Diffeomorphism invariance and critical scaling in quantum gravity

Kevin Falls, Heidelberg University

Based on: arXiv:1501.05331 + arXiv:1503.06233 .

Introduction

- Diffeomorphism invariance underlies quantum gravity.
- Asymptotic safety provides a possible continuum limit for gravity for which we expect universal (i.e. regulator independent) scaling behaviour. Wish to compute:

$$\beta_G = G(d - 2 + \eta_G)$$

- Issues of functional renormalisation group approach:

  - gauge dependence

  - poles due cosmological constant in a massless theory

  - complex critical exponents (qualitatively difference from lattice studies (e.g. Hamber 2000))

- These may all stem from approximations where diffeomorphism invariance is lost.

- Aim: Restore gauge invariance to improve the reliability of perturbative/non-perturbative calculations.

Semi-classical theory

- Aim to find a gauge independent one-loop beta function generated by graviton fluctuations:

$$\beta_G = (d-2)G - b\,G^2$$

- One loop effective action:

$$\Gamma - S = \frac{1}{2}\mathrm{STr}\log S^{(2)}$$

- Standard approach:

$$\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

- Problem: linearised theory,

$$S_{\mathrm{EH},1} = \int d^dx\,\frac{1}{2}h\cdot S^{(2)}_{\mathrm{EH}}\cdot h$$

is not gauge invariant if background metric is off-shell (Deser, Henneaux gr-qc/0611157):

$$\nabla_\mu G_1^{\mu\nu} \neq 0$$

Semi-classical theory

- This leads to the off-shell one-loop effective action being gauge dependent for the linear parameterisation.
- Gauge independence is restored only by going on shell

$$R = \frac{2d}{d-2}\bar{\lambda}$$

- However this obstructs the derivation of the beta function for Newton's constant using covariant heat kernel methods.
- The linearised theory (and hence the naive semiclassical theory) is not unique and depends on the parameterisation(s) of the metric fluctuations (see e.g. Nink 2014).

$$\gamma_{\mu\nu} = \gamma_{\mu\nu}(\varphi^A)$$

- Aim to find a parameterisation which restores gauge invariance at the linear level for:

$$R \neq \frac{2d}{d-2}\bar{\lambda}$$

Restoring diffeomorphism invariance

- Essential observation: going to an arbitrary Einstein space solves all but the trace of the Einstein equations. Hence terms in the action

$$S_{EH,1} - S_{EH,1}|_{\text{invar.}} \propto \left(R - \frac{2d}{d-2}\bar{\lambda}\right)$$

which both breaks diffeomorphism invariance and leads to poles in the propagators are proportional to the trace of the Einstein equations.

- To remove these terms we only need to choose a parameterisation for which the cosmological constant is absent from the linearised action e.g. taking the volume element itself as a field:

$$\sqrt{\gamma(x)} = \omega(x) = \sqrt{\bar{g}(x)}\left(1 + \frac{\sigma(x)}{2}\right) \quad \textbf{e.g.} \quad \gamma_{\mu\nu} = \left(1 + \frac{\sigma}{2}\right)^{\frac{2}{d}} \bar{g}_{\mu\lambda}(e^{\hat{h}})^{\lambda}_{\mu}$$

where $\hat{h}_{\mu\nu}$ is a traceless field.

- Gauge invariance under:

$$\hat{h}_{\mu\nu} \to \hat{h}_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu - \frac{1}{d} g_{\mu\nu} \nabla_\alpha \epsilon^\alpha + \ldots \qquad \sigma \to \sigma + 2\nabla_\alpha \epsilon^\alpha + \ldots$$

Gauge independent effective action

- One loop effective action in quantum gravity for differentially constrained fields:

$$\Gamma[g_{\mu\nu}] - S[g_{\mu\nu}] = \frac{1}{2}\mathrm{Tr}_{TT}[\log \Delta_2] - \frac{1}{2}\mathrm{Tr}_{T}[\log \Delta_1]$$

Transverse-traceless fluctuations

Transverse vector fluctuations

- with differential operators:

$$\Delta_2 \varphi_{\mu\nu} = \left(-\nabla^2 \varphi_{\mu\nu} - 2R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\varphi_{\alpha\beta}\right)$$

$$\Delta_1 \varphi_{\mu} = \left(-\nabla^2 \delta_{\mu}^{\nu} - R_{\mu}{}^{\nu}\right)\varphi_{\nu}$$

One-loop beta function

- The first two issues pointed out in the introduction have been resolved by restoring diffeomorphism invariance at the semi-classical level.
- After a suitable normalisation of the cutoff scale one can find a universal one loop beta function

$$\beta_G = (d-2)G - \frac{2}{3}(18 - N_g)\,G^2$$

$$N_g = \frac{d(d-3)}{2}$$

= Number of polarisations of the graviton

- This beta function can be obtained using covariant heat kernel (proper time) background independent regularisation.
- Can also be found by gauge fixing the conformal fluctuations (Percacci, Vacca 2015).
- Using the functional renormalisation group one can obtain the same result with a type II cut-off:

$$\mathcal{R}_{k,s} = k^2 C(\Delta_s/k^2) \quad \text{for} \quad s = 0,1,2$$

Beyond one-loop

- Use the functional renormalisation group to compute non-perturbative beta function.
- Single metric truncation is gauge independent.

