Magnetic Vector Potentials and Helicity in Periodic Domains

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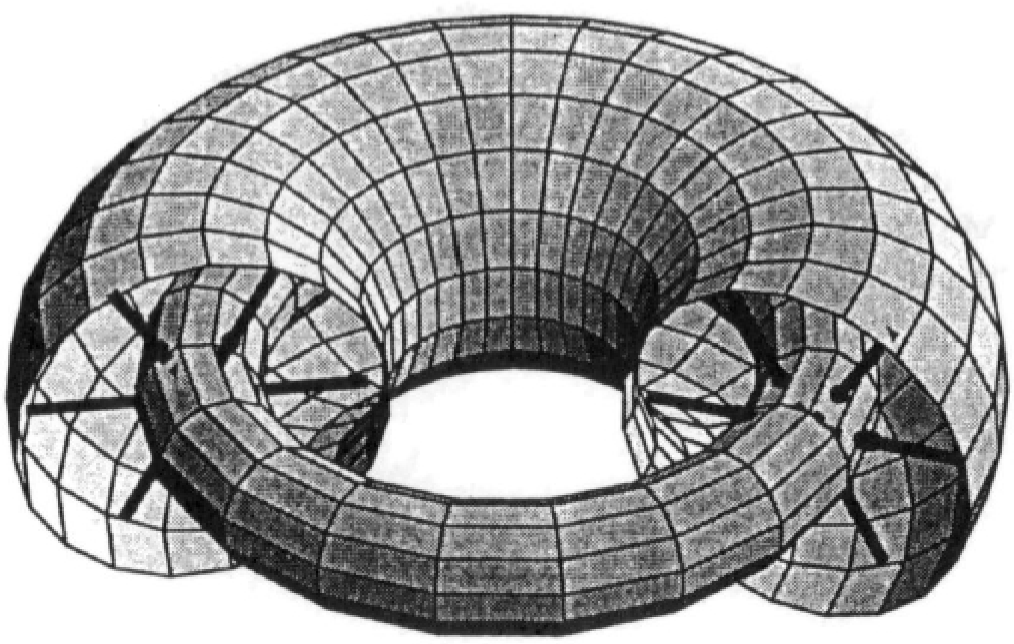


Perfectly Conducting Boundaries

$$H_{\mathrm{m}} = \int_V \mathbf{A} \cdot \mathbf{B} \; \mathrm{d}V$$

boundary conditions: $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$

gauge transformation: $\mathbf{A}' = \mathbf{A} + \nabla \phi$

$$H'_{\mathrm{m}} = \int_V \mathbf{A}' \cdot \mathbf{B} \; \mathrm{d}V = \int_V \mathbf{A} \cdot \mathbf{B} \; \mathrm{d}V + \int_V \nabla \phi \cdot \mathbf{B} \; \mathrm{d}V$$

$$= H_{\mathrm{m}} + \int_V \nabla \cdot (\phi \mathbf{B}) \; \mathrm{d}V = H_{\mathrm{m}} + \int_{\partial V} \phi \mathbf{B} \cdot \hat{\mathbf{n}} \; \mathrm{d}S$$

$$= H_{\mathrm{m}}$$

Periodic Boundaries

boundary conditions: $\mathbf{A}(x_0, y, z) = \mathbf{A}(x_1, y, z)$

$$\mathbf{A}(x, y_0, z) = \mathbf{A}(x, y_1, z)$$

$$\mathbf{A}(x, y, z_0) = \mathbf{A}(x, y, z_1)$$

gauge transformation: $\mathbf{A}' = \mathbf{A} + \nabla\phi$

$\nabla\phi$ periodic

$$H_\mathrm{m}' = H_\mathrm{m} + \int_{\partial V} \phi \mathbf{B} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

with ϕ periodic: $\int_{\partial V} \phi \mathbf{B} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = 0$

