Loop Calculus for Graphical Models of Statistical Inference

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Thanks to V. Chernyak (Wayne State, Detroit) & M. Stepanov (UofA, Tucson)

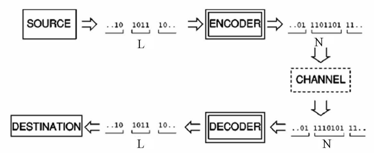
Outline

1. First Part
   - Error Correction. Statistical Inference.
   - Bethe Free Energy and Belief Propagation (BP)
   - Loop Calculus: Gauge Transformations & Loop Series
2. Second Part
   - Error-Floor. Pseudo-Codewords and Instantons.
   - Pseudo-Codeword Search. Spectra.
   - Analysis and Improvement of Decoding with Loop Calculus

Error Correction

Cell phone Hard disk Optical disk Fiber

Scheme:



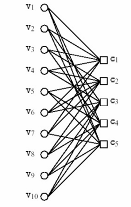
Example of Additive White Gaussian Channel:

$$P(\mathbf{x}_{out}|\mathbf{x}_{in}) = \prod_{i=bits} p(x_{out;i}|x_{in;i})$$

$$p(x|y) \sim \exp(-s^2(x-y)^2/2)$$

- Channel
  is noisy "black box" with only statistical information available
- Encoding:
  use redundancy to redistribute damaging effect of the noise
- Decoding [Algorithm]:
  reconstruct most probable codeword by noisy (polluted) channel

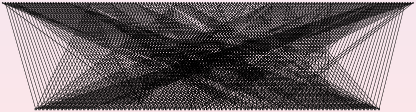
Low Density Parity Check Codes

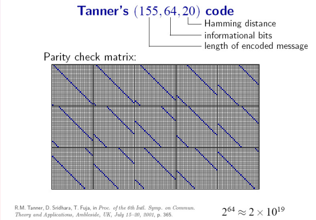


- N bits, M checks, $L = N - M$ information bits
  example: $N = 10$, $M = 5$, $L = 5$
- 2^L codewords of 2^N possible patterns
- Parity check: $\hat{H}\mathbf{v} = \mathbf{c} = \mathbf{0}$
  example:

$$\hat{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

- LDPC = graph (parity check matrix) is sparse





Statistical Models

Ising model $\sigma_i = \pm 1$

$$\mathcal{P}(\boldsymbol{\sigma}) = Z^{-1} \exp\left(\sum_{i,j} J_{ij}\sigma_i\sigma_j\right)$$

J_{ij} define the graph (lattice)

Decoding $\sigma_i = \pm 1$

$$\mathcal{P}(\boldsymbol{\sigma}|\mathbf{x}) = Z^{-1}(\mathbf{x})\prod_{\alpha}\delta\left(\prod_{i\in\alpha}\sigma_i, +1\right)\prod_i p(x_i|\sigma_i)$$

Hard (check) constraints define the graph/code

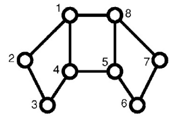
N.Sourlas '89: Error-correction as a Statistical Mechanics

Graphical models

Factorization (Forney '01, Loeliger '01)

$$\mathcal{P}(\boldsymbol{\sigma}|\mathbf{x}) = Z^{-1}\prod_a f_a(\mathbf{x}_a|\boldsymbol{\sigma}_a)$$

$$\underbrace{Z(\mathbf{x}) = \sum_{\boldsymbol{\sigma}}\prod_a f_a(\mathbf{x}_a|\boldsymbol{\sigma}_a))}_{\text{partition function}}$$

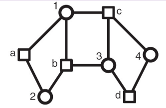


$$f_a \geq 0$$
$$\sigma_{ab} = \sigma_{ba} = \pm 1$$
$$\boldsymbol{\sigma}_1 = (\sigma_{12}, \sigma_{14}, \sigma_{18})$$
$$\boldsymbol{\sigma}_2 = (\sigma_{12}, \sigma_{13})$$

Example: Error-Correction (linear code, bipartite Tanner graph)

$$f_i(h_i|\boldsymbol{\sigma}_i) = \exp(\sigma_i h_i)\cdot\begin{cases} 1, & \forall\alpha,\beta \ni i, \quad \sigma_{i\alpha} = \sigma_{i\beta} \\ 0, & \text{otherwise}\end{cases}$$

$$f_\alpha(\boldsymbol{\sigma}_\alpha) = \delta\left(\prod_{i\in\alpha}\sigma_i, +1\right)$$



h_i - log-likelihoods

Statistical Inference

$\boldsymbol{\sigma}_{\text{orig}}$	$\Rightarrow$	$\mathbf{x}$	$\Rightarrow$	$\boldsymbol{\sigma}$
original *data* $\boldsymbol{\sigma}_{\text{orig}} \in \mathcal{C}$ codeword	noisy channel $\mathcal{P}(\mathbf{x}\|\boldsymbol{\sigma})$	corrupted data: log-likelihood magnetic field	statistical inference	possible preimage $\boldsymbol{\sigma} \in \mathcal{C}$

