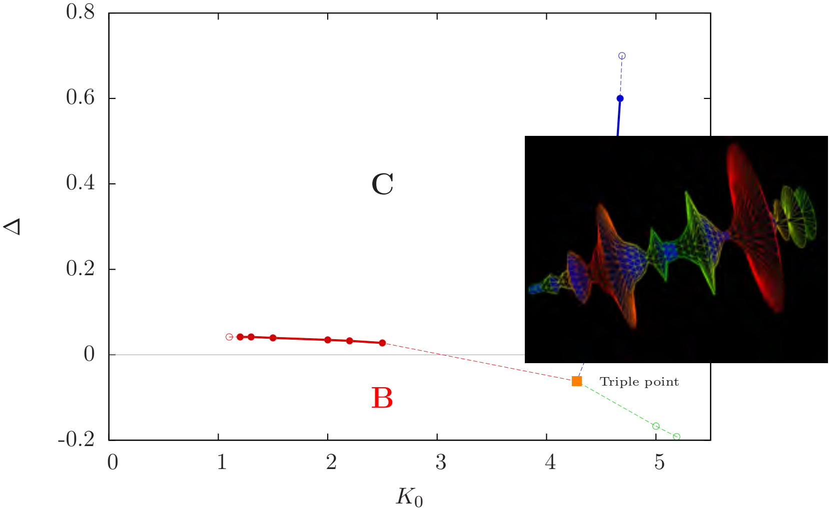

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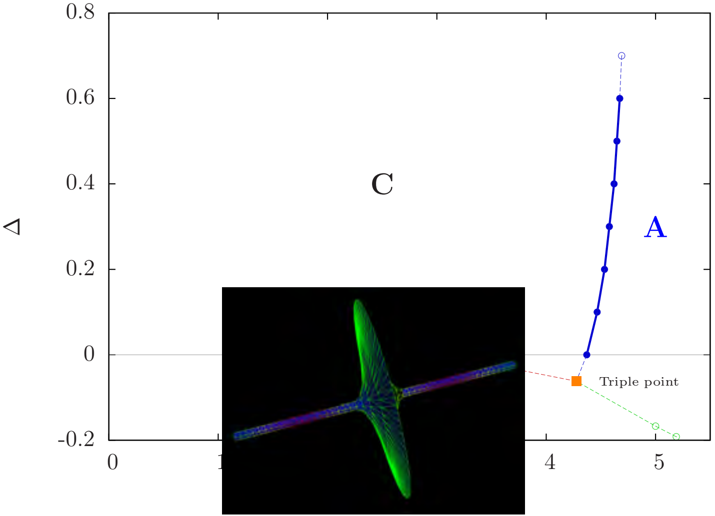

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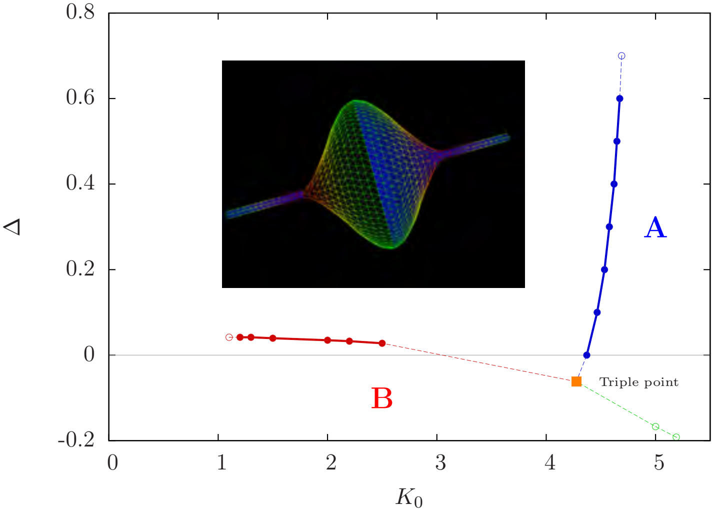

