Understanding Search Trees via Statistical Physics

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Sorting and Search

The Goal: Store data efficiently so that the search time is minimum

Ex: A random sequence of $N = 10$ integers: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$

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Linear Sorting: Store the data sequentially onto a linear table

$$\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$$

Search for 7: Search proceeds sequentially by comparison

$$t_{\text{search}} = 10 \sim O(N) \rightarrow \text{BAD}$$

Tree Sorting: of $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$

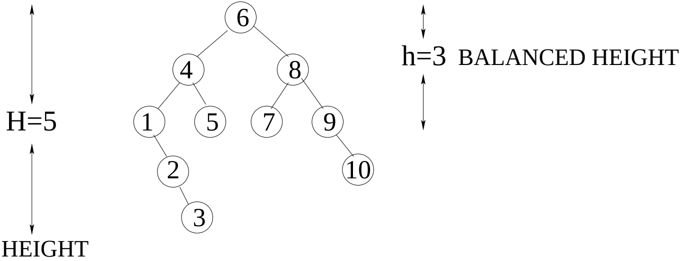

Figure: Binary Search Tree with $N = 10$ Elements.

$t_{\rm search} = {\rm Depth} = D$. Roughly $2^D \sim N$ implying: $t_{\rm search} \sim O(\log N) \rightarrow$ BETTER

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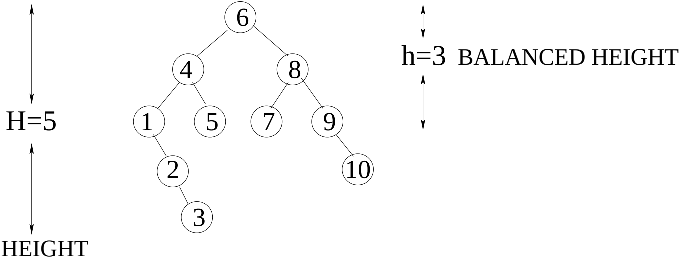

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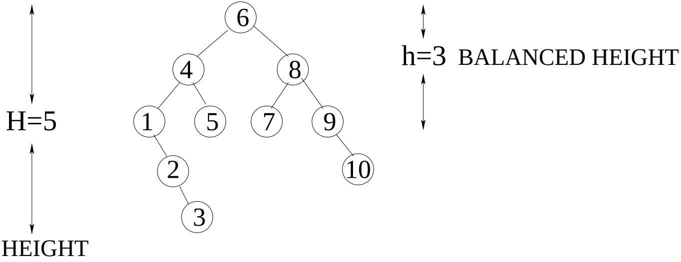

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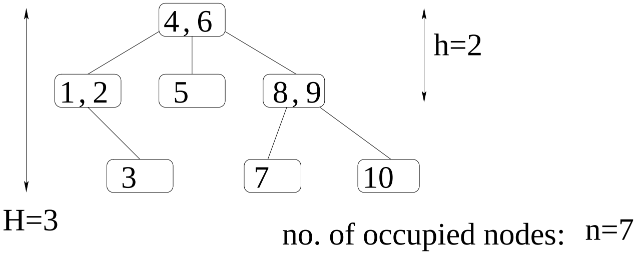
• BALANCED HEIGHT $h = 3$: Depth upto which the tree is balanced

Generalization to m-ary Search Trees

$m = 2 \rightarrow$ Binary Tree

Random Sequence: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$

Each node can contain atmost $(m-1)$ elements.



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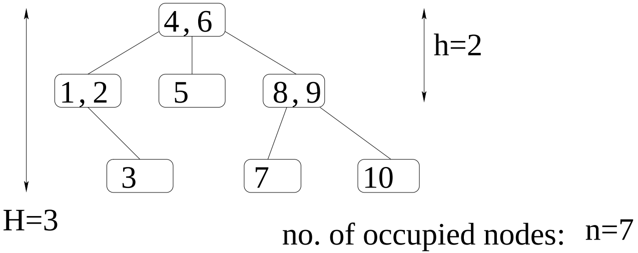