Maximum Likelihood [ground state] Maximum-a-Posteriori [magnetization]

$$\text{ML} = \arg\max_{\boldsymbol{\sigma}} \mathcal{P}(\mathbf{x}|\boldsymbol{\sigma}) \qquad \text{MAP}_i = \arg\max_{\sigma_i} \sum_{\boldsymbol{\sigma}\setminus\sigma_i} \mathcal{P}(\mathbf{x}|\boldsymbol{\sigma})$$

Exhaustive search is generally expensive:
complexity of the algorithm $\sim 2^N$

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$$\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N), \quad N \ \textit{finite}, \quad \sigma_i = \pm 1 \ (\textit{example})$$

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Variational Method in Statistical Mechanics

$$P(\boldsymbol{\sigma}) = \frac{\prod_a f_a(\boldsymbol{\sigma}_a)}{Z}, \quad Z \equiv \sum_{\boldsymbol{\sigma}} \prod_a f_a(\boldsymbol{\sigma}_a)$$

Exact Variational Principle — Kullback-Leibler '51

$$F\{b(\boldsymbol{\sigma})\} = -\sum_{\boldsymbol{\sigma}} b(\boldsymbol{\sigma}) \sum_a \ln f_a(\boldsymbol{\sigma}_a) + \sum_{\boldsymbol{\sigma}} b(\boldsymbol{\sigma}) \ln b(\boldsymbol{\sigma})$$

$$\left.\frac{\delta F}{\delta b(\boldsymbol{\sigma})}\right|_{b(\boldsymbol{\sigma})=p(\boldsymbol{\sigma})} = 0 \quad \text{under} \quad \sum_{\boldsymbol{\sigma}} b(\boldsymbol{\sigma}) = 1$$

Variational Ansatz

- Mean-Field: $p(\boldsymbol{\sigma}) \approx b(\boldsymbol{\sigma}) = \prod_i b_i(\sigma_i)$
- Belief Propagation:

$$p(\boldsymbol{\sigma}) \approx b(\boldsymbol{\sigma}) = \frac{\prod_a b_a(\boldsymbol{\sigma}_a)}{\prod_{(a,b)} b_{ab}(\sigma_{ab})} \quad \text{(exact on a tree)}$$

$$b_a(\boldsymbol{\sigma}_a) = \sum_{\boldsymbol{\sigma} \setminus \boldsymbol{\sigma}_a} b(\boldsymbol{\sigma}), \quad b_{ab}(\sigma_{ab}) = \sum_{\boldsymbol{\sigma} \setminus \sigma_{ab}} b(\boldsymbol{\sigma})$$

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Bethe free energy: variational approach
(Yedidia,Freeman,Weiss '01 - inspired by Bethe '35, Peierls '36)

$$F = \underbrace{-\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a)}_{\text{self-energy}} + \underbrace{\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln b_a(\boldsymbol{\sigma}_a) - \sum_{(a,c)} b_{ac}(\sigma_{ac}) \ln b_{ac}(\sigma_{ac})}_{\text{configurational entropy}}$$

$$\forall\ a;\ c \in a: \quad \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) = 1, \quad b_{ac}(\sigma_{ac}) = \sum_{\boldsymbol{\sigma}_a \setminus \sigma_{ac}} b_a(\boldsymbol{\sigma}_a)$$

$\Rightarrow$ Belief-Propagation Equations: $\left.\frac{\delta F}{\delta b}\right|_{\text{constr.}} = 0$

MAP≈BP=Belief-Propagation (Bethe-Pieirls): iterative ⇒ Gallager '61; MacKay '98

- Exact on a tree ▸ Derivation Sketch
- Trading optimality for reduction in complexity: $\sim 2^L \to \sim L$
- BP = solving equations on the graph:

$$\eta_{\alpha j} = h_j + \sum_{\beta \neq \alpha}^{j \in \beta} \tanh^{-1}\left(\prod_{i \neq j}^{i \in \beta} \tanh \eta_{\beta i}\right) \quad \Leftarrow \text{LDPC representation}$$

- Message Passing = iterative BP
- Convergence of MP to minimum of Bethe Free energy can be enforced

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Linear Programming version of Belief Propagation

In the limit of large SNR, $\ln f_a \to \pm\infty$: BP→LP

Minimize $F \approx E = -\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a)$ = self energy

under set of linear constraints

LP decoding of LDPC codes Feldman, Wainwright, Karger '03

- ML can be restated as an LP over a codeword polytope
- LP decoding is a "local codewords" relaxation of LP-ML
- Codeword convergence certificate
- Discrete and Nice for Analysis
- Large polytope $\{b_\alpha, b_i\} \Rightarrow$ Small polytope $\{b_i\}$

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BP does not account for Loops

Questions:

- Is BP just a heuristic in a loopy case?
- Why does it (often) work so well?
- Does exact inference allow an expression in terms of BP?
- Can one correct BP systematically?

