Mariehamn, Åland July 16th, 2007

**Statistical mechanics of distributed information systems**

# Evolutionary games in finite populations


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# Outline

- Introduction to evolutionary games
- Stochastic game dynamics in finite populations
- From finite to infinite populations
- The evolution of cooperation
- The emergence of punishment

# Beauty contest

- *N* players choose a number between *1* and *100*
- Calculate the average *a*
- The number closest to *a/2* wins

Example

Player

1 2 3 4 5 6


$N = 6$
$a \approx 20$
$a/2 \approx 10$

# What is the optimal choice?

- **Classical game theory**
  - Strategic interactions of humans
  - Assumption of rationality

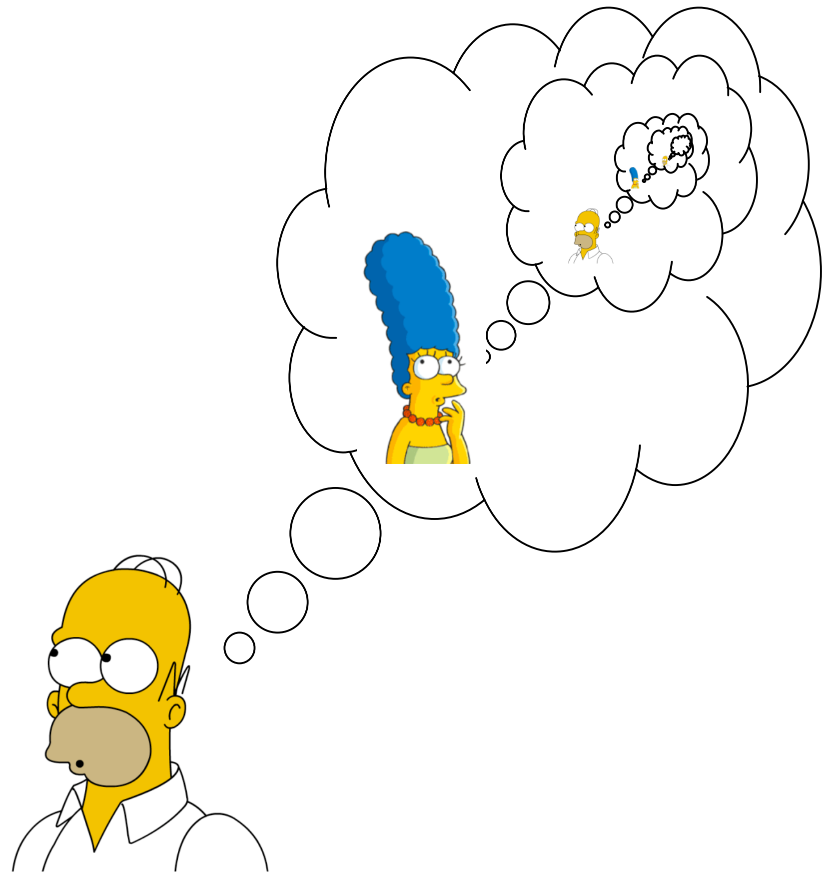

John von Neumann


Oskar Morgenstern

- Random numbers => a = 50 => 25
- Everybody knows => a = 25 => 12
- Everybody knows => a = 12 => 6
- ... => 1

Von Neumann & Morgenstern, The theory of games and economic behavior (1953)

# Rationality



# Coordination games

| |  |  |
|---|---|---|
|  | **1** | **-100** |
|  | **-100** | **1** |

# Matrix games




$a_{AA}$ $a_{AB}$

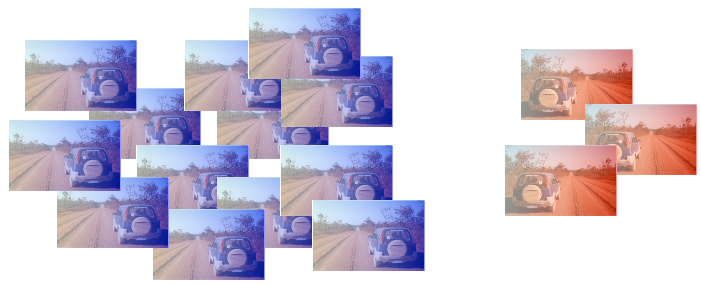

$a_{BA}$ $a_{BB}$

Left player gets

# Evolutionary game theory

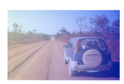
- Populations
- Reproductive fitness = payoff
- Evolutionary dynamics