$$\beta_G = G(d-2+\eta_G) \qquad \eta_G = \frac{2\,(N_g-18)\,G\,\mathcal{I}_{d/2-1}}{3\,(4\pi)^{\frac{d-2}{2}}\,\Gamma\left(\frac{d}{2}-1\right)+(N_g-18)\,G\,\tilde{\mathcal{I}}_{d/2-1}}\,,$$

- Fixed point for positive Newton's constant in d<8 dimensions
- Critical exponent:

$$1/\nu \equiv -\left.\frac{\partial\beta_G}{\partial G}\right|_{G=G_*} = (d-2)+(d-2)^2\,\frac{\tilde{\mathcal{I}}_{d/2-1}}{2\,\mathcal{I}_{d/2-1}}$$

- Involves the integrals

$$\tilde{\mathcal{I}}_n = \int_0^\infty dz\,\frac{z^{n-1}C(z)}{z+C(z)}\,, \qquad \mathcal{I}_n = \tilde{\mathcal{I}}_n - \int_0^\infty dz\,z^n\,\frac{C'(z)}{z+C(z)}\,.$$

Critical scaling

- Regulator independence close to two dimensions:

$$1/\nu = \epsilon + \frac{1}{2}\epsilon^2 + ...$$

- Both gauge and regulator independent to second order

- Comparison with two loop calculation (Aida and Kitazawa (1997), hep-th/9609077)

$$1/\nu = \epsilon + \frac{3}{5}\epsilon^2 + ...$$

Critical scaling

- d=4 dimensions known result from lattice calculation by Hamber (2000):

$$\langle R \rangle \equiv \left\langle \int d^d x \sqrt{\gamma}\, R \right\rangle / V \qquad \langle R \rangle \sim |G_* - G_b|^{4\nu - 1}$$

$$\nu \approx 0.335(4)$$

- Relation to beta function comes from the scaling of the free energy

$$\langle R \rangle = 16\pi G_b^2 \frac{\partial}{\partial G_b} F$$

- Scaling of the free energy can be obtained by integrating the RG flow:

$$F = \Gamma_{k=0}/V \sim |G_b - G_*|^{4\nu}$$

Optimised scaling

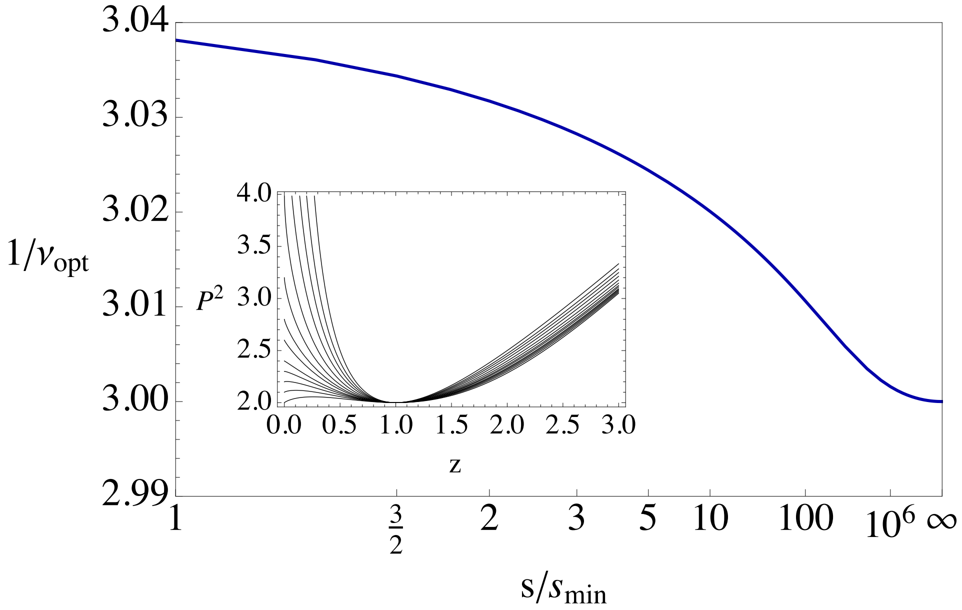
- To obtain a best estimate I apply Litim's optimisation criteria:

  maximise the gap in the inverse propagator under the regulator scheme

$$P^2_{\rm gap}[C] = P^2(z_{\rm min}) \equiv z_{\rm min} + C(z_{\rm min}) \qquad P^2_{\rm gap}[C_{\rm opt}] = \max_{\rm RS}(P^2_{\rm gap}[C])$$

- Optimisation aims for better convergence of approximate solutions.
- Class of optimised regulators:

$$C_{\rm opt}(z) = s z^b \frac{1}{(1+s)^{z^b}-1}\Bigg|_{b=b_{\rm opt}}, \qquad b_{\rm opt}(s) = \frac{s}{(1+s)\log(1+s)-s}$$



Optimised FRG: $\nu \approx 1/3$

Lattice : $\nu \approx 0.335(4)$

Summary

- Restoring diffeomorphism invariance for simple approximations:
  - Gauge independence.
  - No poles (from cosmological constant) in propagators/beta functions.
  - Real critical exponent in quantitative agreement with lattice studies
- Obtaining universal results is surely dependent on approximations and regularisations which respect diffeomorphism invariance.

- To do: Vertex expansion/Bi-metric and higher curvature invariants.