Phase Transitions in Constraint Satisfaction Problems

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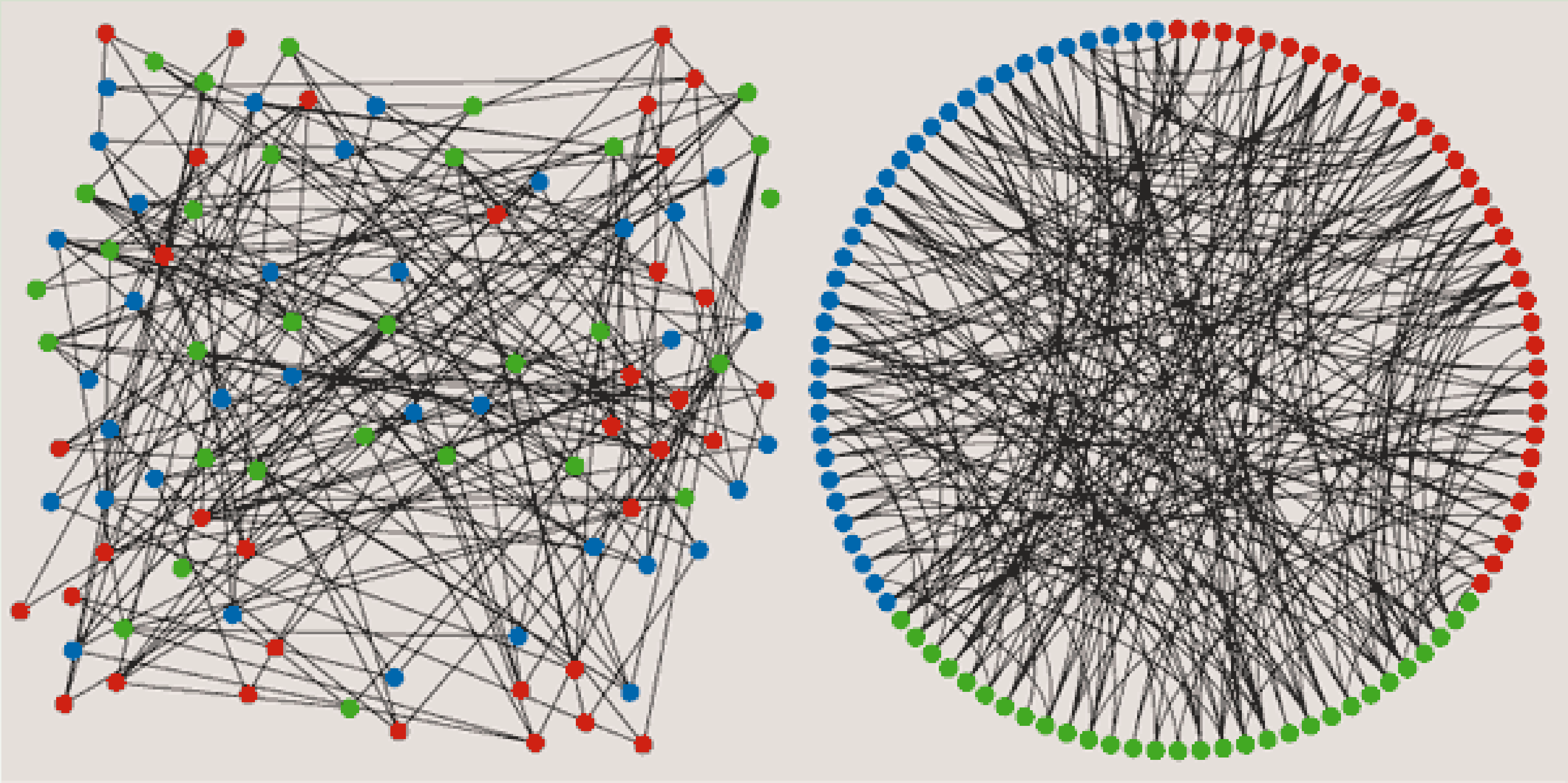
Example of CSP: Coloring of random graphs

Coloring = Antiferromagnetic Potts model at zero temperature

N=100 vertices, M=218 edges, average degree c=2M/N=4.36



Why is random graph coloring interesting?

- Colorable threshold at average degree c_s
  - a.s. colorable for $c < c_s$ and a.s. uncolorable for $c > c_s$
  - (A part of) Proof of existence (Friedgut 1997, Achlioptas, Friedgut, 1999)
  - Exact (but not rigorous) location via cavity method (Mezard, Zecchina, 2002 + Mulet, Pagnani, Weight, Zecchina, 2002)

- Computationally hard region near to the colorable threshold

Why are the typical instances near to the threshold hard?

Is there a way how to make them easy?