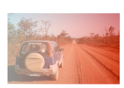
John Maynard Smith



Left side drivers get extinct

~~Focal driver makes decision~~

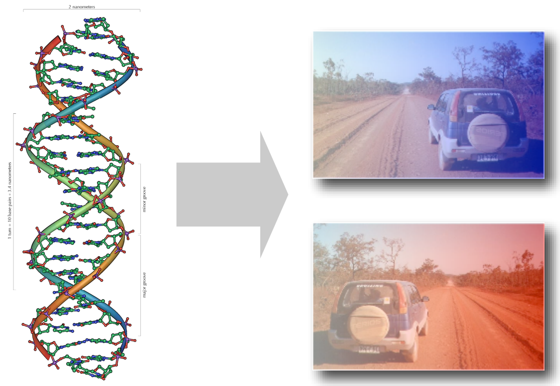
 **1** **-100**

 **-100** **1**

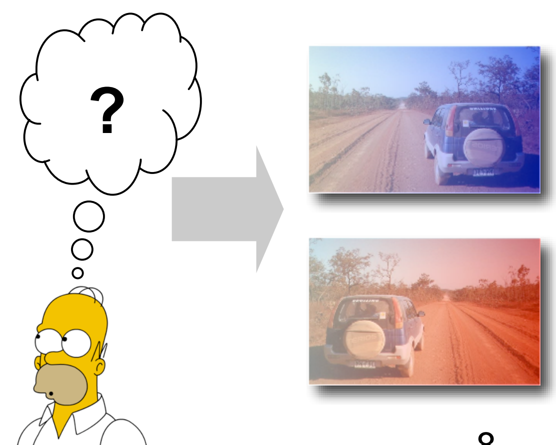
Maynard Smith, Evolution and the theory of games (1982)

# Interpretations of evolutionary game dynamics

- Genetic
  - Strategies are determined genetically
  - More successful individuals produce more offspring
  - Less successful strategies go extinct



- Cultural
  - Strategies are choices of individuals
  - More successful strategies are imitated more often



# Replicator dynamics

$$\dot{x}_i = x_i \left( \pi_i - \langle \pi \rangle \right)$$

Abundance of strategy *i*

Payoff of strategy *i*

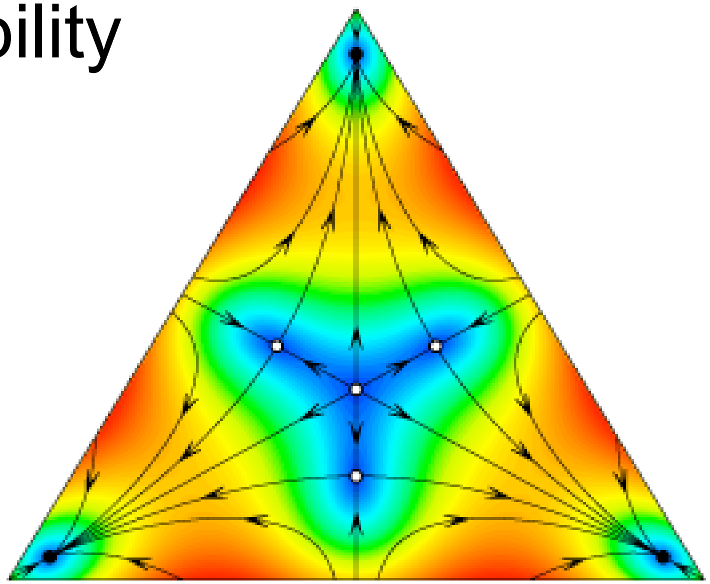
Average payoff in the population

Taylor & Jonker, Math. Biosci. (1978); Weibull, Evolutionary Game Theory (1995); Hofbauer & Sigmund, Evolutionary Games and Population Dynamics (1998)

