# Probing the "fundamental" nature of fluctuating membranes with the Renormalization Group

O. Zanusso
TPI Jena

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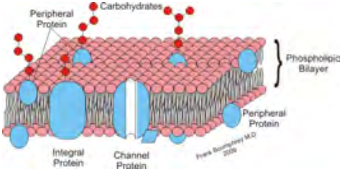
NORDITA

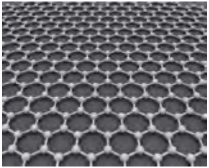
## Why membranes?

Biological:




- Phospholipid bilayer
- Cytoskeleton

Non-biological:



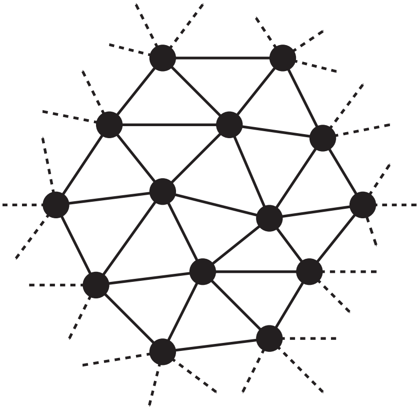
- Two-dimensional crystals

**Why here?**

- Ideal testing ground for RG of random geometry:
  - Physically realized examples at accessible energies.
  - Reference scale can be meaningfully provided by the embedding.
- Reparametrization invariance (not Diff!)
- Broken symmetries generate extension.
- Nonlinear theory.

## A microscopic model

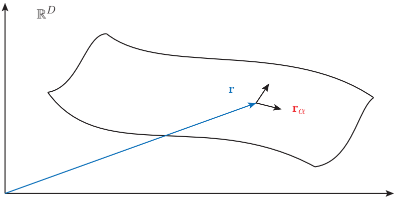
Imagine a membrane as realized through the bonding of *monomers*:



To fix a scale, let the *bonds* length be of order $1\mu m$.

## A macroscopic model

*Effective* continuous description:
$\mathbf{r} : \mathbb{R}^d \to \mathbb{R}^D$




Induced metric:

$$g_{ab} = \partial_a r^\mu \partial_b r^\nu \delta_{\mu\nu}$$

The geometry is characterized by both intrinsic and extrinsic curvatures:

$$\partial_a \partial_b r^\mu = K^i_{ab} n^\mu_i + \Gamma_a{}^c{}_b \partial_c r^\mu$$

## $\epsilon = 2$ just like in gravity

Rigidity of the membrane is generally controlled by

$$\partial_\alpha \partial_\beta r^\mu$$

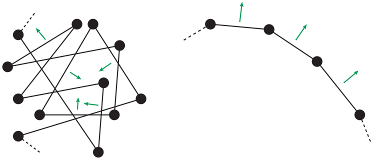
Therefore in perturbation theory and for the physical case $\epsilon = 2$, because the upper itical dimension is $d = 4$.

We need a non-perturbative method to draw phase-diagrams.

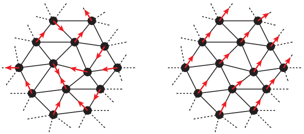
$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2)} + R_k\right)^{-1} k\partial_k R_k$$

## Order parameters

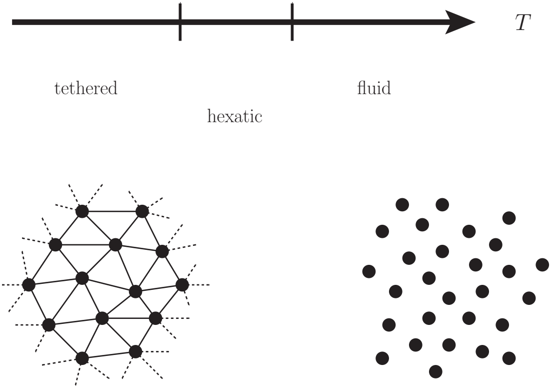
Flat vs Crumpled:
$K^2 \sim (\partial^2 r^\mu)^2 \sim (\partial n^i)^2$