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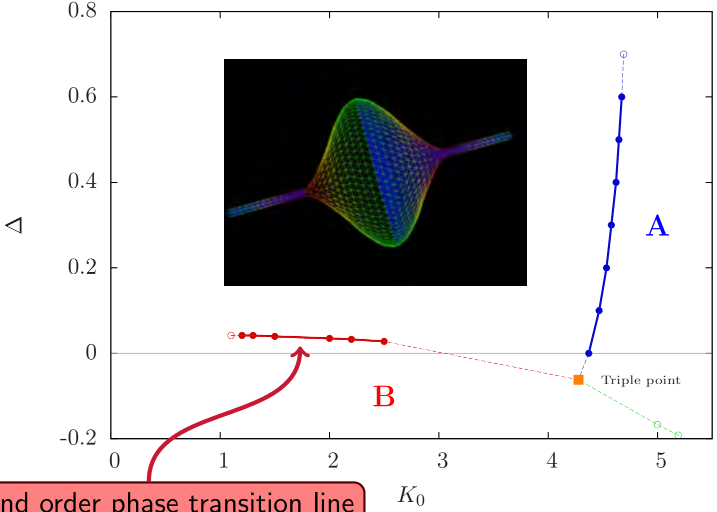
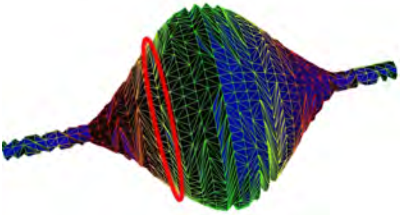


Second order phase transition line

De Sitter phase - properties

- Phase **C**, the so called De Sitter phase, is physically most interesting.



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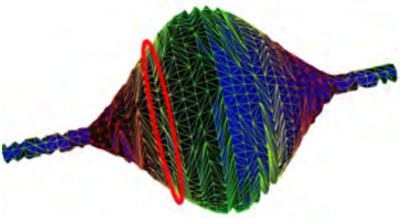
- A **four-dimensional** background geometry emerges dynamically. The background geometry corresponds to Euclidean **de Sitter** space (S^4), a classical **vacuum solution**.



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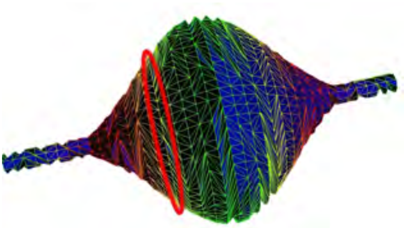

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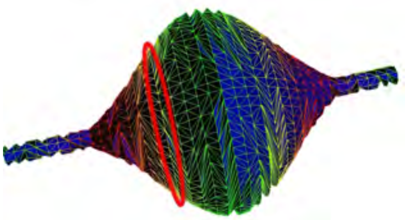
Consider the three-volume $N(i)$ defined as the number of tetrahedra building slice i, $i = 1, \ldots, T$.

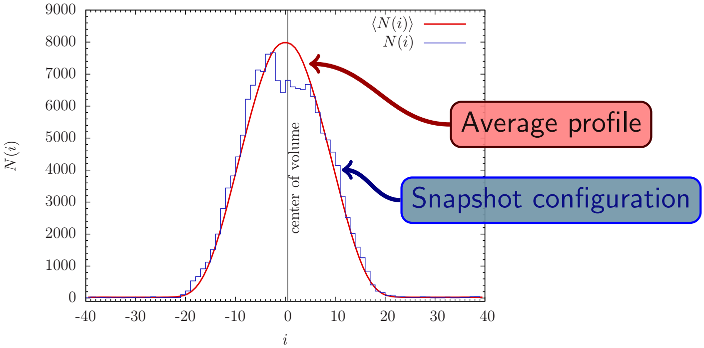
$$N_4 = \sum_i N(i)$$

Total four-volume



De Sitter phase - background geometry

- In phase **C** the time translation symmetry is spontaneously broken and the three-volume profile $N(i)$ is bell-shaped.