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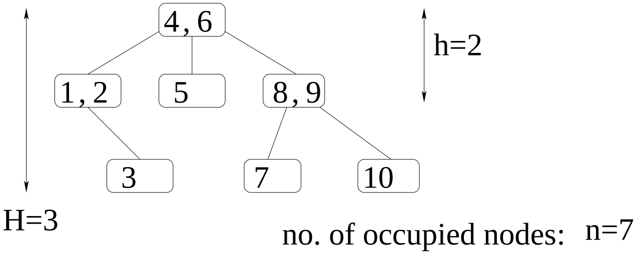

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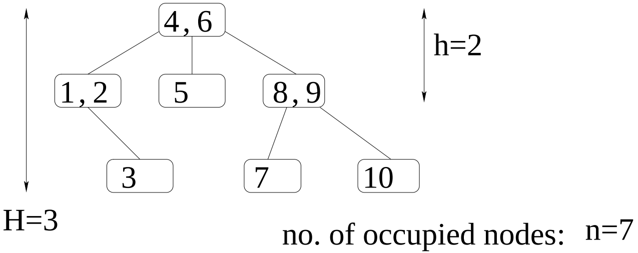

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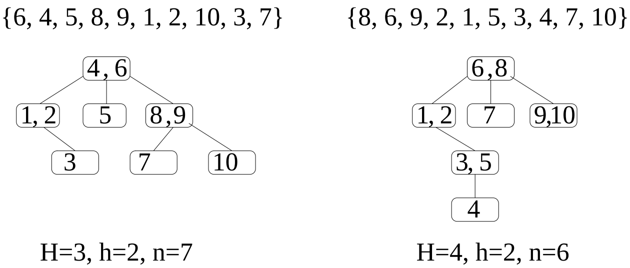
No. of NON-EMPTY nodes: $n = 7 \rightarrow$ No. of nodes required to store the data

Random m-ary Search Tree Model: $RmST$

$N = 10$ data elements: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
Each permutation $\rightarrow$ an m-ary tree.



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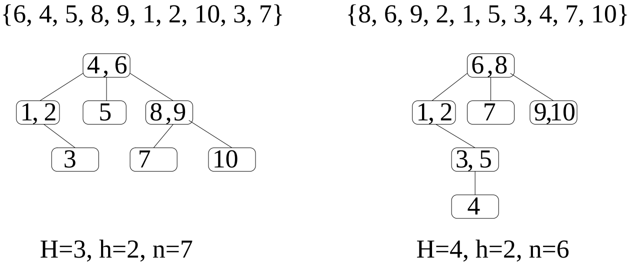

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Q: Statistics of HEIGHT H_N, BALANCED HEIGHT h_N and the no. of NON-EMPTY NODES n_N for RANDOM data of size N?

Asymptotic Results for RmST: for large data size N

(1) Height H_N:

- $\langle H_N \rangle \approx a_m \log(N) + b_m \; \log(\log(N))$ (??) $+ \ldots$

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Binary Tree ($m = 2$): $a_2 = 4.31107\ldots$ and $c_2 = 0.3733\ldots$ (Devroye, 87).

The correction terms $\rightarrow$ conjectured by Hattori and Ochiai (simulations, 2001).

Other results by Robson (2001-), Reed (2001-), Drmota (2001-), Flajolet, ...

Asymptotic Results for RmST: for large data size N...continued

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$$\langle n_N \rangle \approx \alpha_m N + \ldots .$$

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A striking PHASE TRANSITION occurs for the Variance: $\nu_N = \langle \left(n_N - \langle n_N \rangle\right)^2 \rangle$.

$$\begin{aligned} \nu_N &\sim N && \text{for } m \leq 26 \\ &\sim N^{2\theta(m)} && \text{for } m > 26 \end{aligned}$$

(Chern & Hwang, 2001).

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Q: Why 26? What is the mechanism of this Phase Transition and how generic is it? Can one calculate $\theta(m)$ exactly ?

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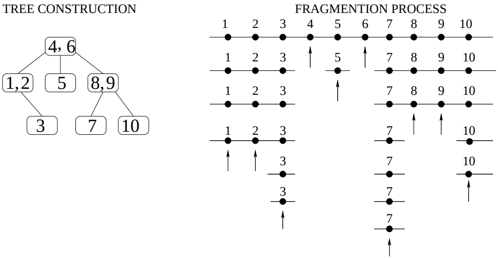
- Various other generalizations: Vector Data

The Mapping to a Fragmentation Process

Construction of the Tree → Dynamical Fragmention Process: Split an interval into m pieces with the break points chosen randomly. An interval can split iff it contains atleast one point.