Insight From Statistical Physics of Spin Glasses

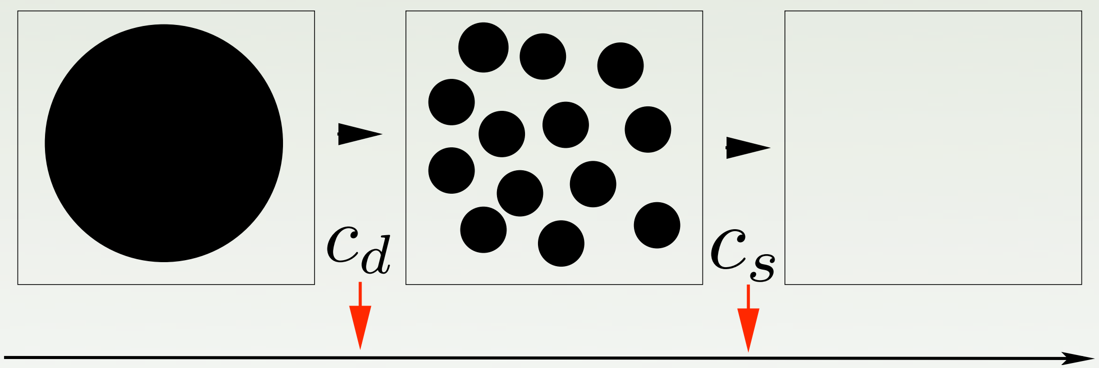
Prediction of a glassy (clustered) phase in the colorable region Mezard, Zecchina, Parisi, 2002, Biroli, Monasson, Weigt, 1999)

Clusters (roughly said): groups of nearby solutions which are in some sense disconnected from each other (more precisely later in the presentation).

Suggestion: Clustering responsible for the onset of hardness.

Accomplishments (Mezard, Zecchina, Parisi, 2002)**:**

- ✓ **The exact colorable threshold computed.**
- ✓ **Survey Propagation algorithm designed.**



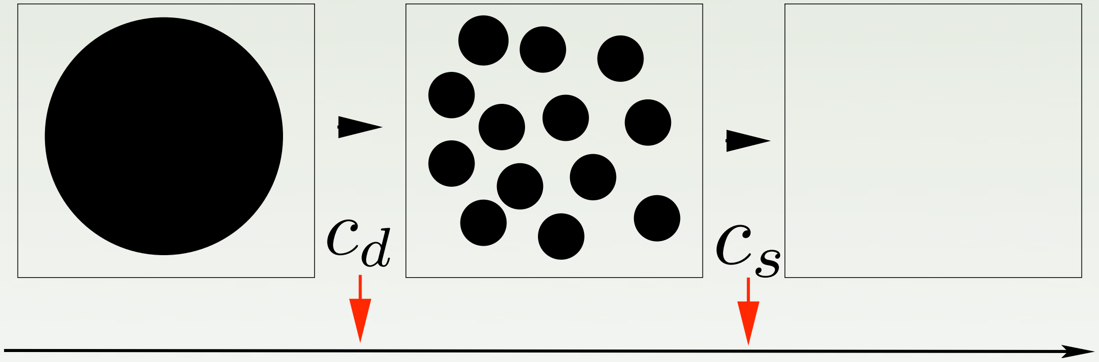
Loose ends to be tied up

- **Computed value of the clustering transition did not correspond to the empirically obtained values of the onset of hardness (linear performance of Walk-SAT etc.)**

- **Clustering transition continuous or discontinuous?**

- **Frozen variables present in clusters or not?** *(Frozen variable takes the same color in all the solutions belonging to the cluster.)*

- **SP works with frozen variables, but finds solutions without them.**



What do we mean by clusters?

- **Roughly said:** Lumps (groups) of nearby solutions which are in some sense disconnected from each other.
- **For mathematical physicist:** "Extremal Gibbs measures = pure states".
- **For computer scientist:** Fixed points of belief propagation.
- **For spin glass physicist:** Solutions of TAP equations.

Why do we need to speak about clusters above c_d?

- The Gibbs measure cease to be extremal. (Mézard, Montanari, 2005)
- The point-to-set correlations do not decay.

➡ **Different from:** A non-trivial solution of the survey propagation exists!

Refining the structure of clusters

- **Entropy (size) of a cluster *s*: logarithm of the number of solutions belonging to the cluster (divided by the number of variables).**

- **Complexity function $\Sigma(s)$: logarithm of the number of clusters of size *s* (divided by the number of variables).**

- **The zero temperature entropic 1RSB cavity method allows us to compute the Legendre transform $\Phi(m)$ of $\Sigma(s)$. Main idea** (Mézard, Palassini, Rivoire, 2005) **weight each cluster by its size to the power m:**

$$e^{N\Phi(m)} = \sum_{\alpha} (e^{Ns_\alpha})^m = \int e^{N[ms+\Sigma(s)]} \mathrm{d}s$$

$$\Phi(m) = ms + \Sigma(s), \quad \frac{\partial \Sigma(s)}{\partial s} = -m$$

Note: the approach of Mézard, Zecchina, Parisi 2002; Mulet, Pagnani, Weigt, Zecchina 2002 **was at m=0.**