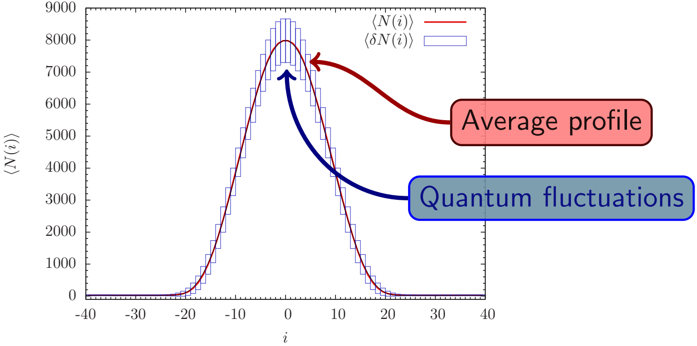


De Sitter phase - background geometry

- In phase **C** the time translation symmetry is spontaneously broken and the three-volume profile $N(i)$ is bell-shaped.
- The average volume $\langle N(i) \rangle$ is with high accuracy given by formula

$$\langle N(i) \rangle = H \cos^3 \left(\frac{i}{W} \right)$$

a classical **vacuum solution**.

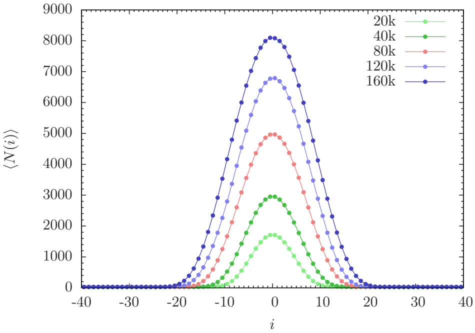


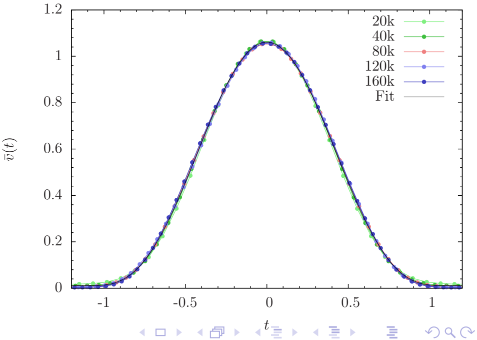
De Sitter phase - scaling

- For different total volumes N_4, $\langle N(i)\rangle$ scales as a genuine **four-dimensional** *Universe*.

$$\langle N(i)\rangle = \frac{3}{4\omega} N_4^{3/4} \cdot \cos^3\left(\frac{i}{\omega N_4^{1/4}}\right)$$

$$\langle \delta N(i)\rangle = \gamma N_4^{1/2} \cdot F\left(\frac{i}{\omega N_4^{1/4}}\right)$$

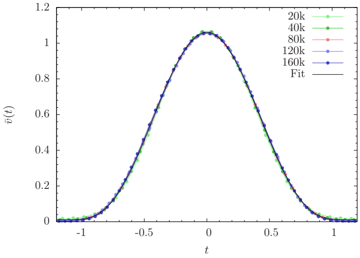




De Sitter phase - scaling

- Scaling the lattice cut-off a with Hausdorff dimension $d_H = 4$, $a \propto N_4^{-1/4}$, gives physical quantities t and $v(t)$.

$$\begin{aligned} i &\longrightarrow t &&= N_4^{-1/4} \cdot i \\ \langle N(i) \rangle &\longrightarrow v(t) &&= N_4^{-3/4} \langle N(i) \rangle &&= \frac{3}{4\omega} \cos^3\left(\frac{t}{\omega}\right) \\ \langle \delta N(i) \rangle &\longrightarrow \delta v(t) &&= N_4^{-3/4} \langle \delta N(i) \rangle &&= \gamma N_4^{-1/4} \cdot F\left(\frac{t}{\omega}\right) \end{aligned}$$



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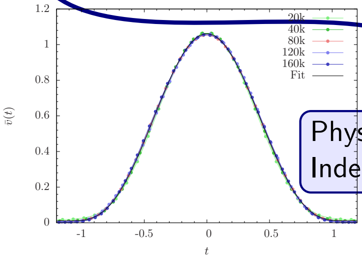
Expressed in lattice units. Directly measured.



Physical quantities. Independent of cut-off a.