Ex: Consider the data: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$ on a $(m = 3)$-ary tree



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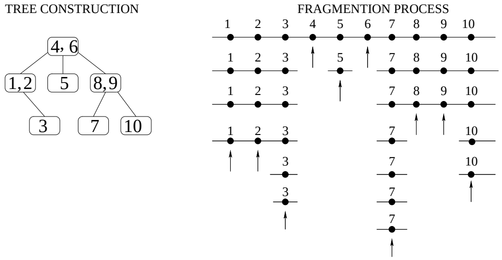
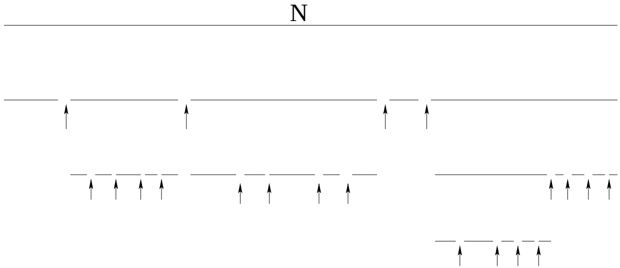


NOTE:

No. of NONEMPTY nodes n=7= No. of SPLITTING EVENTS

Fragmentation Process: Continuum Limit



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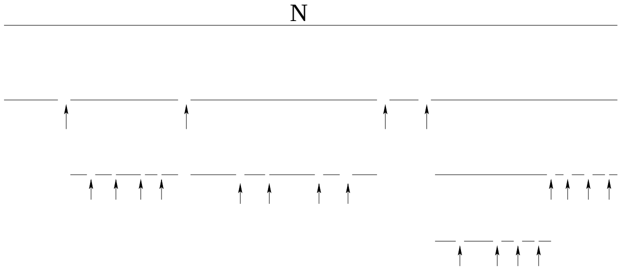

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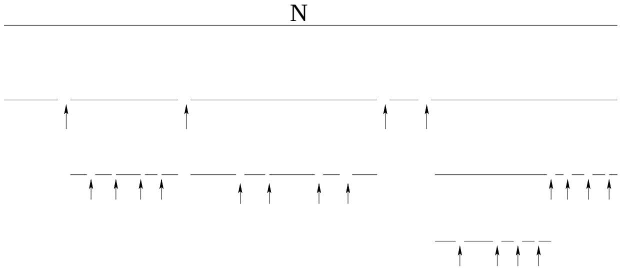

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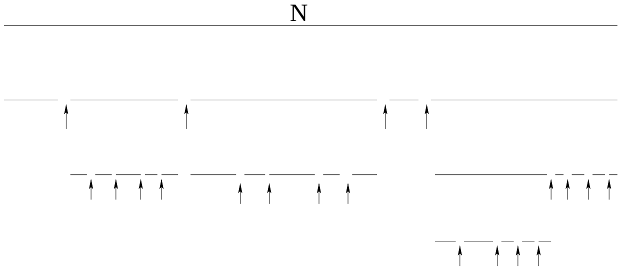

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4. Repeat the process till all pieces have length < 1 and then STOP.

DICTIONARY Between the Search Tree and the Fragmentation Process:

m-ary SEARCH TREE $\equiv$ FRAGMENTATION PROCESS

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Height H_N:

- Prob[$H_N < n$]= Prob[$l_1 < 1$, $l_2 < 1$, ... after n steps]

Balanced Height h_N:

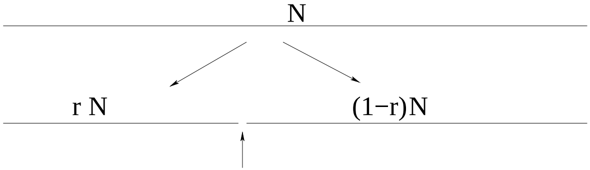
- Prob[$h_N > n$]=Prob[$l_1 > 1$, $l_2 > 1$, ... after n steps]

Number of Nonempty Nodes n_N ($m > 2$):

- Prob[$n_N = n$]= Prob[there are n SPILLITING EVENTS till the end of the Fragmentation process].