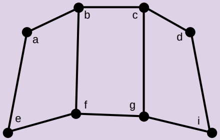
Previous Considerations:

- Rizzo, Montanari '05 - Corrections to BP approximation
- Parisi, Slanina '05 - BP as a saddle-point + corrections

Gauge Transformations

Chertkov, Chernyak '06

Local Gauge, G, Transformations



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The partition function is invariant under any G-gauge!

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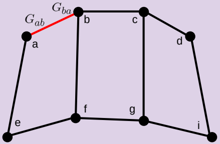

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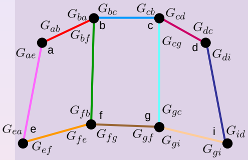

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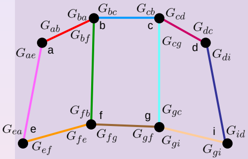

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Gauge Transformations: Binary Representation

$$Z=\sum_{\boldsymbol\sigma}\prod_a f_a(\boldsymbol\sigma_a)=\sum_{\boldsymbol\sigma'}\prod_a f_a(\boldsymbol\sigma_a)\prod_{bc}\frac{1+\sigma_{bc}\sigma_{cb}}{2},\quad \sigma_{bc}\neq\sigma_{cb}$$

The binary trick: $1+\sigma_{bc}\sigma_{cb}=$

$$\frac{\exp(\sigma_{bc}\eta_{bc}+\sigma_{cb}\eta_{cb})}{\cosh(\eta_{bc}+\eta_{cb})}\left(1+(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{bc})(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{cb})\cosh^2(\eta_{bc}+\eta_{cb})\right)$$

$$\tilde f_a(\boldsymbol\sigma_a)=f_a(\boldsymbol\sigma_a)\prod_{b\in a}\exp(\eta_{ab}\sigma_{ab})$$

$$V_{bc}(\sigma_{bc},\sigma_{cb})=1+(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{bc})(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{cb})\cosh^2(\eta_{bc}+\eta_{cb})$$

Graph Coloring

$$Z=\Big(\prod_{bc}2\cosh(\eta_{bc}+\eta_{cb})\Big)^{-1}\sum_{\boldsymbol\sigma'}\prod_a\tilde f_a(\boldsymbol\sigma_a)\prod_{bc}V_{bc}$$

$$Z=\underbrace{Z_0(\eta)}_{\substack{\text{ground state}\\ \boldsymbol\sigma=+1}}+\underbrace{\sum_{\text{all possible colorings of the graph}}Z_c(\eta)}_{\text{excited states}}$$

Gauges and BP

Fixing the gauges ⇒ BP equations!!

Two alternative ways to understand BP-gauges:

▸ BP equations

Color Principe:
no loose ends

$$Z = Z_0(\eta) + \sum_{c=colorings} Z_c(\eta)$$

$$Z_c(\eta) = \prod_{a\in C} \Psi_{a;C}(\eta)$$

Variational Principe:
ground state is η-independent

$$Z \to Z_0(\eta)$$

$$\left.\frac{\partial Z_0}{\partial \eta_{ab}}\right|_{\eta^{(bp)}} = 0$$

Related to Wainwright, Jaakkola, and Willsky '03
Reparametrization Framework

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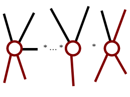

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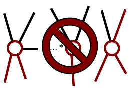

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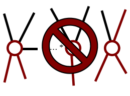

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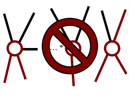

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Reparametrization Framework

Loop Series: Chertkov,Chernyak '06

Exact (!!) expression in terms of BP

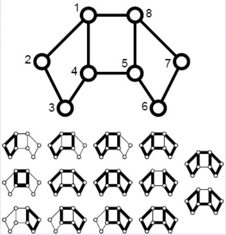
$$Z = \sum_{\sigma_\sigma} \prod_a f_a(\sigma_a) = Z_0 \left(1 + \sum_C r(C)\right)$$

$$r(C) = \frac{\prod_{a\in C} \mu_a}{\prod_{(ab)\in C} (1 - m_{ab}^2)} = \prod_{a\in C} \tilde{\mu}_a$$