# Replicator dynamics

$$\dot{x}_i = x_i \left(\pi_i - \langle \pi \rangle\right)$$

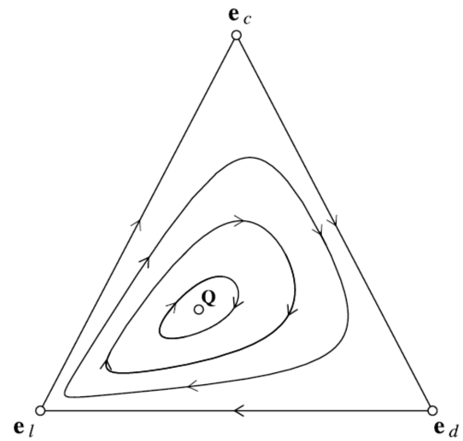
- Fixed points & their stability



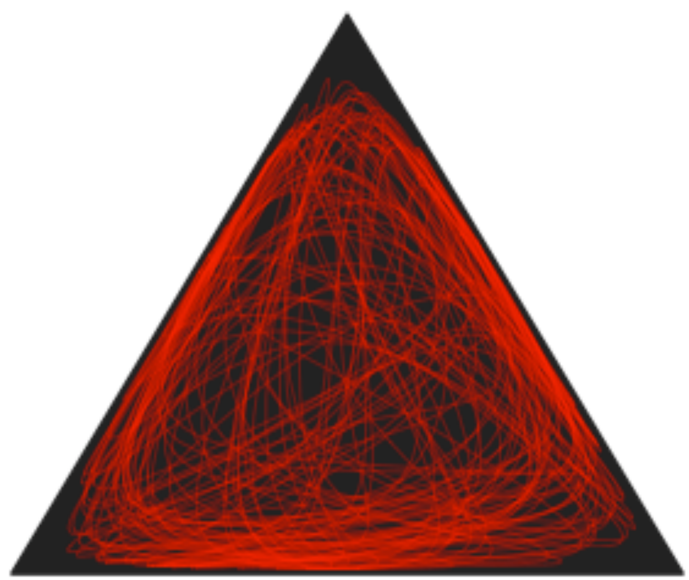
# Replicator dynamics

$$\dot{x}_i = x_i \left(\pi_i - \langle \pi \rangle\right)$$

- Constants of motion




- Deterministic Chaos



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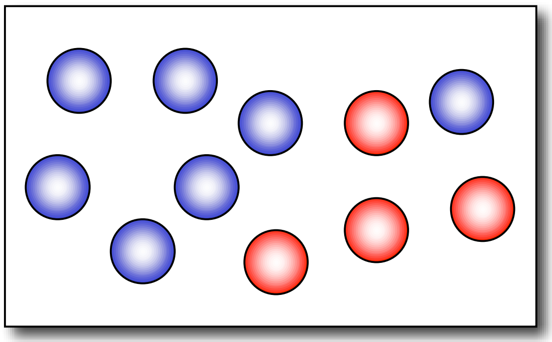
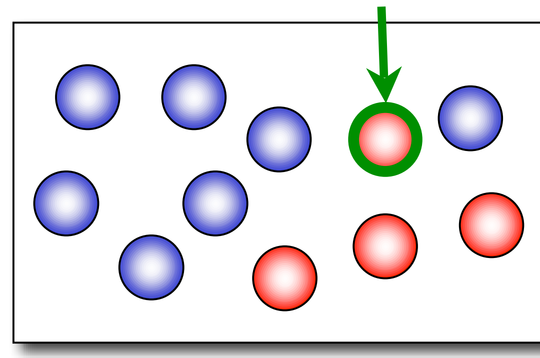
# Pairwise comparison

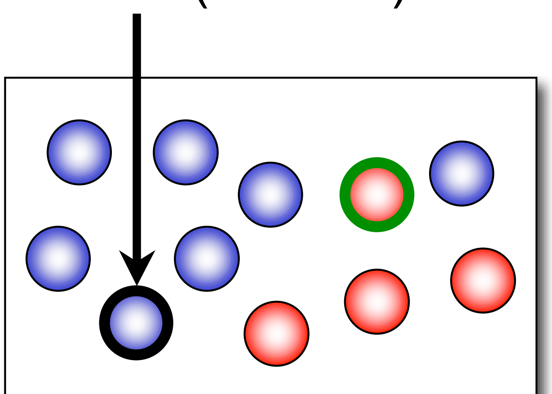
Initial population



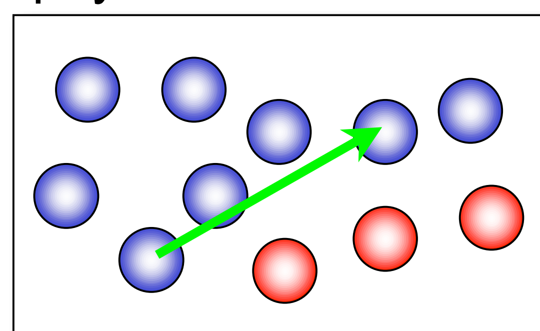
Select focal individual
(at random)



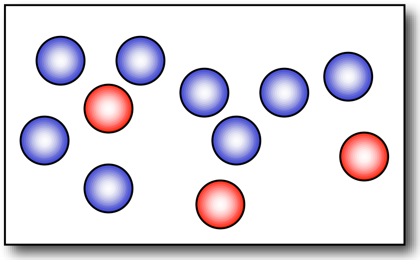
Select (random) role model



Adopt strategy according to payoff difference



# The "Fermi" process



$$T_{i,i+1} = \frac{N-i}{N}\,\frac{i}{N}\,\underbrace{\frac{1}{1+e^{-\beta(\pi_A-\pi_B)}}}_{\text{Probability to switch to } \mathbf{A}}$$

A **B** player updates its strategy

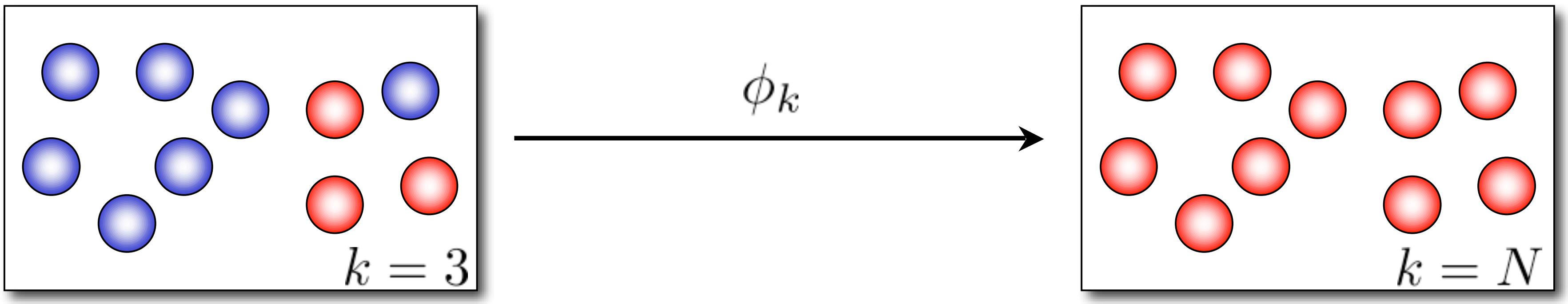
The role model is **A**

Inverse temperature of selection

$$T_{i,i-1} = \frac{i}{N}\,\frac{N-i}{N}\,\frac{1}{1+e^{+\beta(\pi_A-\pi_B)}}$$

Blume, Games and Econ. Behav. 5:387 (1993); Szabo & Töke, PRE 58:69 (1998); Traulsen, Pacheco & Nowak, JTB 246:522 (2007).