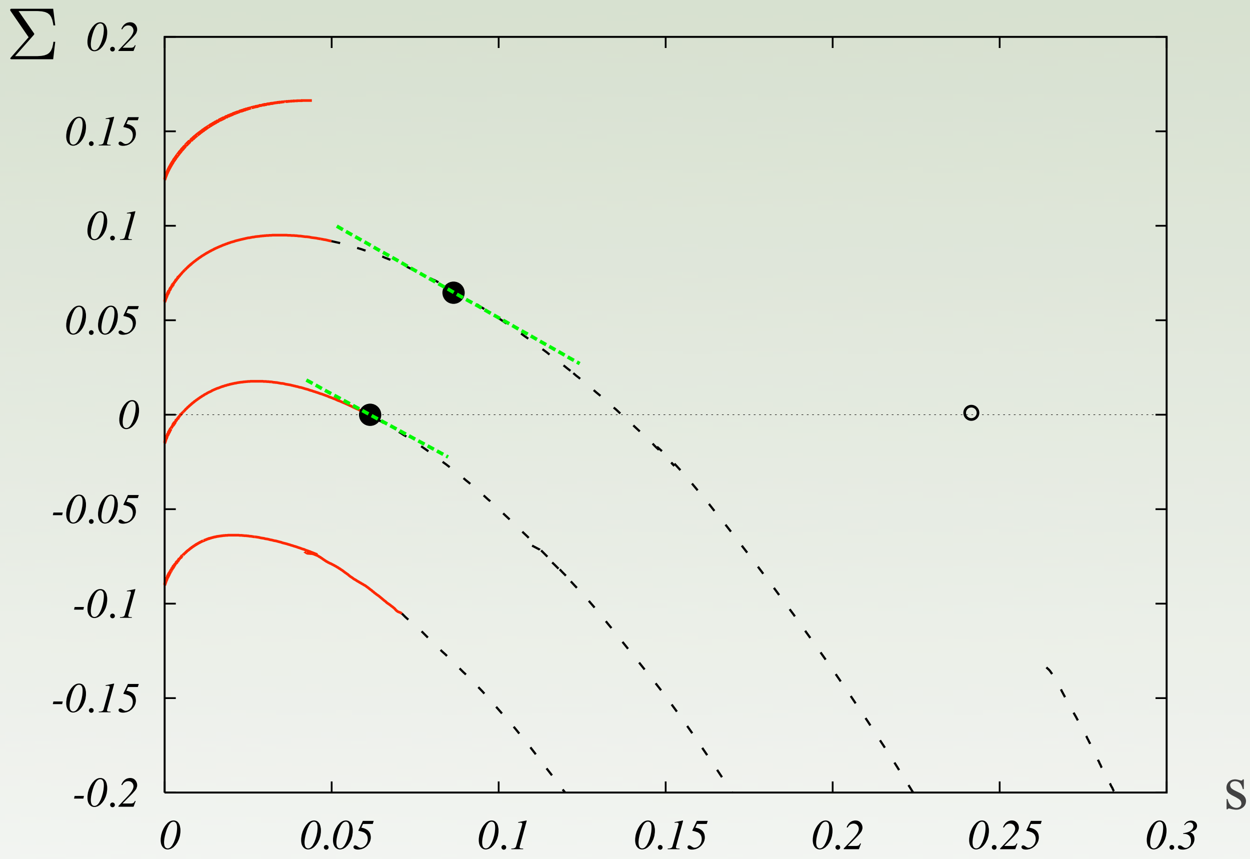
Solve (mostly numerically) the 1RSB cavity equations

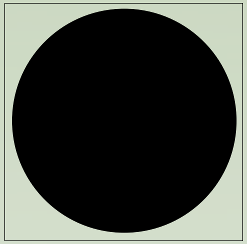
+ Work out the several special cases when the equations simplify (m=1, m=0, frozen variables, regular graphs ...)

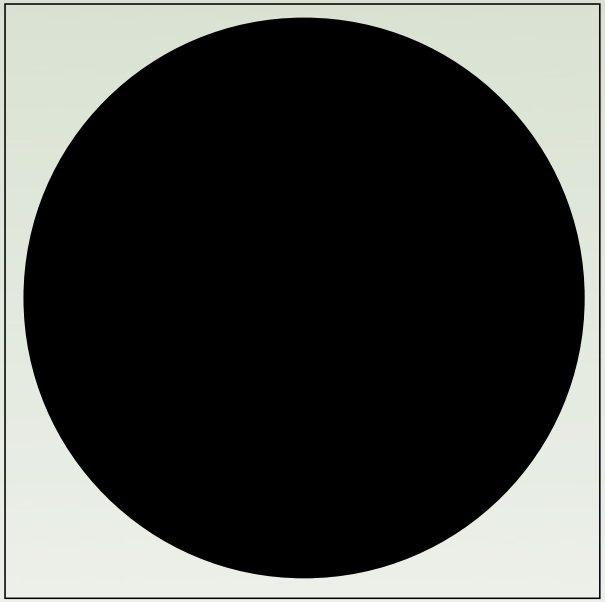
$\Rightarrow$ **Our results (+ their meaning)**