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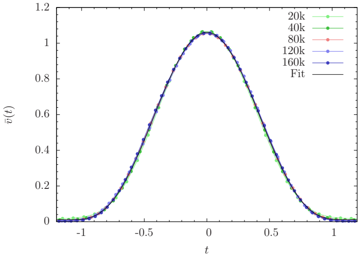
$$

Proper time



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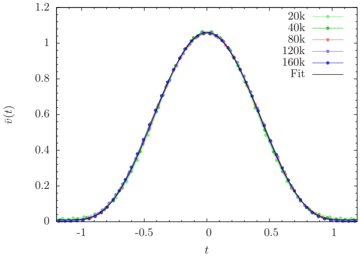
Physical volume



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Quantum fluctuations

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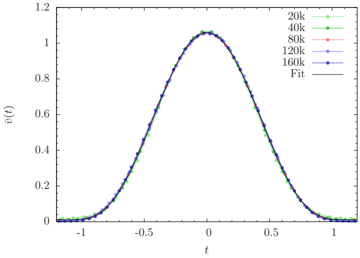

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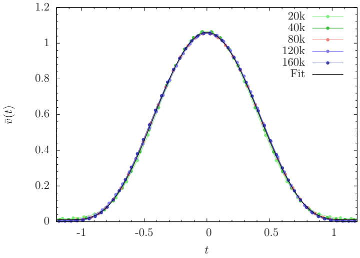
Width, shape parameter

Amplitude of fluctuations. **Note** $N_4^{-1/4}$!



Lines of constant physics

- Taking a **continuum limit** is achieved by sending the lattice spacing $a \to 0$, while keeping physical quantities fixed.
- We assume that the scaling holds everywhere in phase C, and that the three-volume profile $v(t)$ as well as its fluctuations $\delta v(t)$ are physical quantities.
- The width parameter ω and fluctuations γ are functions of the coupling constants, but do not depend on N_4,

$$\omega = \omega(K_0, \Delta), \quad \gamma = \gamma(K_0, \Delta)$$

- This leads us to the conditions for a *path of constant physics*:

$$\left.\begin{aligned} V_4 &= N_4 a^4 \\ v(t) &= \frac{3}{4\omega} \cos^3\left(\frac{t}{\omega}\right) \\ \delta v(t) &= \gamma N_4^{-1/4} \cdot F\left(\frac{t}{\omega}\right) \end{aligned}\right\} \text{fixed} \Rightarrow \left\{\begin{aligned} N_4 &\to +\infty \\ \omega &= \text{const} \\ \gamma \cdot N_4^{-1/4} &= \text{const} \end{aligned}\right.$$

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Fixed total volume

$$\omega = \omega(K_0, \Delta), \quad \gamma = \gamma(K_0, \Delta)$$

Fixed shape

- This leads us to the condition for a path of constant *physics*:

Fixed amplitude of fluctuations (Newton's constant)

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Renormalization group flow

- To approach the continuum limit, we follow a path in the coupling constants plane $(K_0, \Delta) = (K_0(N_4), \Delta(N_4))$, parametrized by N_4 or a, such that physical properties are fixed.
- While the lattice spacing a goes to zero, the lattice size (number of simplices) has to grow to infinity ($a \propto N_4^{-1/4}$).
- From the conditions, it follows that the fluctuation parameter γ has to approach infinity in the continuum limit. A second-order (or higher) phase transition is needed.

Conditions for *the line of constant physics*

$$\omega(K_0(N_4), \Delta(N_4)) = \mathrm{const}, \quad \gamma(K_0(N_4), \Delta(N_4)) \propto N_4^{1/4}$$

$$N_4 \to \infty$$

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- $\omega(K_0, \Delta) = \mathrm{const}$
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- $V_4 = N_4 \cdot a^4$
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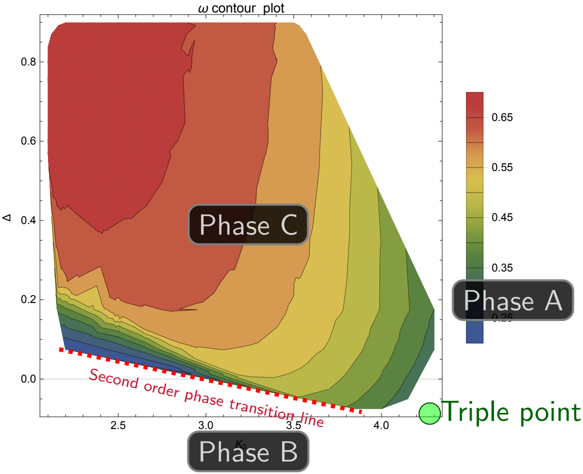