$C \in$ Generalized Loops $=$ Loops without loose ends

$$m_{ab} = \int d\sigma_a b_a^{(bp)}(\sigma_a)\sigma_{ab}$$

$$\mu_a = \int d\sigma_a b_a^{(bp)}(\sigma_a) \prod_{b\in a,C} (\sigma_{ab} - m_{ab})$$

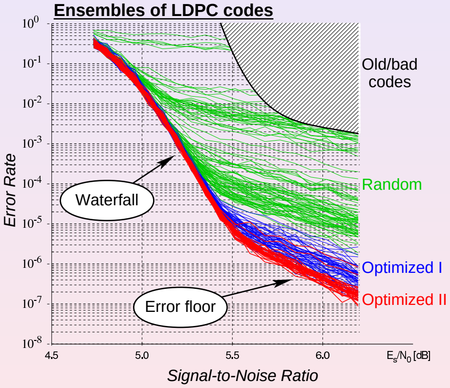


- The Loop Series is finite
- All terms in the series are calculated within BP
- BP is exact on a tree
- BP is a Gauge fixing condition. Other choices of Gauges would lead to different representation.
- Features of the Loop Calculus/Series

Summary (first part)

- BP eqs. solve Gauge fixing conditions
- BP eqs also explains no-loose-end coloring constraints
- BP minimizes gauge dependence in the ground state
- Loop series expresses partition function in terms of a sum of terms, each associated with a generalized loop of the graph
- Each term in the Loop Series depends explicitly on the BP solution

Error-Floor



- BER vs SNR = measure of performance
- Finite size effects
- Waterfall $\leftrightarrow$ Error-floor
- Error-floor typically emerges due to sub-optimality of decoding
- Monte-Carlo is useless at FER $\lesssim 10^{-8}$
- Need an efficient method to analyze error-floor

Pseudo-codewords and Instantons

Error-floor is caused by Pseudo-codewords:

Wiberg '96; Forney et.al'99; Frey et.al '01; Richardson '03; Vontobel, Koetter '04-'06

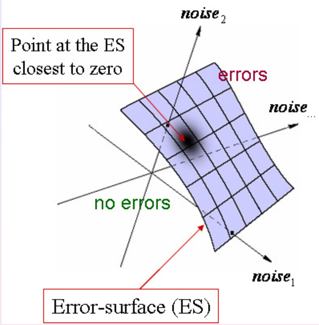
Instanton = optimal conf of the noise

$$BER = \int d(noise)\ WEIGHT(noise)$$

$$BER \sim WEIGHT\left(\begin{array}{c} optimal\ \ conf \\ of\ \ the\ \ noise \end{array}\right)$$

$$\begin{array}{c} optimal\ \ conf \\ of\ \ the\ \ noise \end{array} = \begin{array}{c} Point\ \ at\ \ the\ \ ES \\ closest\ \ to\ \ "0" \end{array}$$

Instantons are decoded to Pseudo-Codewords



Instanton-amoeba

= optimization algorithm

Stepanov, et.al '04,'05

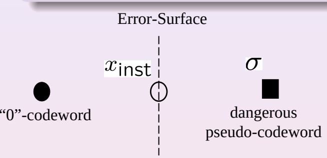
Stepanov, Chertkov '06

Pseudo-Codeword Search Algorithm

LP decoding ($\sigma_i = 0, 1$ AWGN channel)

Minimize, $E = \sum_\alpha \sum_{\sigma_\alpha} b_\alpha(\sigma_\alpha) \sum_{i \in \alpha} \sigma_i (1 - 2x_i)/q_i$, under $0 \leq b_i(\sigma_i), b_\alpha(\sigma_\alpha) \leq 1$

$\forall\ \alpha: \ \sum_{\sigma_\alpha} b_\alpha(\sigma_\alpha) = 1$, & $\forall\ i\ \forall\ \alpha \ni i: \ b_i(\sigma_i) = \sum_{\sigma_\alpha \setminus \sigma_i} b_\alpha(\sigma_\alpha)$



Weighted Median:

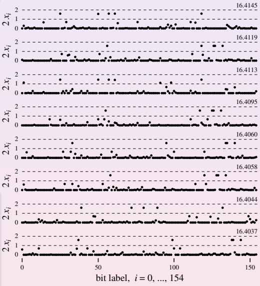
$$\mathbf{x}_{\text{inst}} = \frac{\boldsymbol{\sigma}}{2} \frac{\sum_i \sigma_i}{\sum_i \sigma_i^2}, \quad d = \frac{(\sum_i \sigma_i)^2}{\sum_i \sigma_i^2}$$

$$\text{FER} \sim \exp(-d \cdot s^2/2)$$

Wiberg '96; Forney et.al '01
Vontobel, Koetter '03,'05

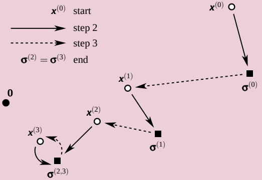
(155, 64, 20), AWGN test:

- Fast Convergence



$\sim$ 200 pseudo-codewords within $16.4037 < d < 20$

PCS Algorithm Chertkov, Stepanov '06



- **Start:** Initiate $\mathbf{x}^{(0)}$.
- **Step 1:** $\mathbf{x}^{(k)}$ is decoded to $\boldsymbol{\sigma}^{(k)}$.
- **Step 2:** Find $\mathbf{y}^{(k)}$ - weighted median between $\boldsymbol{\sigma}^{(k)}$, and "0"
- **Step 3:** If $\mathbf{y}^{(k)} = \mathbf{y}^{(k-1)}$, $k_* = k$ End. Otherwise go to **Step 2** with $\mathbf{x}^{(k+1)} = \mathbf{y}^{(k)} + 0$.

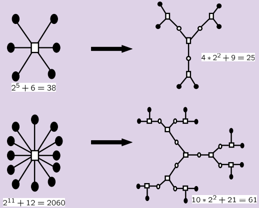
Reducing complexity of LP

Complexity of the bare LP grows exponentially with check degree

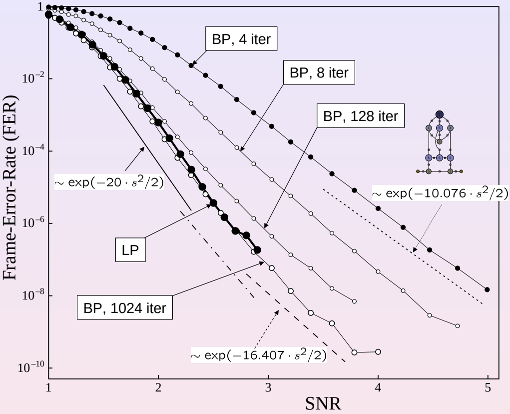
Current solutions:

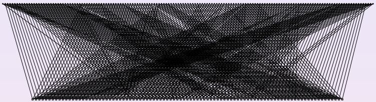
- Adaptive LP (Taghavi, Siegel '06)
- BP-style relaxation of LP (Vontobel, Koetter '06)

Dendro-trick = Graph Modification (our solution) Chertkov,Stepanov'07



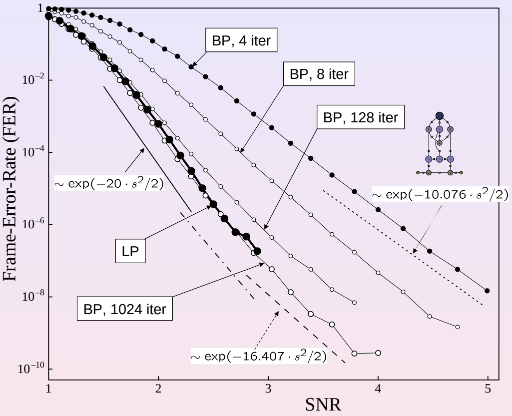
- MAP solutions are identical
- Set of Pseudo-codewords are identical
- Instanton spectra are very alike, $\approx$

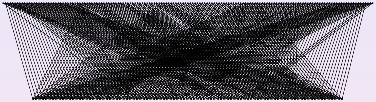




Instanton-amoeba:
Stepanov, et.al '04,'05,'06
LP-search:
Chertkov, Stepanov '06,'07

What does Loop Calculus show for dangerous Pseudo-codewords?





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Loop Calculus & Pseudo-Codeword Analysis

Chertkov, Chernyak '06

Single loop truncation

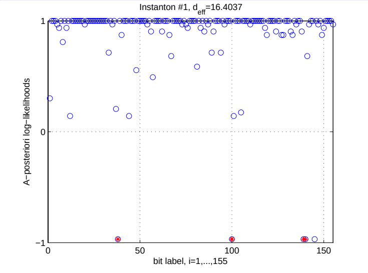
$$Z = Z_0(1+\sum_C r_C) \approx Z_0(1+r(\Gamma))$$

Synthesis of Pseudo-Codeword Search Algorithm (Chertkov, Stepanov '06) & Loop Calculus