in general: $H_\mathrm{m}' \neq H_\mathrm{m}$

Wikipedia

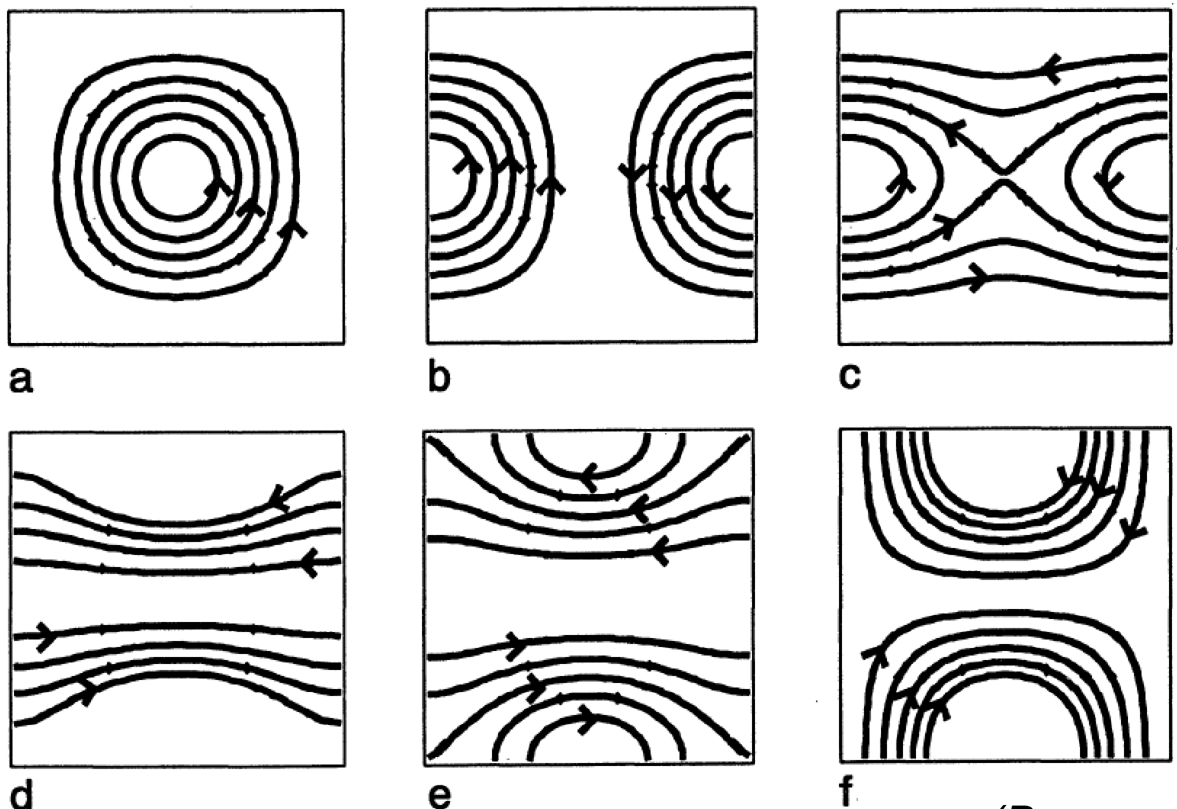
Magnetic helicity is a gauge-dependent quantity, because $\mathbf{A}$ can be redefined by adding a gradient to it (gauge transformation). However, for perfectly conducting boundaries or periodic systems without a net magnetic flux, the magnetic helicity is gauge invariant.

- (cur | prev) 16:39, 3 February 2007 AxelBrandenburg (talk | contribs) . . (1,834 bytes) **(+578)** . . (undo)

+ However, for periodic or perfectly conducting boundaries the magnetic helicity is gauge invariant.

Periodic Domains with Mean Flux

Mean magnetic field in z-direction



(Berger 1996)

⇨ inversion of magnetic helicity

But: Problem starts with magnetic vector potential.