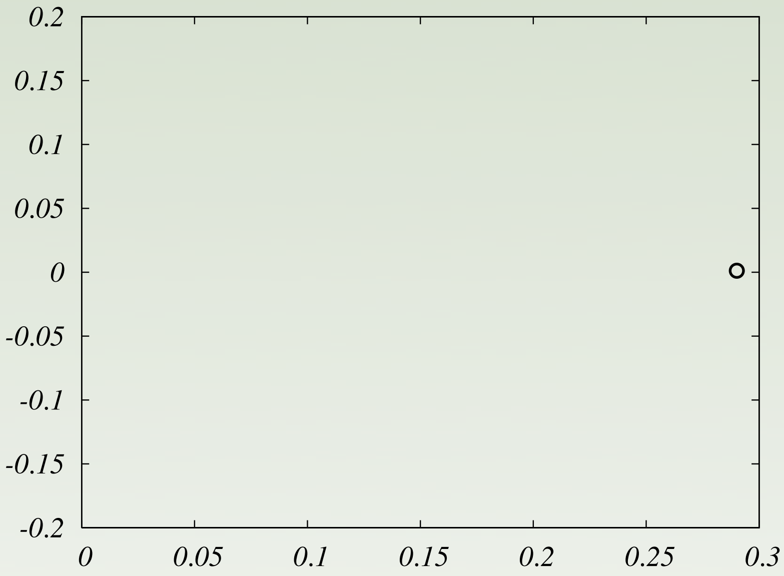
Learning from $\Sigma(s)$

Example of 6-coloring, regular graphs, degrees 17, 18, 19, 20



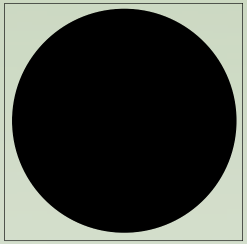


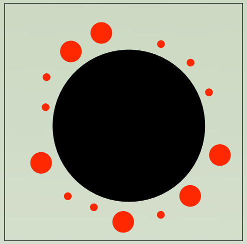


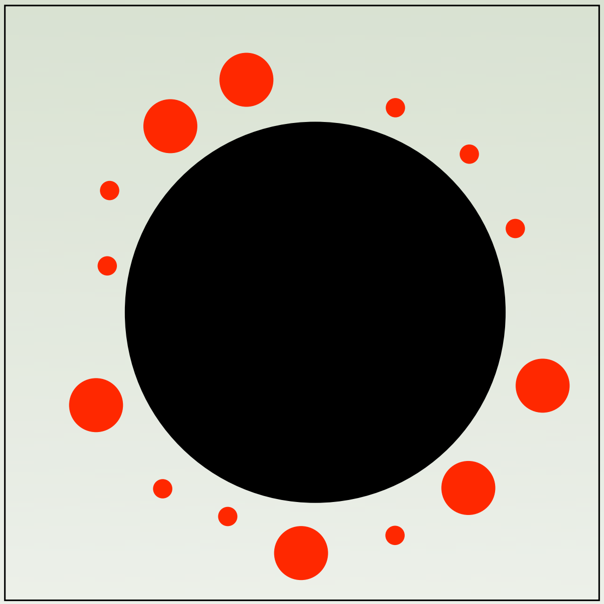


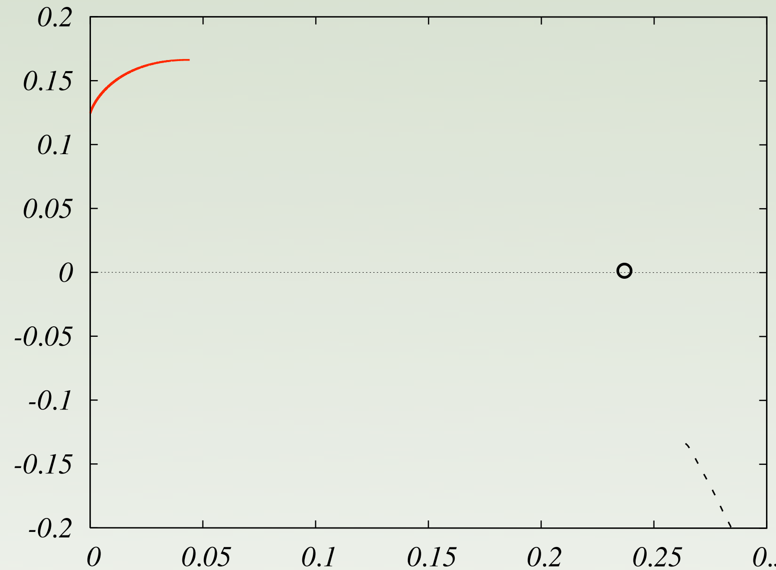
6 coloring of regular random graph

very low degree



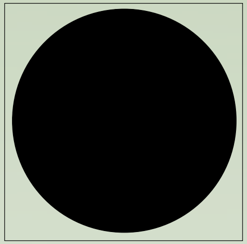


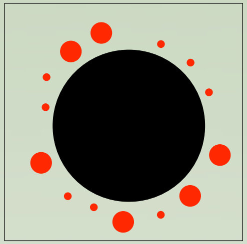


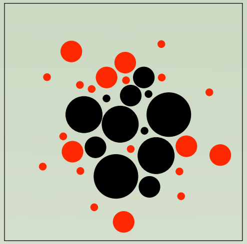