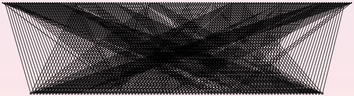
- Consider pseudo-codewords one after other
- For an individual pseudo-codeword/instanton identify a critical loop, Γ, giving major contribution to the loop series.
- Hint: look for single connected loops and use local "triad" contributions as a tester: $r(\Gamma) = \prod_{\alpha\in\Gamma} \tilde{\mu}_\alpha^{(bp)}$

Proof-of-Concept test [(155, 64, 20) code over AWGN]

- $\forall$ pseudo-codewords with $16.4037 < d < 20$ ($\sim$ 200 found) there always exists a simple single-connected critical loop(s) with $r(\Gamma) \sim 1$.
- Pseudo-codewords with the lowest d show $r(\Gamma) = 1$
- Invariant with respect to other choices of the original codeword



Bigger Set



Extended Variational Principle & Loop-Corrected BP

Bare BP Variational Principle: $\left.\frac{\partial Z_0}{\partial \eta_{ab}}\right|_{\eta^{(bp)}} = 0$

New choice of Gauges guided by the knowledge of the critical loop Γ

$$\left.\frac{\partial \exp(-\mathcal{F})}{\partial \eta_{ab}}\right|_{\eta_{\text{eff}}} = 0, \quad \mathcal{F} \equiv -\ln(Z_0 + Z_\Gamma)$$

BP-equations are modified along the critical loop Γ

$$\left.\frac{\sum_{\sigma_a}(\tanh(\eta_{ab}+\eta_{ba})-\sigma_{ab})P_a(\sigma_a)}{\sum_{\sigma_a} P_a(\sigma_a)}\right|_{\eta_{\text{eff}}} = \text{explicitly known contribution}\Big|_{\eta_{\text{eff}}} \neq 0 \quad [\text{along } \Gamma]$$

Loop-Corrected BP Algorithm

- **1.** Run bare BP algorithm. Terminate if BP succeeds (i.e. a valid code word is found).
- **2.** If BP fails find the most relevant loop Γ that corresponds to the maximal $|r_\Gamma|$. Triad search is helping.
- **3.** Solve the modified-BP equations for the given Γ. Terminate if the improved-BP succeeds.
- **4.** Return to **Step 2** with an improved Γ-loop selection.

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- **3.** Modify the log-likelihoods along the loop Γ introducing a shift towards zero, i.e. introduce a complete or partial erasure of the log-likelihoods at the bits. Run LP with modified log-likelihoods. Terminate if the modified LP succeeds.
- **4.** Return to **Step 2** with an improved selection principle for the critical loop.

$(155, 64, 20)$ Test

- **IT WORKS!**
  All troublemakers ($\sim$ 200 of them) previously found by LP-based Pseudo-Codeword-Search Algorithm method were successfully corrected by the LP-erasure algorithm.
- Method is invariant with respect the choice of the codeword (used to generate pseudo-codewords).

General Conjecture:

- Loop-erasure algorithm is capable of reducing the error-floor
- Local adjustment of the algorithm, anywhere along the critical loop, in the spirit of the Facet Guessing (Dimakis, Wainwright '06), may be sufficient $\Rightarrow$

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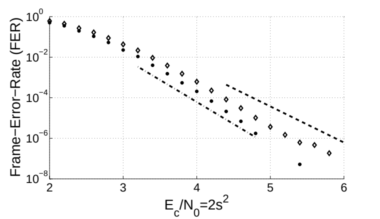
Breaking the critical loop locally

Chertkov '07

- Exhaustive Bit Guessing (simplified version of [Dimakis, Wainwright '06]) corrects all the $\sim$ 200 dangerous pseudo-codewords !!
- Set of "successful" bits correlates strongly with the set of bits forming the critical loop

$[155, 64, 20]$ test of LGG



Loop Guided Guessing (LGG)

- **1.** Run the LP algorithm. Terminate if LP succeeds.
- **2.** If LP fails, find the critical loop, Γ.
- **3.** Pick any bit along the critical loop and "fix the bit" running two two corrected LP schemes. Terminate if any of LPs succeeds.
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- Draper, Yedidia, Wang ISIT'07: ML decoder = fixing enough of constraints
- Our statement = fixing one constraint can be enough

- Complexity of LGG is the same as of LP
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- Error Floor is Reduced !!