# Fixation probabilities




$$\phi_k = T_{k,k-1}\phi_{k-1} + T_{k,k+1}\phi_{k+1} + T_{k,k}\phi_k$$

$$\phi_k = \frac{\sum_{i=0}^{k-1} \prod_{j=1}^{i} T_{j,j-1}/T_{j,j+1}}{\sum_{i=0}^{N-1} \prod_{j=1}^{i} T_{j,j-1}/T_{j,j+1}}$$

Nowak, Sasaki, Taylor & Fudenberg, Nature 428:646 (2004)

# Ratio of transition probabilities

$$\phi_k = \frac{\sum_{i=0}^{k-1} \prod_{j=1}^{i} \boxed{T_{j,j-1}/T_{j,j+1}}}{\sum_{i=0}^{N-1} \prod_{j=1}^{i} \boxed{T_{j,j-1}/T_{j,j+1}}}$$

$$\begin{aligned}
\boxed{\frac{T_{i,i-1}}{T_{i,i+1}}} &= \frac{\frac{i}{N}\frac{N-i}{N}\frac{1}{1+e^{+\beta(\pi_A-\pi_B)}}}{\frac{N-i}{N}\frac{i}{N}\frac{1}{1+e^{-\beta(\pi_A-\pi_B)}}} \\
&= \frac{1+e^{-\beta(\pi_A-\pi_B)}}{1+e^{+\beta(\pi_A-\pi_B)}} \\
&= e^{-\beta(\pi_A-\pi_B)}
\end{aligned}$$

# Fixation probability for the Fermi process

$$
\begin{aligned}
\phi_k &= \frac{\sum_{i=0}^{k-1} \prod_{j=1}^{i} \frac{T_{j,j-1}}{T_{j,j+1}}}{\sum_{i=0}^{N-1} \prod_{j=1}^{i} \frac{T_{j,j-1}}{T_{j,j+1}}} \\
&= \frac{\sum_{i=0}^{k-1} \prod_{j=1}^{i} e^{-\beta(\pi_A - \pi_B)}}{\sum_{i=0}^{N-1} \prod_{j=1}^{i} e^{-\beta(\pi_A - \pi_B)}} \\
&= \frac{\sum_{i=0}^{k-1} e^{-\beta \sum_{j=1}^{i} (\pi_A - \pi_B)}}{\sum_{i=0}^{N-1} e^{-\beta \sum_{j=1}^{i} (\pi_A - \pi_B)}} \\
&\approx \frac{\int_{i=-0.5}^{k-0.5} e^{-\beta \sum_{j=1}^{i} (\pi_A - \pi_B)} di}{\int_{i=-0.5}^{N-0.5} e^{-\beta \sum_{j=1}^{i} (\pi_A - \pi_B)} di}
\end{aligned}
$$

General equation

Fermi process

Large populations

# Calculation of payoffs

$$\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

$$\boxed{\begin{aligned} \pi_A &= a\frac{i-1}{N-1} + b\frac{N-i}{N-1} \\ \pi_B &= c\frac{i}{N-1} + d\frac{N-i-1}{N-1} \end{aligned}}$$

$$\pi_A - \pi_B = \underbrace{\frac{(a-b-c+d)}{N-1}}_{u} i + \underbrace{\frac{-a+bN-dN+d}{N-1}}_{v}$$

$u = 0$ "equal gains from switching"

Nowak & Sigmund, Acta Appl. Math 20:247 (1990)