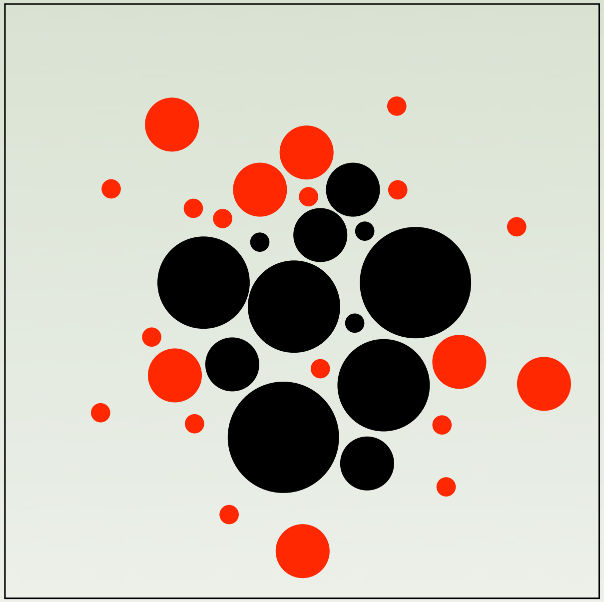
6 coloring of regular random graph

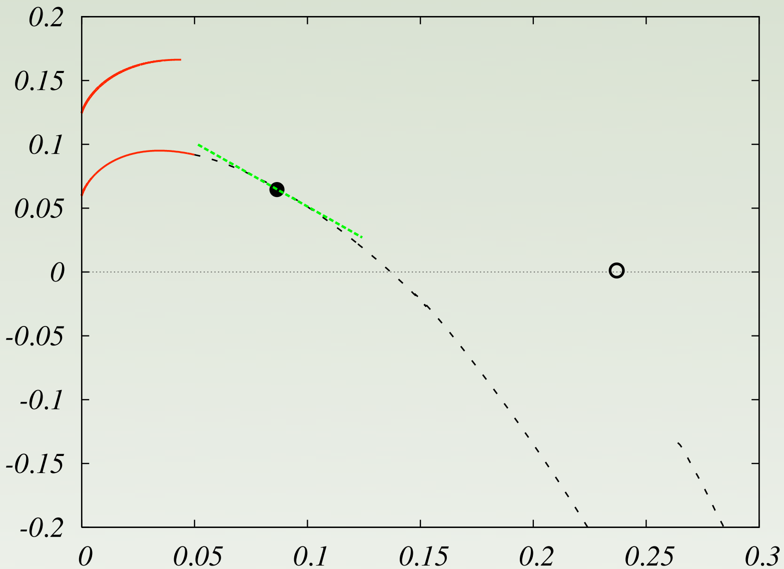
degree c=17





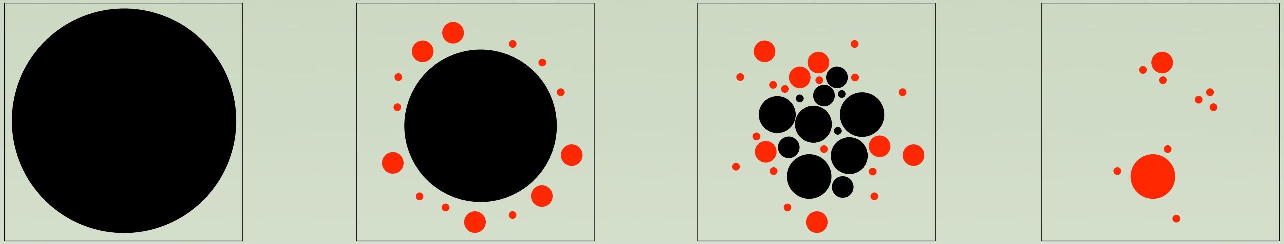


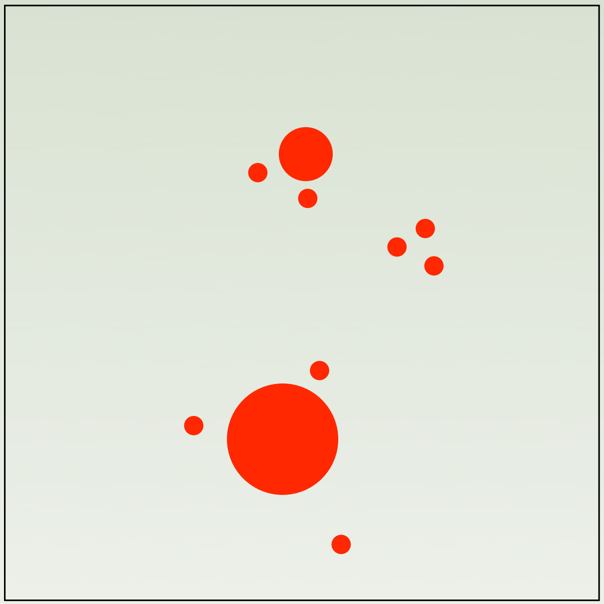


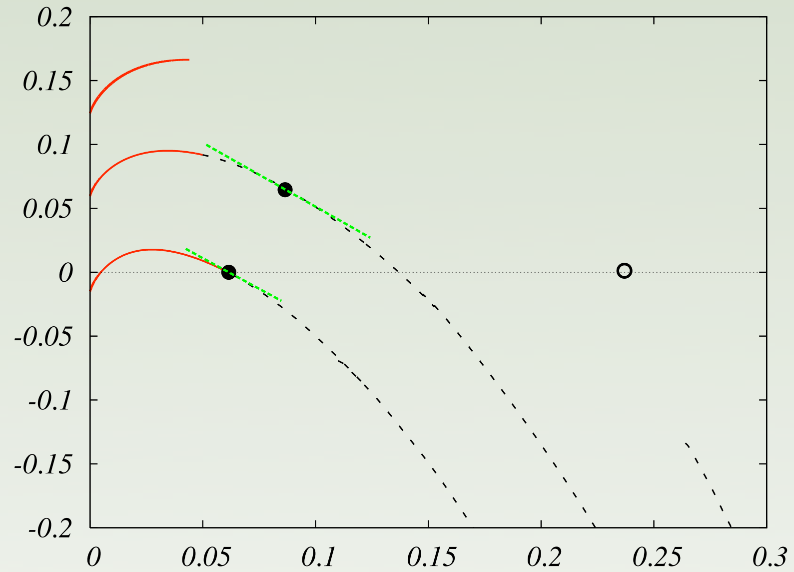


6 coloring of regular random graph

degree c=18

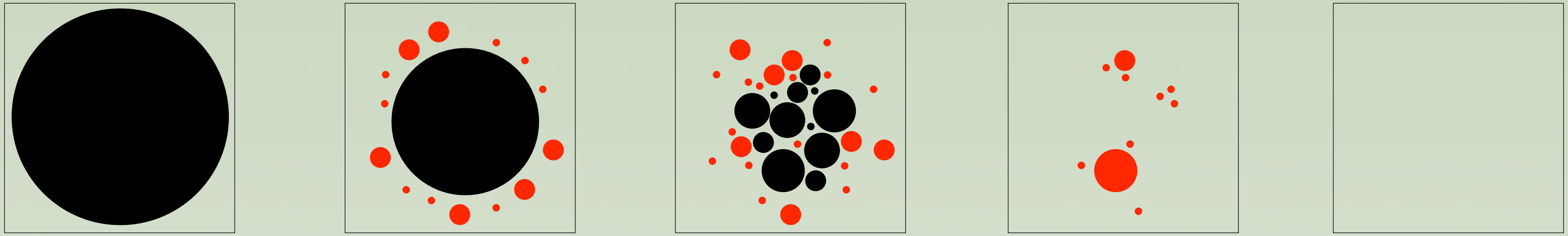


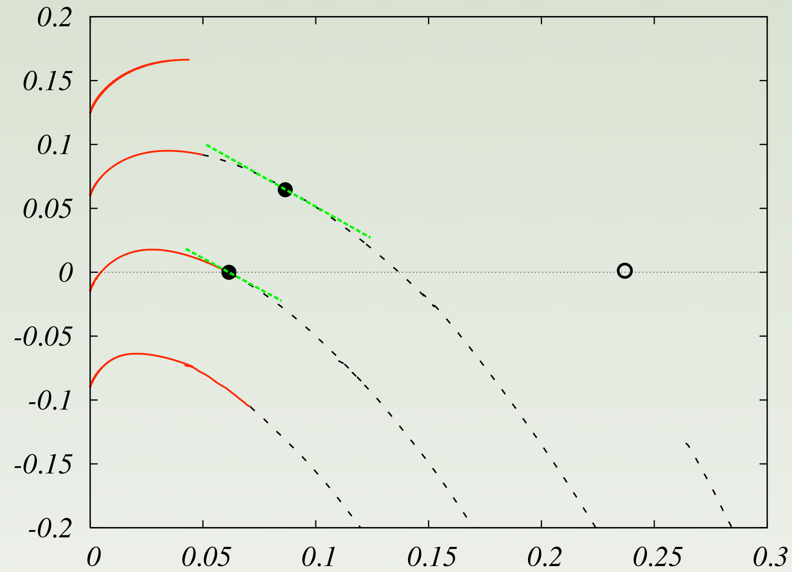




6 coloring of regular random graph

degree c=19

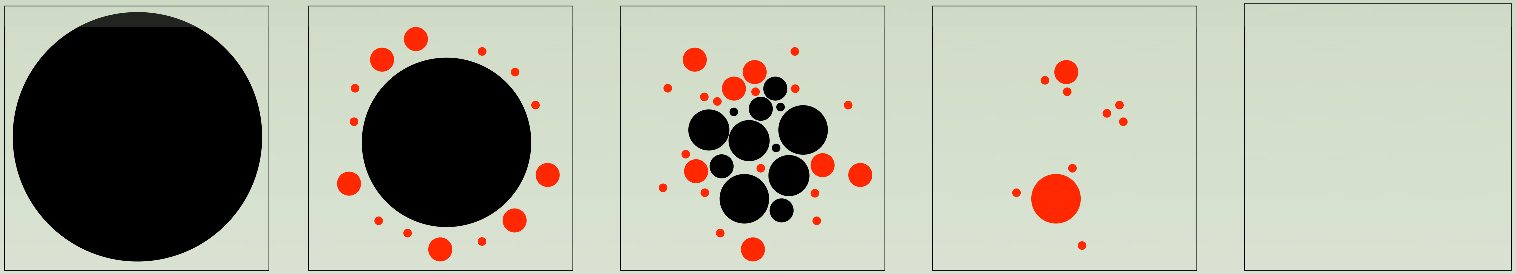




6 coloring of regular random graph