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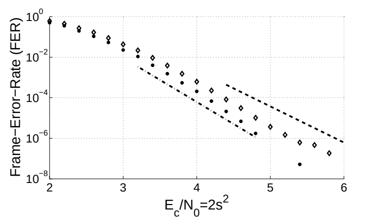

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Summary (second part)

- Error floor is typically due to rare but dangerous pseudo-codewords.
- Instanton-amoeba and, especially, Pseudo-Codeword Search Algorithm offer efficient methods of the error-floor exploration.
- Loop Series for the factor functions of a dangerous pseudo-codeword can be accurately approximated by a sum of the leading BP term and a critical loop term. [Experimentally verified conjecture.]
- Loop Guided Guessing is an efficient algorithm (of the same complexity as LP) seriously outperforming the bare LP and overall reducing the error-floor. LGG "brakes" the critical loop locally at any bit of the critical loop.

Results

- BP is better then just a heuristic in the loopy case ... BP is the special Gauge condition eliminating all contributions but loops.
- ML allows explicit Loop Series expression in terms of a solution of the BP equations.
- Truncation and/or Re-summation of the Loop Series provide hierarchy of systematically improvable approximations/algorithms. Standard BP/LP is a first member in the hierarchy.
- Finding a critical loop, or a small number of critical loops, is algorithmically sufficient for reducing effect of the decoding sub-optimality in the error-floor domain.

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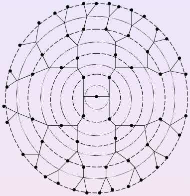
- Loop calculus based analysis of graph ensembles, e.g. understanding and improving the cavity method [Mézard, Parisi '85-'03]
- Extending the list of Loop Calculus Applications, e.g. SAT and cryptography
- Non-BP gauges, e.g. for stat problems on regular and irregular lattices. Loop calculus for "near easy" problems on dense graphs.
- Relation to graph ζ-functions [Koetter, Li, Vontobel, Walker '05]
- ...

Thank You !!

BP is Exact on a Tree (LDPC)



$$Z(\mathbf{h}) = \sum_{\boldsymbol{\sigma}} \prod_{\alpha=1}^{M} \delta\left(\prod_{i\in\alpha} \sigma_i, 1\right) \exp\left(\sum_{i=1}^{N} h_i\sigma_i\right)$$

h_i is a log-likelihood at a bit (outcome of the channel)

$$Z^{\pm}_{j\alpha}(\mathbf{h}^{>}) \equiv \sum_{\boldsymbol{\sigma}^{>}}^{\sigma_j=\pm 1} \prod_{\beta>} \delta\left(\prod_{i\in\beta} \sigma_i, 1\right) \exp\left(\sum_{i>} h_i\sigma_i\right)$$

$$Z^{\pm}_{j\alpha} = \exp(\pm h_j) \prod_{\beta\neq\alpha}^{j\in\beta} \frac{1}{2}\left(\prod_{i\neq j}^{i\in\beta}(Z^+_{i\beta} + Z^-_{i\beta}) \pm \prod_{i\neq j}^{i\in\beta}(Z^+_{i\beta} - Z^-_{i\beta})\right)$$

$$\eta_{j\alpha} \equiv \frac{1}{2}\ln\left(\frac{Z^+_{j\alpha}}{Z^-_{j\alpha}}\right), \quad \eta_{j\alpha} = h_j + \sum_{\beta\neq\alpha}^{j\in\beta} \tanh^{-1}\left(\prod_{i\neq j}^{i\in\beta} \tanh \eta_{i\beta}\right)$$

◂ Bethe Free Energy

Gauges and BP equations

Partition function in the colored representation

$$Z = (\prod_{bc} 2\cosh(\eta_{bc}+\eta_{cb}))^{-1} \sum_{\boldsymbol{\sigma}'} \prod_a \tilde{f}_a \prod_{bc} V_{bc}, \quad \tilde{f}_a(\boldsymbol{\sigma}_a;\boldsymbol{\eta}_a) = f_a(\boldsymbol{\sigma}_a) \prod_{b\in a} \exp(\eta_{ab}\sigma_{ab})$$

$$V_{bc}(\sigma_{bc},\sigma_{cb}) = 1 + (\tanh(\eta_{bc}+\eta_{cb}) - \sigma_{bc})(\tanh(\eta_{bc}+\eta_{cb}) - \sigma_{cb})\cosh^2(\eta_{bc}+\eta_{cb})$$

Fixing the gauges $\Rightarrow$ BP equations!!

$$\sum_{\boldsymbol{\sigma}_a} \left(\tanh(\eta_{ab}^{(bp)} + \eta_{ba}^{(bp)}) - \sigma_{ab}\right) \tilde{f}_a(\boldsymbol{\sigma}_a;\boldsymbol{\eta}_a) = 0 \quad \Rightarrow \quad \underbrace{\eta_{\alpha j}^{bp} = h_j + \sum_{\beta\neq\alpha}^{j\in\beta} \tanh^{-1}(\prod_{i\neq j}^{i\in\beta} \tanh \eta_{\beta i}^{bp})}_{\text{LDPC case}}$$