# Fixation probabilities for arbitrary N and β

$$\pi_A - \pi_B = u \cdot i + v$$

Frequency dependence

**1. Frequency independent, *u=0***

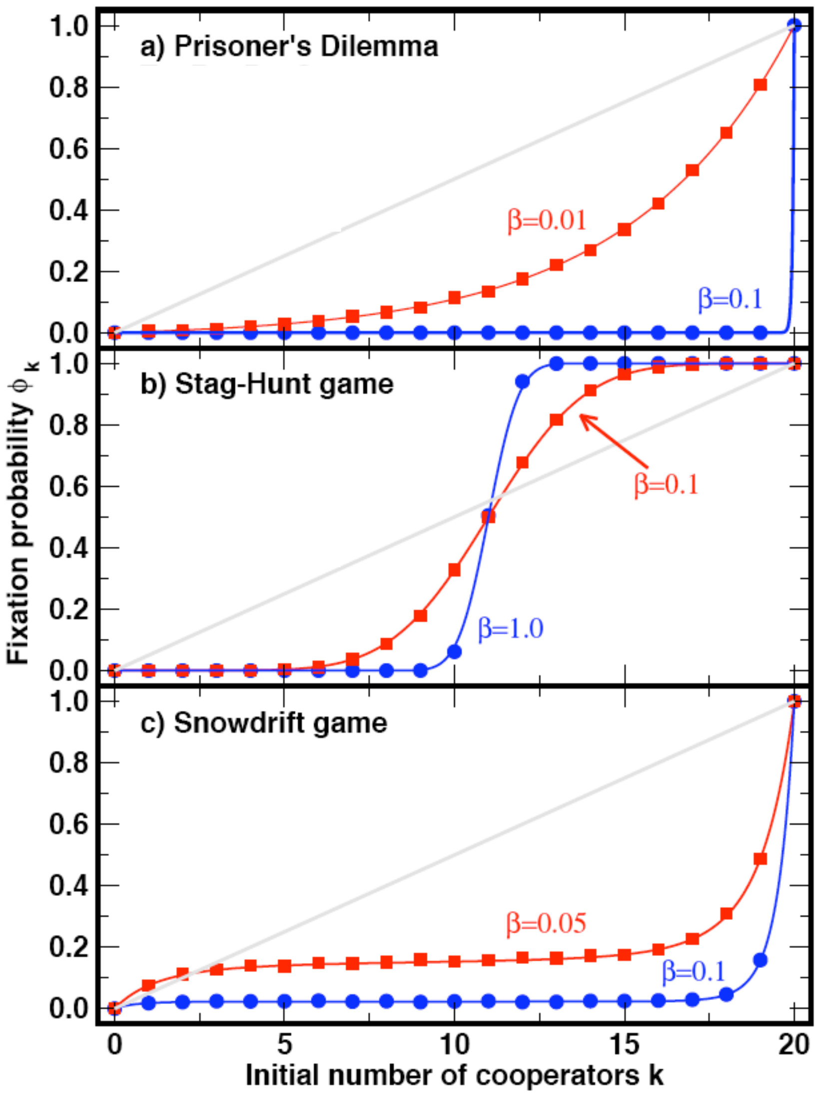
$$\phi_k = \frac{1 - r^{-k}}{1 - r^{-N}} \qquad r = e^{2\beta v}$$

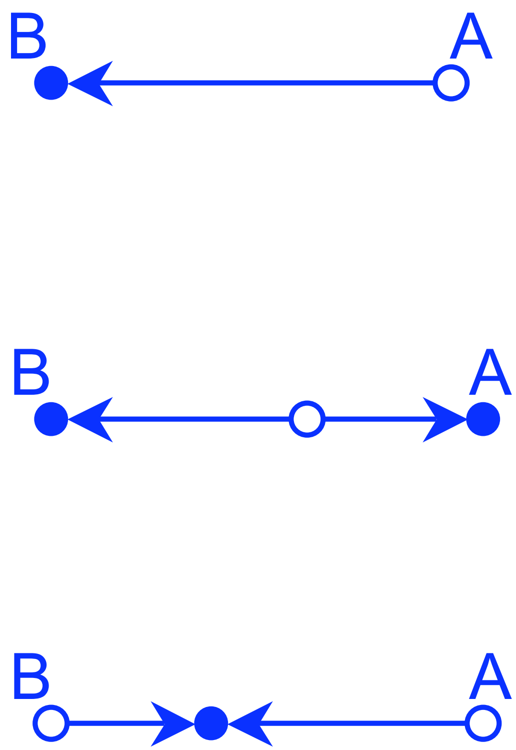
**2. Frequency dependent**

$$\phi_k = \frac{\mathrm{erf}\,[\xi_k] - \mathrm{erf}\,[\xi_0]}{\mathrm{erf}\,[\xi_N] - \mathrm{erf}\,[\xi_0]} \qquad \xi_k = \sqrt{\frac{\beta}{u}}(ku + v)$$

Traulsen, Nowak & Pacheco, PRE 74:11909 (2006)