Local order:
$\mathbb{Z}_6 \to \mathrm{SO}(2)$



## The KTNHY description




**Kosterlitz, Thouless ’73; Nelson, Halperin ’79; Young ’79**

## Coset models

Embedding's isometries:

$$\mathrm{ISO}(D): \quad r^\mu \to R(\alpha)^\mu{}_\nu r^\nu + b^\mu$$

Full symmetry group:

$$\mathrm{ISO}(D) \times G_{\mathrm{int}}$$

Extension is generated breaking (at least) the translations.
Given the unbroken group $H \supset H_0$, all the membrane models are constructed as

$$\mathrm{ISO}(D) \times G_{\mathrm{int}} \big/ H_0$$

**West '00**

## The tethered membrane model

Each monomer breaks translations fully:

$$\mathrm{ISO}(D)/\mathrm{SO}(D)$$

The order parameters of the broken translations are $\partial_\alpha r^\mu$.

$$\begin{aligned} S[\mathbf{r}] &= \int \mathrm{d}^d x \Big( \frac{\kappa}{2} (\partial_\alpha \partial_\alpha \mathbf{r})^2 + \frac{t}{2} (\partial_\alpha \mathbf{r})^2 \\ &\quad + u (\partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r})^2 + v (\partial_\alpha \mathbf{r} \cdot \partial_\alpha \mathbf{r})^2 + \dots \Big) \end{aligned}$$

- $\kappa$: bending rigidity.
- $t$: tension.
- $u$ and $v$: Lamé coefficients.

## The fluid membrane model

Bonds melt, the infinitesimal plaquette $r^\mu + \partial_\alpha r^\mu \mathrm{d}x^\alpha$ breaks:

$$\mathrm{ISO}(D)/\mathrm{SO}(d) \times \mathrm{SO}(D-d)$$

Compared to the tethered model the breaking corresponds to:

$$\mathrm{SO}(D)/\mathrm{SO}(d) \times \mathrm{SO}(D-d)$$

However, one Goldstone field is eliminated by the "inverse Higgs mechanism" $\Longrightarrow$ tangential translations restored inside reparametrization invariance.

$$S[\mathbf{r}] \;=\; \int \mathrm{d}^d x \sqrt{g}\Big(\mu + \frac{\kappa}{2}K^2 + \frac{\bar{\kappa}}{2}R + \dots\Big)$$

- $\kappa$: bending rigidity.
- $\mu$: surface tension.
- $\bar{\kappa}$: Gaussian rigidity.

## Long–range effects from the melting of the tethered model

$$S[\mathbf{r}] \;=\; \int \mathrm{d}^2x \left(\epsilon_0 + \frac{t}{2}\,(\partial_\alpha \mathbf{r})^2 + \dots\right)$$

Field dependent transformation to highlight the Goldstones of the broken rotations of the fluid phase:

$$\partial_\alpha r^\mu \to R(\xi)^\mu{}_\nu \partial_\alpha r^\nu$$

and functionally integrate them:

$$\Gamma[\mathbf{r}] = \int \mathrm{d}^2x \sqrt{g}\left(t_R - \frac{1}{96\pi} R \frac{1}{\Delta} R\right)$$

The fluid description of the tethered model enjoys long range interactions.

**Z. '14**

## The hexatic membrane model

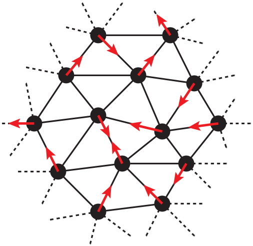
Crystalline structure breaks internal rotations:

$$\mathrm{ISO}(D) \times \mathrm{SO}(d)_{\mathrm{cr}} / \mathrm{SO}(d)_{\mathrm{diag}} \times \mathrm{SO}(D-d)$$

New order parameter $N^\alpha = \cos\theta \, {e_1}^\alpha + \sin\theta \, {e_2}^\alpha$ (d=2).

$$S[\mathbf{r}, \mathbf{N}] \;=\; \int \mathrm{d}^d x \, \sqrt{g} \Big( \mu + \frac{\kappa}{2} K^2 + \frac{\bar{\kappa}}{2} R + \frac{K_A}{2} (\nabla_a \mathbf{N})^2 + \dots \Big)$$

- $K_A$: Hexatic rigidity.



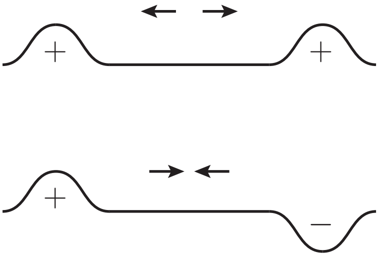
**David, Guitter, Peliti '87; Park, Lubensky '96; ...**

## Long–range effects (again)

**N** is integrated away (finite renormalization for $K_A$):

$$\Gamma[\mathbf{r}] \;=\; \int \mathrm{d}^d x\,\sqrt{g}\Big(\mu + \frac{\kappa}{2}K^2 + \frac{\bar\kappa}{2}R + \frac{K_A}{8}R\frac{1}{\Delta}R + \dots\Big)$$

**N** induces long range interactions among curvatures.




The long range interactions free us from Mermin–Wagner theorem.