Differential p-forms

P-tensor: multilinear map:

$$T : V_1 \otimes V_2 \otimes \dots V_p \to \mathbb{R}$$

$$T \to T(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p)$$

P-form: covariant p-tensor that is antisymmetric:

$$\alpha(\dots, \mathbf{v}_r, \dots, \mathbf{v}_s, \dots) = -\alpha(\dots, \mathbf{v}_s, \dots, \mathbf{v}_r, \dots)$$

Differential 1-form: $\mathrm{d}f = \frac{\partial f}{\partial x_i} \mathrm{d}x_i \qquad \mathrm{d}x_i(\mathbf{v}) = v_i$

Construct every differential p-form from 1-forms with the wedge product: $\gamma = \alpha \wedge \beta = -\beta \wedge \alpha$

$\wedge =$ anti-symmetrized tensor product

$$\alpha \in \Lambda^1$$
$$\beta \in \Lambda^1$$
$$\gamma \in \Lambda^2$$

Differentiation of forms

$$\mathrm{d} : \Lambda^p \rightarrow \Lambda^{p+1}$$

$$\mathrm{d}\alpha = \frac{\partial \alpha_{i_1,\ldots,i_p}}{\partial x_j} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \ldots \mathrm{d}x^{i_p} \in \Lambda^{p+1}$$

Linearity: $\mathrm{d}(\lambda\alpha + \mu\beta) = \lambda \mathrm{d}\alpha + \mu \mathrm{d}\beta \qquad \lambda, \in \mathbb{R}$

Double differentiation: $\mathrm{d}(\mathrm{d}\alpha) = 0$

⇒ $\nabla \cdot \nabla \times \mathbf{v} = 0 \qquad \nabla \times \nabla \phi = 0$

Associated vectors in $\mathbb{R}^3$ (Euclidean metric):

$$\mathbf{v} \leftrightarrow \nu = \langle \mathbf{v}, . \rangle = g_{ij} v^i \mathrm{d}x^j$$

$$\mathbf{v} \leftrightarrow \sqrt{\det(g)}(v^1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 + v^2 \mathrm{d}x^3 \wedge \mathrm{d}x^1 + v^3 \mathrm{d}x^1 \wedge \mathrm{d}x^2)$$

$$\boxed{\mathrm{d}\omega^1_{\mathbf{A}} = \omega^2_{\nabla \times \mathbf{A}}} \qquad \omega^1 \in \Lambda^1, \omega^2 \in \Lambda^2$$

Potentials

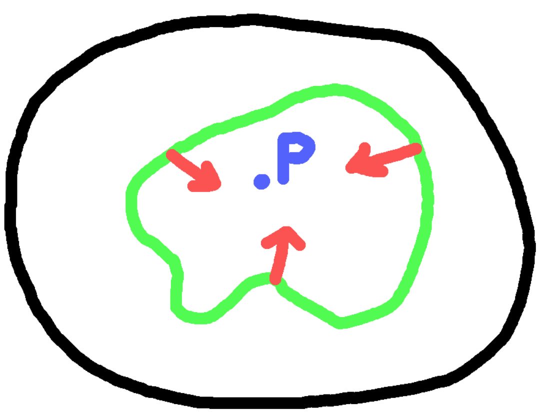
A p-1 form α is a potential for a given p-form β if $\beta = \mathrm{d}\alpha$.

need: $\mathrm{d}(\mathrm{d}\alpha) = \mathrm{d}\beta = 0$ (exact form)

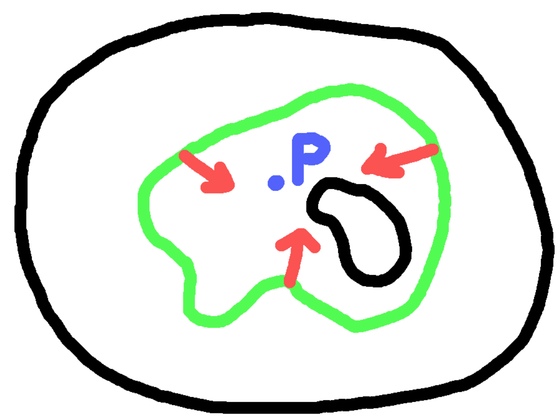
Poincaré Lemma:

On any contractible manifold, if $\mathrm{d}\beta = 0$ *then there exists an* α *such that* $\beta = \mathrm{d}\alpha$ *.*

Contractible manifold:



Non-contractible manifold:

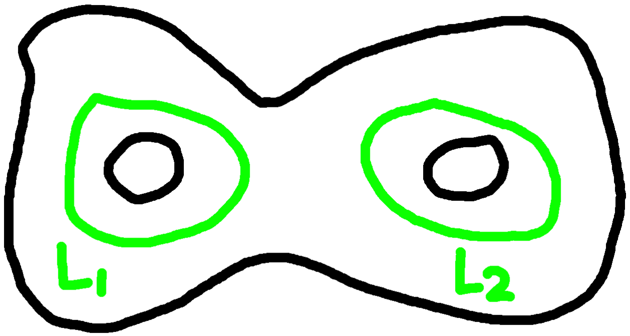


Integral along closed loops: $\int_L \beta = \int_L \mathrm{d}\alpha = \int_{\partial L} \alpha = 0$

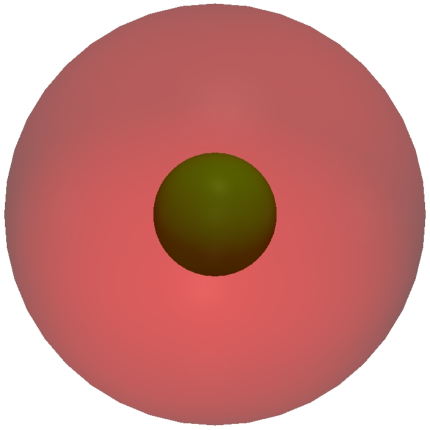
Fundamental Periods, Betti Numbers

Fundamental periods: Closed hypersurfaces (e.g. loops) which cannot be continuously transformed into each other.

Betti number: number of such loops.



$$b_1 = 2$$



$$b_1 = 0$$

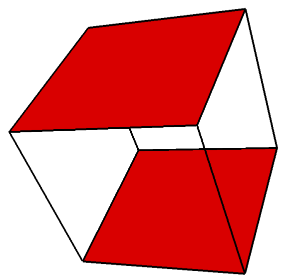
$$b_2 = 1$$

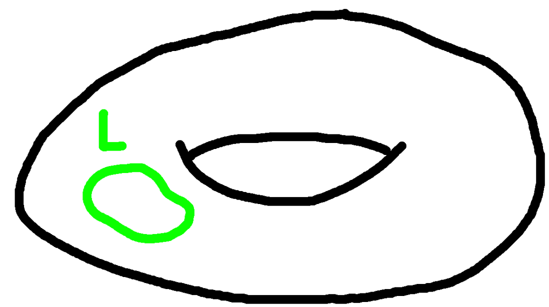
For a potential to exist the Betti number must vanish or integrals along the fundamental periods must vanish.

Periodic Domains

$$\int_L \omega_B^2 = \int_L \mathbf{B} \cdot \hat{\mathbf{n}} = 0 \quad \Rightarrow \quad \mathbf{A} \text{ exists}$$

1 periodic side:

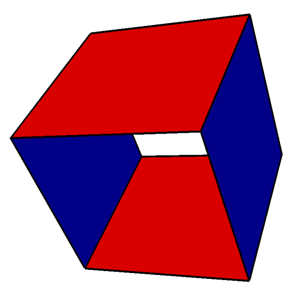


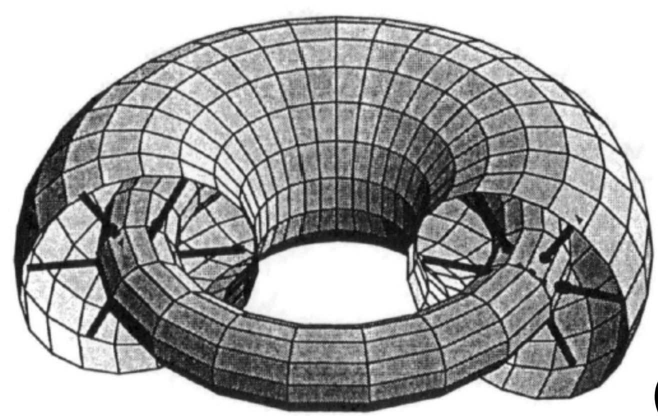


$b_2 = 0$

$\mathbf{A}$ exists

2 periodic sides:





$b_2 = 1$

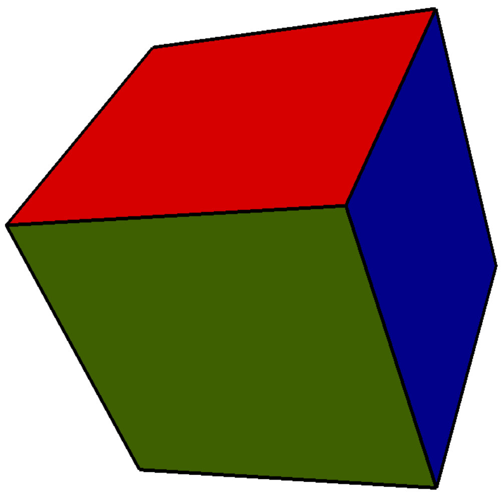
(Berger 1996)

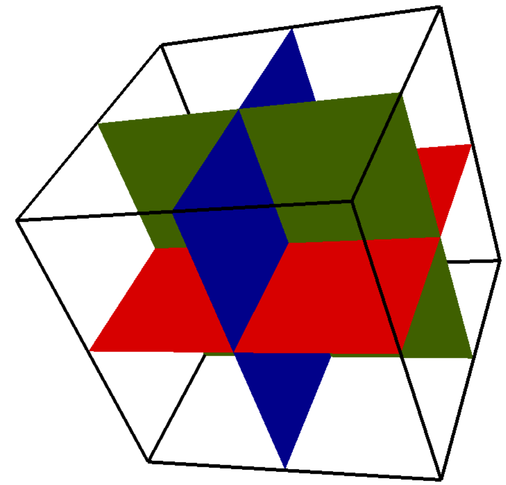
Mean magnetic field along periodic directions: $\mathbf{A}$ exists

Mean magnetic field along non-periodic direction: $\mathbf{A}$ does not exist

Periodic Domains

3 periodic sides:





$$b_2 = 3$$

3 fundamental periods

⇒ Mean magnetic field through all sides must vanish.

Non-Periodic Domains

Problem: known: $\mathbf{B}$ find: $\mathbf{A}$

Solution: $\mathbf{B} = \nabla \times \mathbf{A}$

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A}$$

$\mathbf{J} = -\nabla^2 \mathbf{A}$ Coulomb gauge: $\nabla \cdot \mathbf{A} = 0$

$\hat{\mathbf{J}} = -\mathbf{k}^2 \hat{\mathbf{A}}$!assumed periodicity!

Remedy: Volterra's formula

$$\mathbf{A}(\mathbf{x}) = \int_0^1 \tau \mathbf{B}(\tau \mathbf{x}) \times \mathbf{x} \, \mathrm{d}\tau$$

Need star-shaped domain.

Conclusions

- Magnetic helicity gauge dependent in periodic domains.
- Vector potential does not always exist in periodic domains.
- Volterra's formula for non-periodic domains.

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