# Numerical results for fixation probabilities




Very good agreement between theory and simulation

$$\sum_i \ldots \to \int \ldots di$$

- Introduction to evolutionary games
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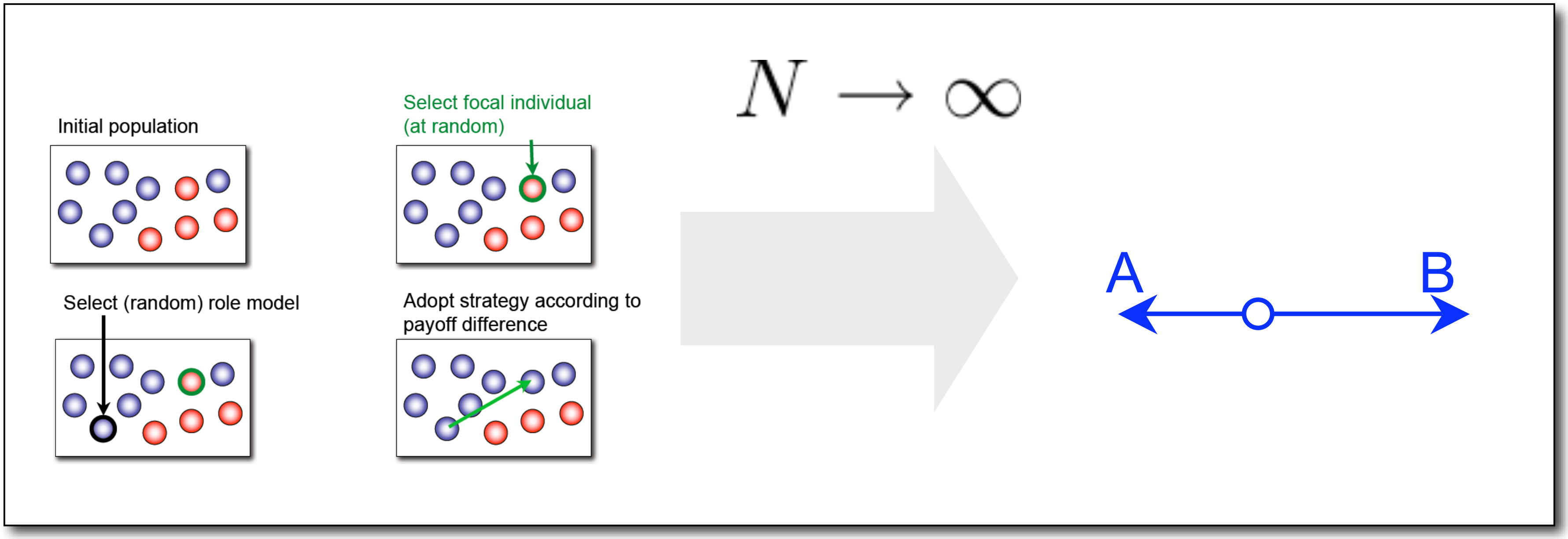
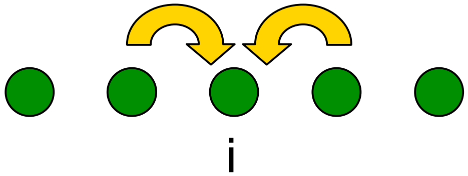
- $\boxed{\text{From finite to infinite populations}}$
- The evolution of cooperation
- The emergence of punishment

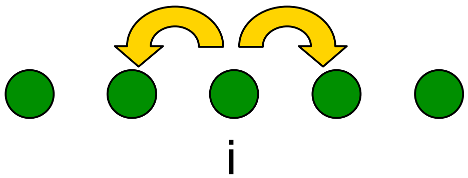
# From finite to infinite populations

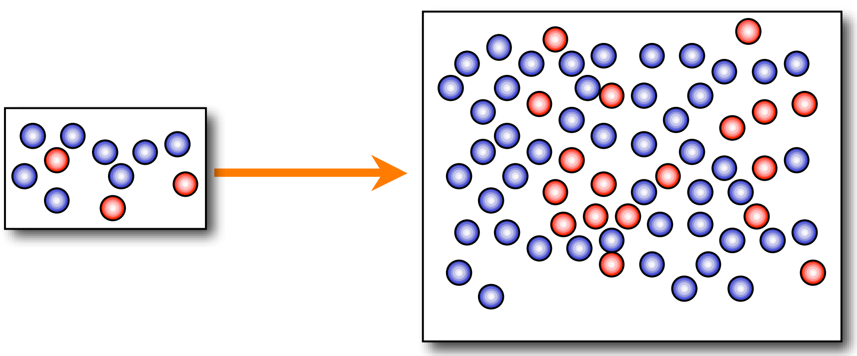


# Temporal dynamics

$$
\begin{aligned}
P_i^{\tau+1} &= P_i^{\tau} \\
&+ P_{i-1}^{\tau} T_{i-1}^{+} + P_{i+1}^{\tau} T_{i+1}^{-} \\
&- P_i^{\tau} T_i^{+} - P_i^{\tau} T_i^{-}
\end{aligned}
$$



__Large populations:__
Approximate Master equation by Fokker-Planck equation

# Fokker-Planck equation

$$\dot{\rho}(x,t) \approx -\frac{\partial}{\partial x}\left[a(x)\rho(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[b^2(x)\rho(x,t)\right]$$

**Drift**

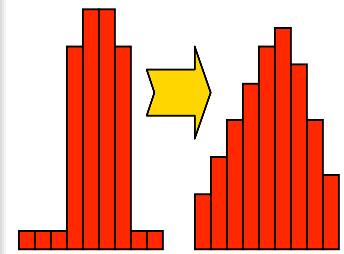
$$a(x) = T^{+}(x) - T^{-}(x)$$

**Diffusion**

$$b(x) = \sqrt{\frac{T^{+}(x) + T^{-}(x)}{N}}$$

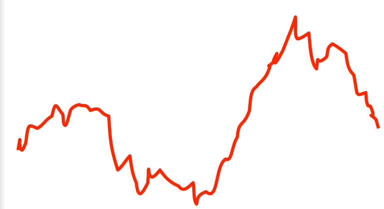
# From Fokker-Planck to Langevin

$$\dot{\rho}(x,t) \approx -\frac{\partial}{\partial x}\left[a(x)\rho(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[b^2(x)\rho(x,t)\right]$$



Stochastic Calculus

$$\dot{x} = a(x) + b(x)\xi$$



Gardiner, Handbook of Stochastic Methods (1981).