Analysis of HEIGHT H_N

P(n,N)= Prob[$H_N < n$]= Prob[$l_1 < 1,\ l_2 < 1,\ \ldots$ after n steps starting with initial length N]



Recursion: $P(n,N) = \int_0^1 P\,(n-1, rN)\, P\,(n-1,(1-r)N)\, dr$ starting with $P(n,1) = \theta(n-1)$.

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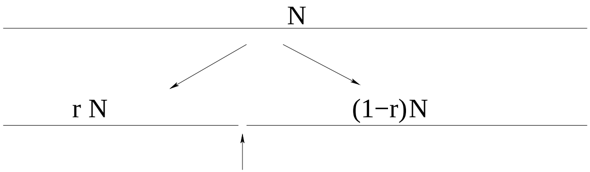

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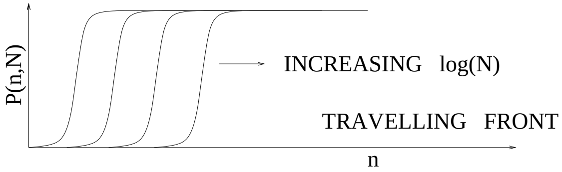

Travelling Front in Fisher Equation

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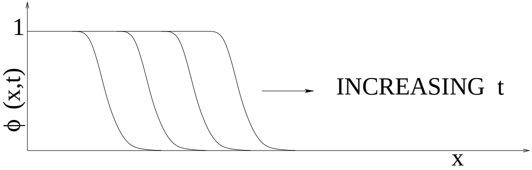

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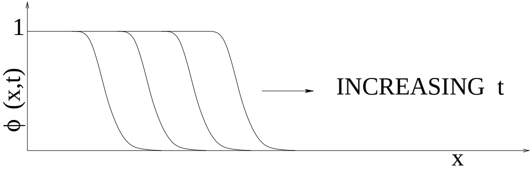

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$$x_f(t) \sim \mathrm{v}\; t + \ldots.$$

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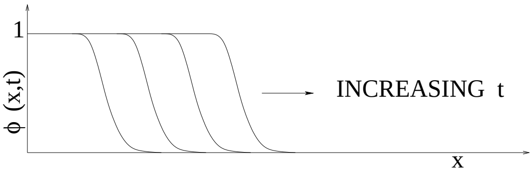

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Q: How to determine the Front Velocity v?

Kolmogorov's Velocity Selection Principle:

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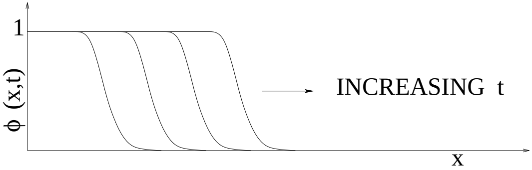

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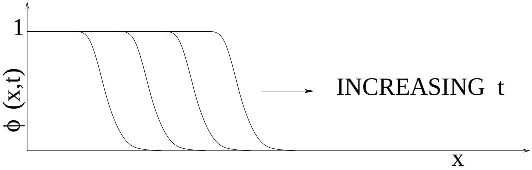

Linearize near the tail $\rightarrow$ $\phi(x,t) \sim \exp[-\lambda(x - vt)]$

DISPERSION RELATION: $v(\lambda) = \lambda + \frac{1}{\lambda}$ $\rightarrow$ minimum at $\lambda^* = 1$.

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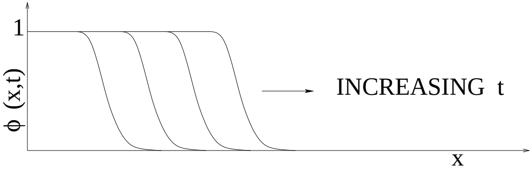

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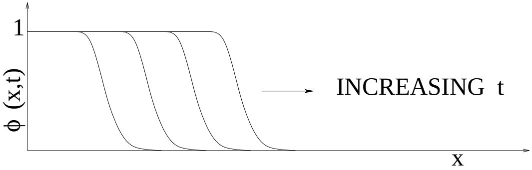

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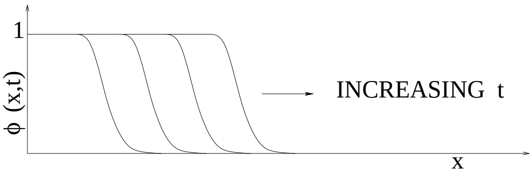

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(Bramson, Brunet & Derrida, van Saarloos, ...)