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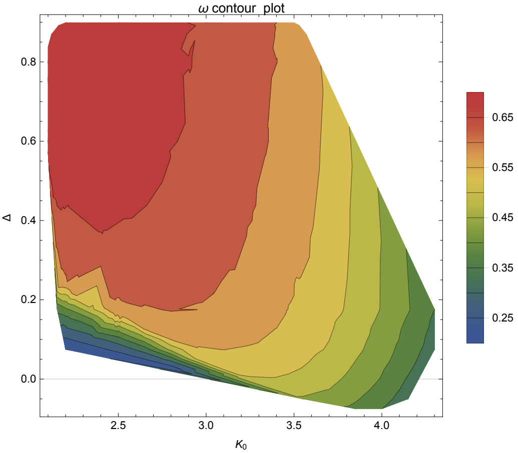

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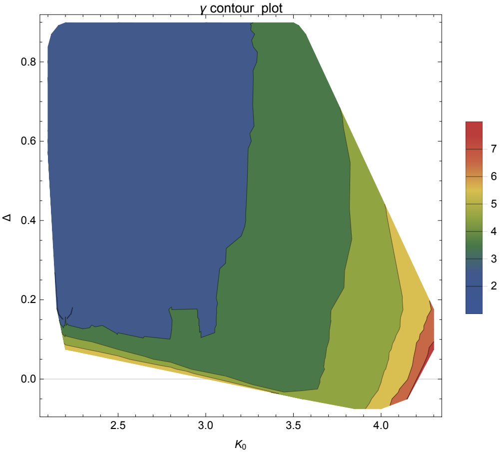

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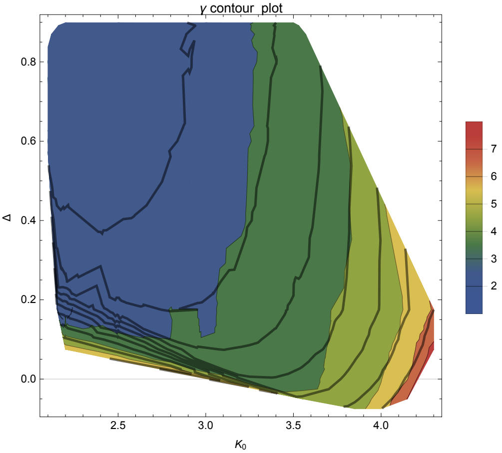

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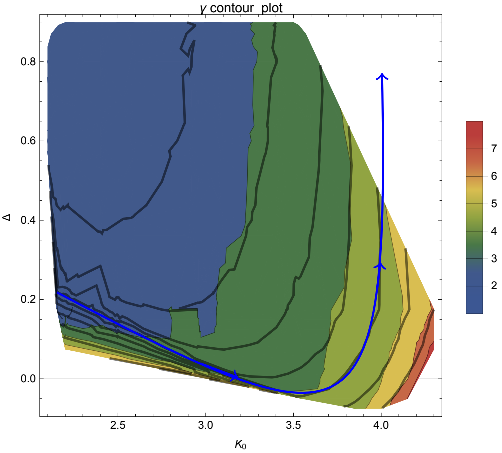


Effective action

- Simulations show that inside phase C, there exists an effective action for three-volume $N(i)$,

$$P(\{N(i)\}) = \frac{1}{Z} e^{-S[N]}$$

$$S[N] = \frac{1}{\Gamma} \sum_i \left(\frac{(N(i+1) - N(i))^2}{N(i+1) + N(i)} + \mu N(i)^{1/3} - \lambda N(i) \right)$$

- It agrees with the discretization of the minisuperspace action

$$S[v] = \frac{1}{G} \int \frac{\dot{v}^2}{v} + v^{1/3} - \lambda v \,\mathrm{d}t$$

obtained from the Einstein-Hilbert action by „freezing" all degrees of freedom except the scale factor.

- Classical trajectory corresponds to Euclidean de Sitter space,

$$v(t) = \frac{3}{4\omega} \cos^3\left(\frac{t}{\omega}\right)$$

- The form of $S[v]$ gives the previously described scaling.