# Deterministic replicator equation for large *N*

$$\dot{x} = a(x) + \cancel{b(x)\xi}$$

$$b(x) = \sqrt{\frac{T^+(x) + T^-(x)}{N}}$$

$$N \to \infty$$

$$\begin{aligned}
\dot{x} &= a(x) \\
&= T^+(x) - T^-(x) \\
&= (1-x)x\frac{1}{1+e^{-\beta(\pi_A - \pi_B)}} - x(1-x)\frac{1}{1+e^{+\beta(\pi_A - \pi_B)}} \\
&= x(1-x)\tanh\left[\frac{\beta}{2}(\pi_A - \pi_B)\right]
\end{aligned}$$

Traulsen, Claussen & Hauert, PRL 95:238701 (2005); Traulsen, Nowak & Pacheco, PRE 74:11909 (2006)

# Back to the usual replicator equation

$$
\begin{aligned}
\dot{x} &= x(1-x)\tanh\left[\frac{\beta}{2}(\pi_A - \pi_B)\right] \\
&\approx x(1-x)\frac{\beta}{2}(\pi_A - \pi_B) \\
&= x(1-x)(\pi_A - \pi_B) \\
&= x\left[\pi_A - x\pi_A - (1-x)\pi_B)\right] \\
&= x\left[\pi_A - \langle\pi\rangle\right]
\end{aligned}
$$

Weak selection

New timescale

$$
\boxed{\dot{x}_i = x_i\left(\pi_i - \langle\pi\rangle\right)}
$$

# Advantages of the Fermi process

- Arbitrary intensity of selection


- Closed expression for fixation probabilities

$$\phi_k = \frac{\mathrm{erf}\,[\xi_k] - \mathrm{erf}\,[\xi_0]}{\mathrm{erf}\,[\xi_N] - \mathrm{erf}\,[\xi_0]}$$

- Simple replicator equation for large *N*

$$\dot{x} = x(1-x)\tanh\left[\frac{\beta}{2}(\pi_A - \pi_B)\right]$$

Traulsen, Nowak & Pacheco, PRE 74:11909 (2006)

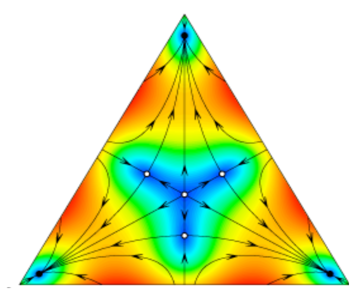
# Generalization to more strategies

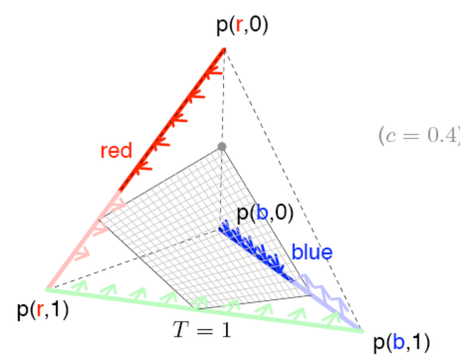
- Use d instead of 2 strategies

$$\dot{\rho}(\boldsymbol{x}) = -\sum_{k=1}^{d-1} \frac{\partial}{\partial x_k} \rho(\boldsymbol{x}) a_k(\boldsymbol{x}) + \frac{1}{2} \sum_{j,k=1}^{d-1} \frac{\partial^2}{\partial x_k \partial x_j} \rho(\boldsymbol{x}) b_{jk}(\boldsymbol{x})$$

$$b_{jk}(\boldsymbol{x}) = \frac{1}{N}\left[-T_{jk}(\boldsymbol{x}) - T_{kj}(\boldsymbol{x}) + \delta_{jk} \sum_{l=1}^{d} T_{jl}(\boldsymbol{x}) + T_{lj}(\boldsymbol{x})\right]$$

$$a_k(\boldsymbol{x}) = \sum_{j=1}^{d} T_{jk}(\boldsymbol{x}) - T_{kj}(\boldsymbol{x})$$