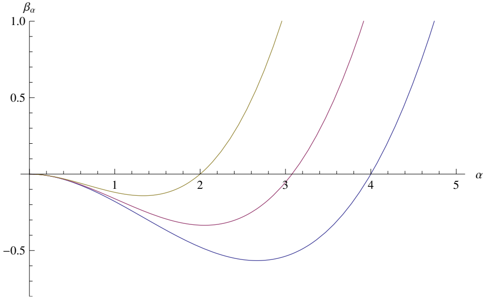
## Beta functions in the FRG scheme, $\alpha = 1/\kappa$

$$
\begin{aligned}
k\partial_k\tilde{\mu}_k &= -2\tilde{\mu}_k - \frac{D-2}{2\pi\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}}\log\frac{2+\tilde{\mu}_k-\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}}{2+\tilde{\mu}_k+\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}} \\
k\partial_k\alpha_k &= \frac{\alpha_k\left(D-\frac{3}{4}K_A\alpha_k\right)}{2\pi\left(4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}\right)}\left[\frac{2(1-\alpha_k)+\alpha_k\tilde{\mu}_k}{1+\alpha_k\tilde{\mu}_k}\right. \\
&\quad \left. + \frac{\tilde{\mu}_k}{\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}}\log\frac{2+\tilde{\mu}_k-\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}}{2+\tilde{\mu}_k+\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}}\right]
\end{aligned}
$$

**Codello, Z. '13**

## **A 2nd order PT** in $\alpha = 1/\kappa$

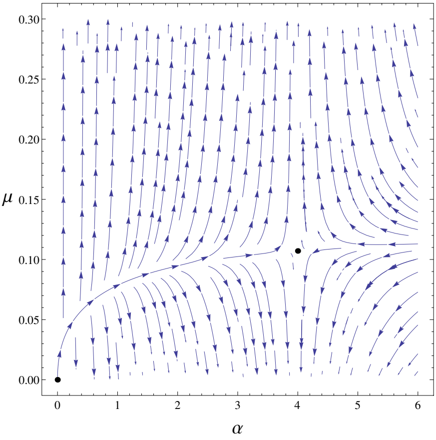
Non-trivial fixed point $\alpha^* \sim 1/K_A$.
Critical exponent: $\nu \simeq 0.37$.
Beta function of $\alpha$ for $K_A = 1/2, 1, 2$ from bottom:




$1/K_A$ is the natural "small" expansion parameter of an asymptotically safe theory.

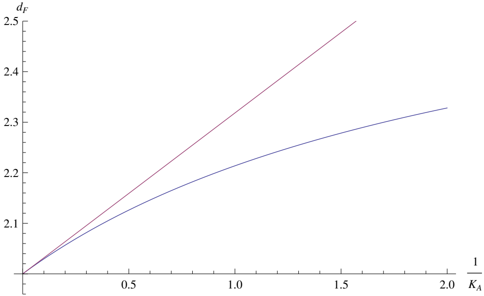
## The phase–diagram

Non-trivial fixed point $\alpha^* \sim 1/K_A$ and critical exponent $\nu \simeq 0.37$.
For $K_A = 1$:



## It is extended but not quite

By decreasing $K_A$ the fractal dimension saturates to 2.71 in our estimate.




*Crinkled phase*: the long–range interactions stack positive and negative curvatures together making the (ground state) surface very fuzzy.

## Minimally coupled $\mathbb{Z}_2$-scalar in the LPA'

$$\Gamma_k[r] = \int \mathrm{d}^d y \, \sqrt{g} \left\{ \frac{Z}{2} g^{ab} \partial_a \varphi \partial_b \varphi + V(\varphi) + \frac{\kappa}{2} K^2 + \frac{\bar{\kappa}}{2} R \right\}$$

$$\begin{aligned} k\partial_k v(\varphi) =& -2v(\varphi) + \frac{1}{2}\eta\varphi v'(\varphi) \\ & + \frac{4-\eta}{16\pi\left(1+v''(\varphi)\right)} + \frac{\kappa \log\left(1+\frac{v(\varphi)}{\kappa}\right)}{2\pi v(\varphi)}, \\ \eta =& \frac{v'''(\varphi_0)^2}{4\pi\left(1+v''(\varphi_0)\right)^4} + \frac{\kappa}{2\pi v(\varphi_0)} - \frac{\kappa^2 \log\left(1+\frac{v(\varphi_0)}{\kappa}\right)}{2\pi v(\varphi_0)^2} \end{aligned}$$