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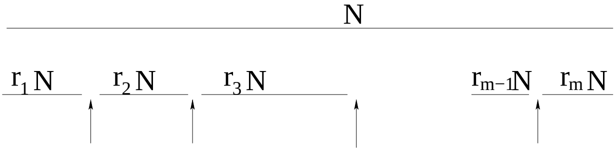
No of Non-Empty Nodes:



$$r_1 + r_2 + r_3 + \ldots\ldots + r_m = 1$$

No. of Non-empty nodes $n(N)$ in the tree $\equiv$ Total no. of Splitting Events in the fragmentation process till the end, starting with the initial length N

Recursion:

$$n(N) \equiv n(r_1 N) + n(r_2 N) + n(r_3 N) + \cdots + n(r_m N) + 1; \qquad \sum_i^n r_i = 1$$

The marginal distribution of any fragment: $\eta_m(r) = (m-1)(1-r)^{m-2}$

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Variance: $\nu(N) = \langle (n(N) - \mu(N))^2 \rangle$ satisfies another integral equation:

$$\nu(N) = m \int_{1/N}^{1} \nu(rN)\eta(r)dr + \langle (S - \langle S \rangle)^2 \rangle$$

where the Source Function $S = \sum_{i=1}^{m} \mu(r_i N)$.

These integral equations can be solved analytically:

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For large N:

Mean: $\mu(N) \sim \alpha_0 + \alpha_1 N + \sum_{k=2}^{\infty} \alpha_K N^{\lambda_k}$

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For $m > m_c = 26.0461...$, $\theta(m) = \lambda_2(m)$. (Dean and S.M., 2002).

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Vector Data: Quadtree Structure (Finkel and Bentley, Flajolet and Richmond)

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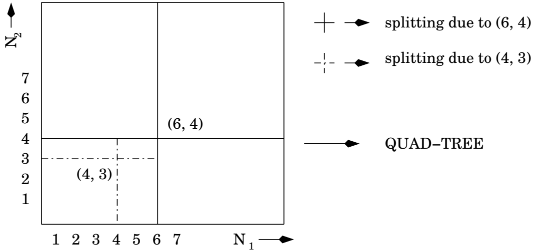

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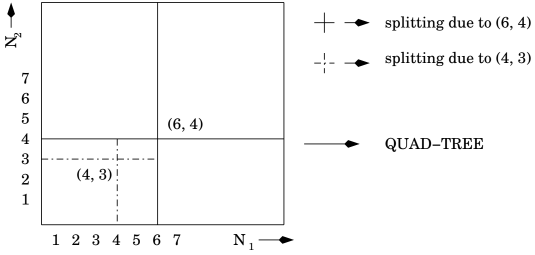

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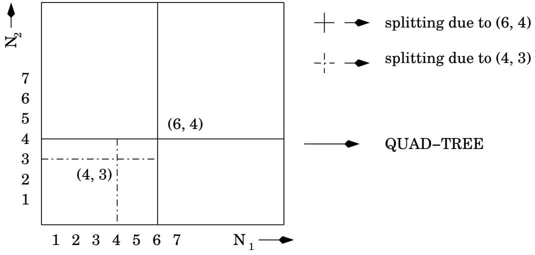

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Is there a PHASE TRANSITION in the variance of n_N?

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$\theta(D) = 2\cos(\frac{2\pi}{D}) - 1 \rightarrow$ increases continuously with D for $D > D_c$

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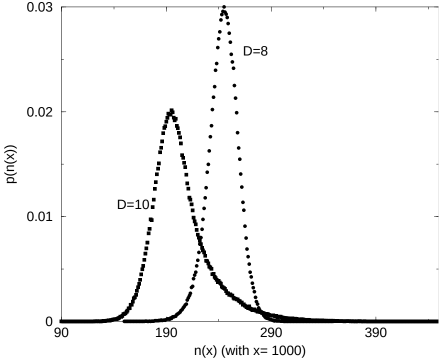

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Perspectives: Lots of beautiful open problems in Sorting and Search that may be possible to handle by using statistical physics techniques.

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