◂ Gauges and BP

Features of the Loop Calculus

$$Z = Z_0(1 + \sum_C r_C),\ r_C = \prod_{a\in C} \tilde{\mu}_a$$

- Bethe Free Energy is related to the "ground state" term in the partition function: $F(b^*(\eta)) = -\ln Z_0(\eta)$, where
$$b^*_a(\boldsymbol{\sigma}_a) = \frac{f_a(\boldsymbol{\sigma}_a)\exp(\sum_{b\in a}\eta_{ab}\sigma_{ab})}{\sum_{\boldsymbol{\sigma}_a} f_a(\boldsymbol{\sigma}_a)\exp(\sum_{b\in a}\eta_{ab}\sigma_{ab})}, \quad b^*_{ab}(\sigma_{ab}) = \frac{\exp((\eta_{ab}+\eta_{ba})\sigma_{ab})}{2\cosh(\eta_{ab}+\eta_{ba})}$$
- Extrema of $F(b)$ are related to extrema of $Z_0(\eta)$
- Loop series can be built around any extremum (minimum, maximum or saddle-point) of the Bethe Free energy
- $-1 \leq r_C, \tilde{\mu}_a \leq 1$
- Linear Programming limit of the Loop Calculus is well defined

Loop Series

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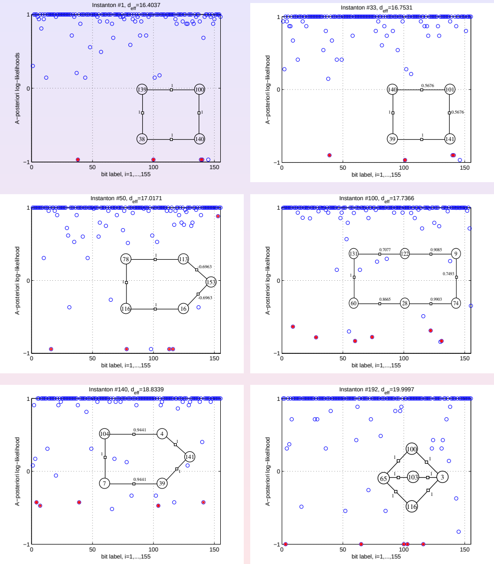
Instanton #1, d_eff=16.4037
Instanton #33, d_eff=16.7531
Instanton #50, d_eff=17.0171
Instanton #100, d_eff=17.7366
Instanton #140, d_eff=18.8339
Instanton #192, d_eff=19.9997
A−posteriori log−likelihood
bit label, i=1,...,155

Back

Relation to the Bethe Free Energy approach

in the spirit of Yedidia, Freeman, Weiss '01

Minimize: $\Phi_B = \sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln\left(\frac{b_a(\boldsymbol{\sigma}_a)}{f_a(\boldsymbol{\sigma}_a)}\right) - \sum_{(ab)} \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) \ln b_{ab}(\sigma_{ab})$

under the conditions: $\forall a$ & $\forall c \in a$

$$0 \le b_a(\boldsymbol{\sigma}_a), b_{ac}(\sigma_{ac}) \le 1$$
$$\sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) = 1$$
$$b_{ac}(\sigma_{ac}) = \sum_{\boldsymbol{\sigma}_a \backslash \sigma_{ac}} b_a(\boldsymbol{\sigma}_a)$$

- $\mathcal{L}_B = \Phi_B + \sum_{(ab)} [\sum_{\sigma_{ab}} \ln(\epsilon_{ab}(\sigma_{ab}))(b_{ab}(\sigma_{ab}) - \sum_{\boldsymbol{\sigma}_a \backslash \sigma_{ab}} b_a(\boldsymbol{\sigma}_a)) + \sum_{\sigma_{ba}} \ln(\epsilon_{ba}(\sigma_{ba}))(b_{ab}(\sigma_{ba}) - \sum_{\boldsymbol{\sigma}_b \backslash \sigma_{ba}} b_b(\boldsymbol{\sigma}_b))]$
- Finding extremum of the Bethe Lagrangian with respect to beliefs, b_{ab} and b_a and expressing the result in terms of ϵ: $\mathcal{L}_B(b,\epsilon) \Rightarrow \mathcal{F}_B(\epsilon)$
- $\mathcal{F}_B(\epsilon)\big|_{\{\forall (a,b):\ \sum_{\sigma_{ab}} \epsilon_{ab}(\sigma_{ab})\epsilon_{ba}(\sigma_{ab})=1\}} = \mathcal{F}_0(\epsilon) = -\ln(Z(\epsilon))$

◂ Variational approach

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