**Symmetry breaking?**

Define a VEV $\varphi_0 = \left\langle \varphi^2 \right\rangle^{1/2}$ of the potential from $v'(\varphi_0) = 0$

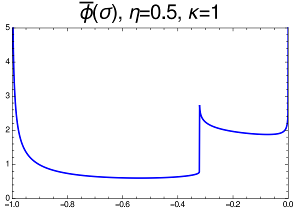
$$k\partial_k \varphi_0 = -\frac{1}{2}\eta\varphi_0 + \frac{(4-\eta)v'''(\varphi_0)}{16\pi v''(\varphi_0)\left(1+v''(\varphi_0)\right)^2}$$

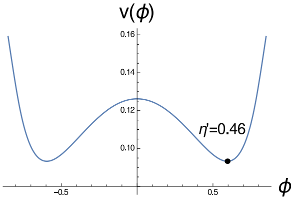
The VEV couples very mildly to $\kappa$! (only through $\eta$)

# Scaling solutions

Divergence:
$v'' \to \infty$
$-2v + \frac{1}{2}\eta\varphi v' + \mathcal{O}(\kappa) \to 0$

Parametrize:
$\sigma \equiv v''(0)$
$\bar{\varphi}(\sigma) \simeq \frac{4v}{\eta v'} + \mathcal{O}(\kappa)$




Iteration: $\eta = 0.4826$

**Shooting the spectrum**

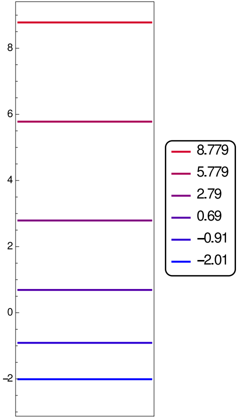
$$v(\varphi) \to v(\varphi) + \epsilon\, \delta v(\varphi) \left(\frac{k}{k_0}\right)^e$$

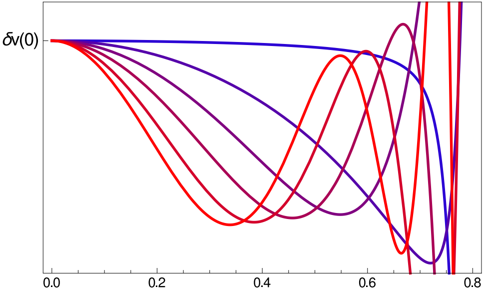
$$\delta v'' - \frac{8\pi\eta\varphi(1+v'')^2}{4-\eta}\delta v' - \frac{8(1+v'')^2}{(4-\eta)v(1+\frac{v}{\kappa})}\delta v$$
$$+\frac{8\kappa(1+v'')^2\log\left(1+\frac{v}{\kappa}\right)}{(4-\eta)v^2}\delta v + \frac{16\pi(2+e)(1+v'')^2}{4-\eta}\delta v = 0$$

# Spectrum

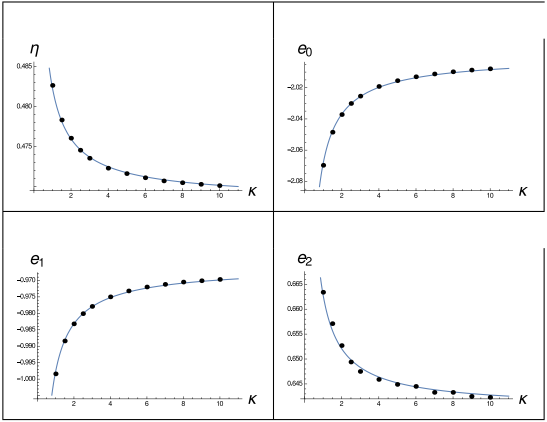
$$v(\varphi) \to v(\varphi) + \epsilon\, \delta v(\varphi) \left(\frac{k}{k_0}\right)^{e}$$

Example at $\kappa = 1$:



# Spectrum as a function of $\kappa$




$$e = e_\infty + a\left(\frac{\kappa}{\kappa_0}\right)^{-b}; \qquad b \simeq 0.85 \pm 0.02$$

## Conclusions

- Rich phase diagram of membranes.
- Symmetry breaking mechanism fully under control.
- Importance of long-range interactions.
- Extrinsic deformations of the Ising critical properties.