degree c=20

The important phase transitions: Overview

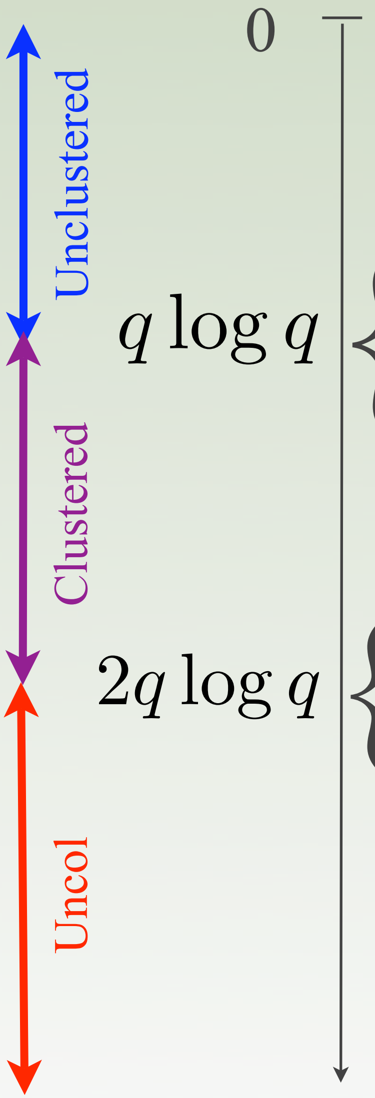


- **Clustering (dynamical) transition** $c_d(3) = 4,\ c_d(4) = 8.35,\ c_d(5) = 12.84$
  - Glassy solution appears at m=1.
  - The replica symmetric entropy still correct, no non-analyticity.
  - Entropy dominated by exponentially many states.
- **Condensation (Kauzmann) transition** $c_c(3) = 4,\ c_c(4) = 8.46,\ c_c(5) = 13.23$
  - Discontinuity in the second derivative of entropy.
  - Entropy dominated by finite number of the largest states.
- **Rigidity transition** $c_r(3) = 4.66,\ c_r(4) = 8.83,\ c_r(5) = 13.55$
  - Frozen variables appears in the dominating states.
  - Frozen clusters disappear abruptly.
  - Minimal rearrangements diverge (Semerjian, 2007).

➡ **Moreover:** The entropically dominating clusters are **1RSB stable in the colorable phase** (at least for q>3)

Colorable threshold: $c_s(3) = 4.69,\ c_s(4) = 8.90,\ c_s(5) = 13.67$

Large number of colors (analytical results)