Effective action

- Simulations show that inside phase C, there exists an effective action for three-volume $N(i)$,

$$P(\{N(i)\}) = \frac{1}{Z} e^{-S[N]}$$

$$S[N] = \frac{1}{\Gamma} \sum_i \left(\frac{(N(i+1) - N(i))^2}{N(i+1) + N(i)} + \mu N(i)^{1/3} - \lambda N(i) \right)$$

- It agrees with the discretization of the minisuperspace action

$$S[v] = \frac{1}{G} \int \frac{\dot{v}^2}{v} + v^{1/3} - \lambda v \, dt$$

Couples only adjacent slices

obtained from the Einstein-Hilbert action by ,,freezing" all degrees of freedom except the scale factor.

- Classical trajectory corresponds to Euclidean de Sitter space,

$$v(t) = \frac{3}{4\omega} \cos^3\left(\frac{t}{\omega}\right)$$

- The form of $S[v]$ gives the previously described scaling.

Effective transfer matrix

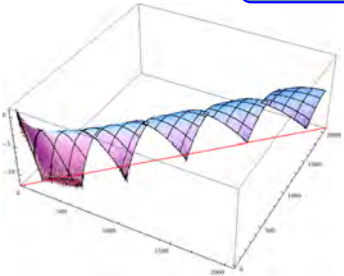
- The effective action suggests existence of an effective transfer matrix M labeled by the scale factor

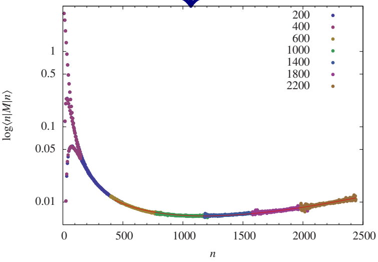
$$P(\{N(i)\}) = \frac{1}{Z} \underbrace{\langle N(1)|M|N(2)\rangle\langle N(2)|M|N(3)\rangle \cdots \langle N(T)|M|N(1)\rangle}_{e^{-S[N]}}$$

$$\langle n|M|m\rangle = \mathcal{N} e^{-\frac{1}{\Gamma}\left[\frac{(n-m)^2}{n+m} + \mu\left(\frac{n+m}{2}\right)^{1/3} - \lambda\frac{n+m}{2}\right]}$$

Product of matrix elements

Potential term





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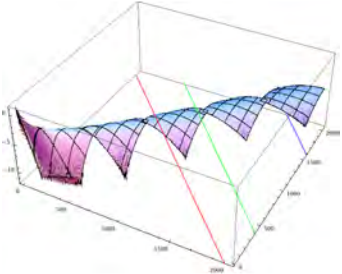
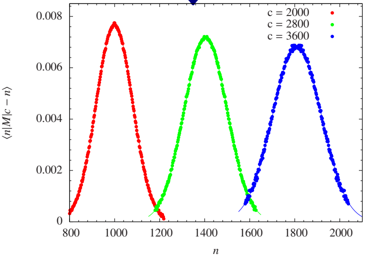
Product of matrix elements

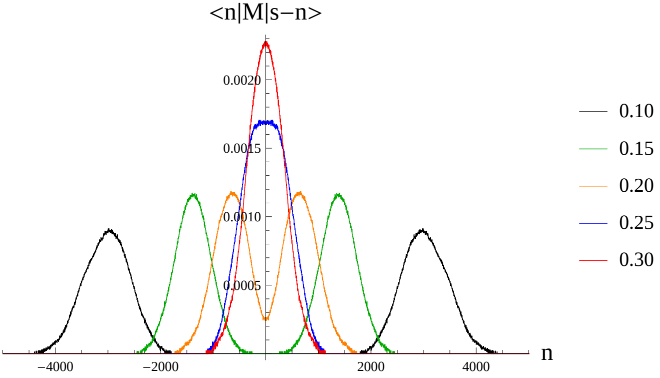
Kinetic term, gaussian





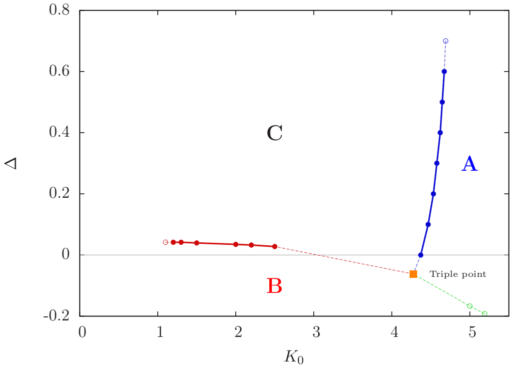
New *bifurcation* phase?