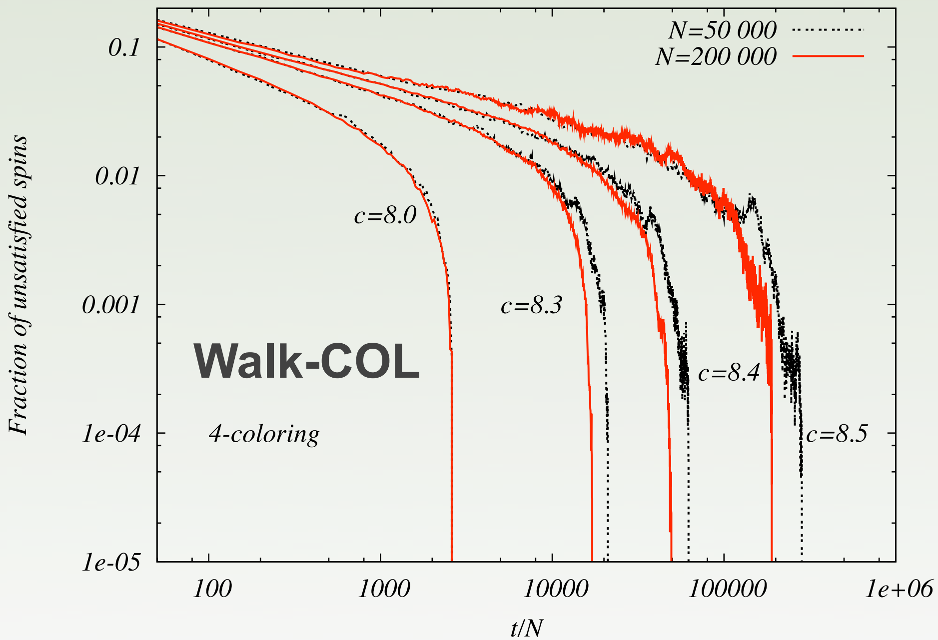
$q \log q$
- SP nontrivial — $c_{\rm SP} \simeq q(\log q + \log\log q + 1 - \log 2)$
- Clustering
- Rigidity — $c_r \simeq q(\log q + \log\log q + 1)$

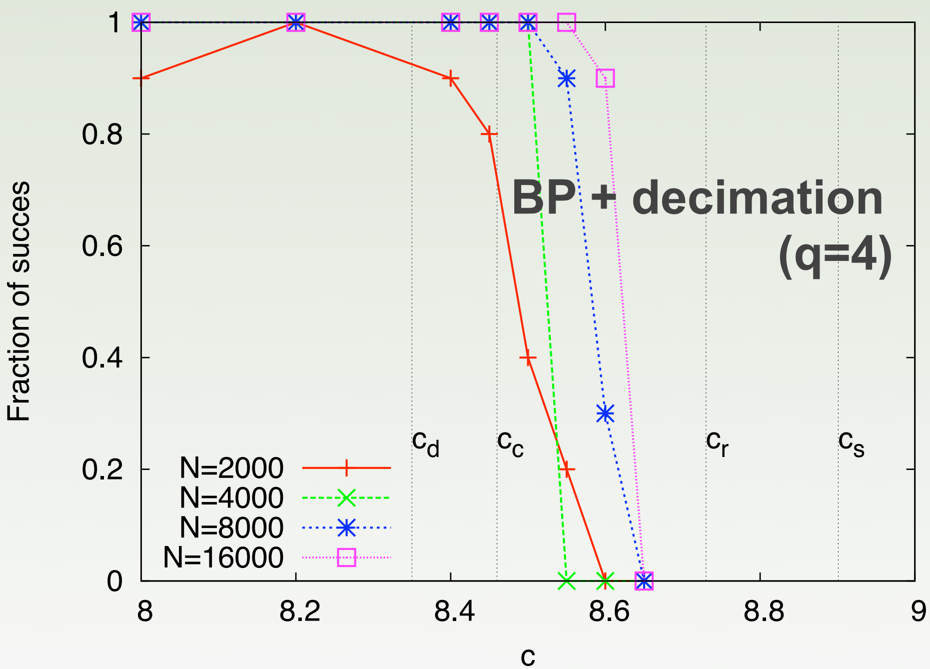
$2q \log q$
- Condensation — $c_g \simeq 2q\log q - \log q - 2\log 2$
- Col/Uncol — $c_q \simeq 2q\log q - \log q - 1$

Algorithms beyond clustering transition

- **Hard to uniformly sample solutions with Monte Carlo (equilibration time diverges).**
- **Not necessarily hard to find solutions.**

- **Belief Propagation gives correct marginals up to the condensation transition. Walk-COL works in linear time (empirical) beyond clustering (clusters without frozen variables cannot “end” at positive energy).**





Is the rigidity more significant for hardness?

Two arguments:

- **All the solutions we are able to find on large graphs (N>10 000) belong to clusters without frozen variables.**

- **Minimal rearrangments** (Semerjian, 2007) **diverge if and only if frozen variables present. Local search should fail.**

Conclusions

- The dynamical transition redefined with respect to the dominating clusters and located.
- The condensation and rigidity transitions revealed and located.
- Algorithmic consequences understood better, but more work have to be done in this direction.

References

- Florent **Krzakala** (ESPCI Paris), Andrea **Montanari** (ENS Paris + Standford), Federico **Ricci-Tersenghi** (Rome), Guilhem **Semerjian** (ENS Paris), Lenka **Zdeborová** (Orsay):

  Gibbs States and the Set of Solutions or Random Constraint Satisfaction Problems,
  Proc. Natl. Acad. Sci. U.S.A., 104, 10318 (2007).

  E-print: arXiv:cond-mat/0612365.

- Lenka **Zdeborová,** Florent **Krzakala**:

  Phase Transitions in the Coloring of Random Graphs,

  Submitted to Phys. Rev. E.

  E-print: arXiv:0704.1269v2.