- In phase **C** the cross-diagonals of the transfer matrix $\langle n|M|s-n\rangle$ are given by a centered Gaussian function.
- In the **bifurcation** region the cross-diagonals split into a sum of two shifted Gaussians.



New *bifurcation* phase?

- Recent results show between phase B and foregoing phase C, there is a new region with different behaviour of the transfer matrix.
- In this region, the scaling arguments seem not to be valid in the whole range of volumes.



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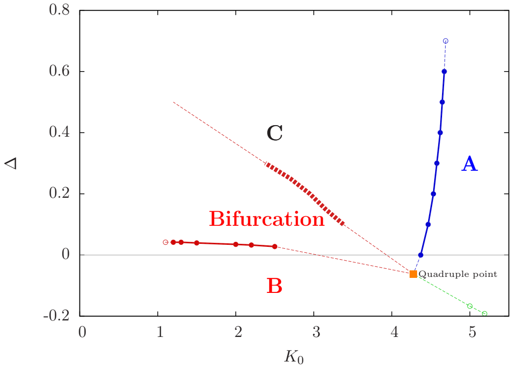


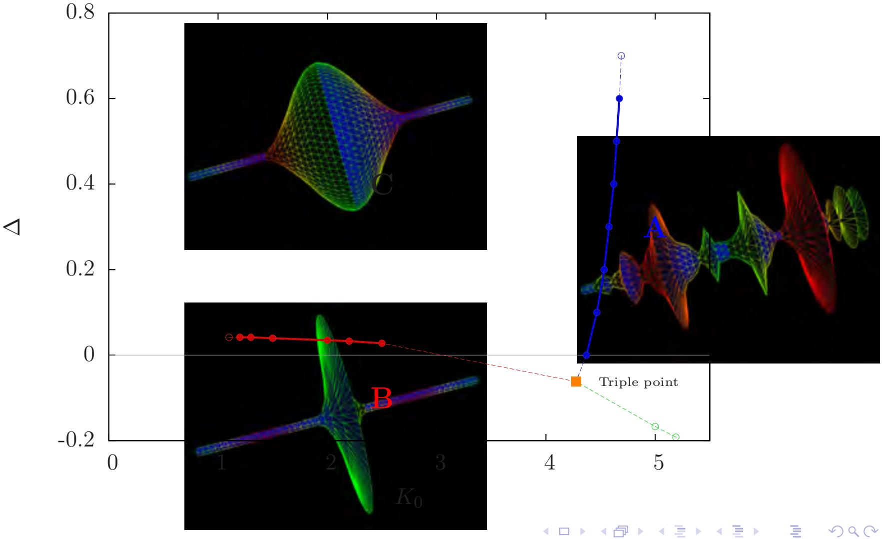
Conclusions

1. In Causal Dynamical Triangulations a **four-dimensional** background geometry emerges dynamically. It corresponds to the Euclidean **de Sitter** space, i.e. **classical solution** of the minisuperspace model.
2. We have presented the most direct way of addressing the renormalization group flow in CDT via defining *lines of constant physics*.
3. There is, however, no unique prescription. Other aspects might influence the outcome:
   - Inclusion of the asymmetry parameter $\alpha = a_t/a_s$ which dependends on K_0 and Δ.
   - Anisotropic scenarios á la Hořava-Lifshitz Gravity.
   - Different way of defining *constant physics*.
   - Different scaling relations.
4. Existence of the new *bifurcation* phase should be also taken into account, while considering the coupling constant flow.

Thank you for your attention!

Lifshitz phase diagram and CDT

$$S[\mathcal{T}] = -K_0 N_0 + K_4 N_4 + \Delta(N_{14} - 6N_0)$$



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