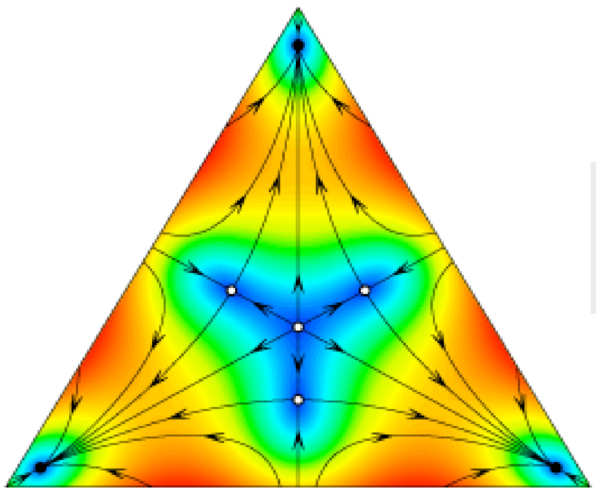


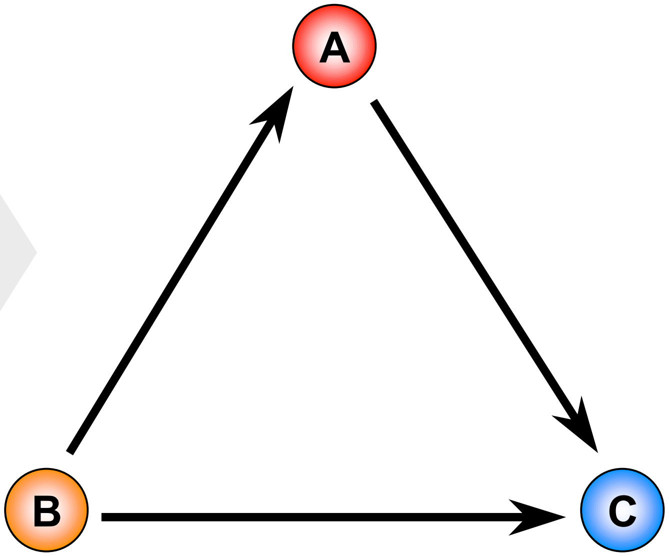



Traulsen, Claussen and Hauert, Phys. Rev. E 74, 011901 (2006)

# More strategies in finite populations

- Small mutation rates

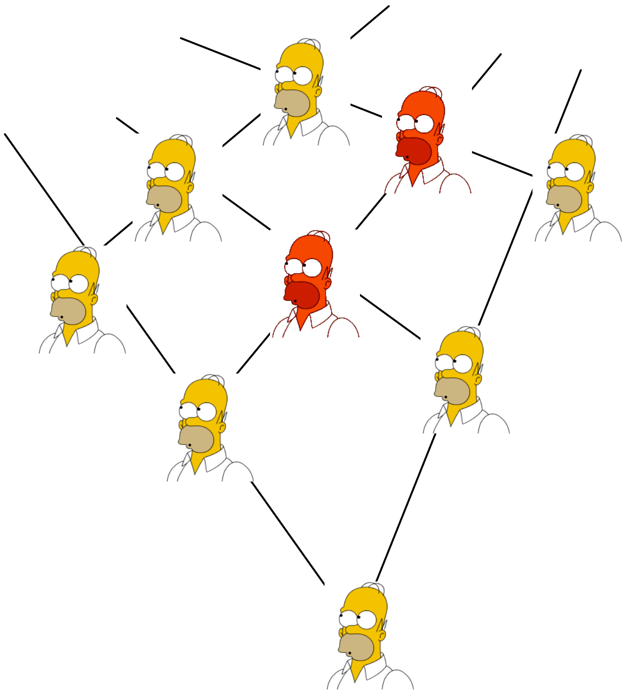





Simulations for larger mutation rates

Imhof, Fudenberg & Nowak, PNAS 102:10797 (2005)

# Population structure



- Cellular automata
- Pair approximation
- Cluster methods
- Mapping to Ising models
- Kin selection
- …

Nowak & May, Nature 359:826 (1992); Szabo and Fath, Physics Reports (2007)

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# The problem of cooperation

- Alarm calls

- Environment preservation

# The Prisoner's Dilemma

| | |
|---|---|
| Player 1 cooperation | *-c* |
| Player 2 cooperation | *+b* |
| Player 1 payoff | *b-c* |

Player 2

| Player 1 | $C$ | $D$ |
|---|---|---|
| $C$ | $b - c$ | $-c$ |
| $D$ | $b$ | $0$ |


Rapoport & Chammah, Prisoner's Dilemma (1965)

# Mechanisms for the evolution of cooperation

- Kin selection
- Direct reciprocity
- Indirect reciprocity
- Group selection
- Spatial reciprocity
- ...

Nowak, Science 314:1560 (2006)

# Kin selection

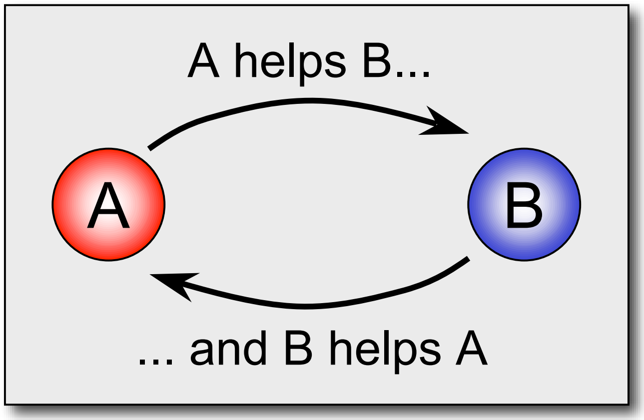
J.B.S Haldane

"I will jump into the river to save two brothers or eight cousins."

$$\frac{b}{c} > \frac{1}{r}$$

relatedness

# Direct reciprocity



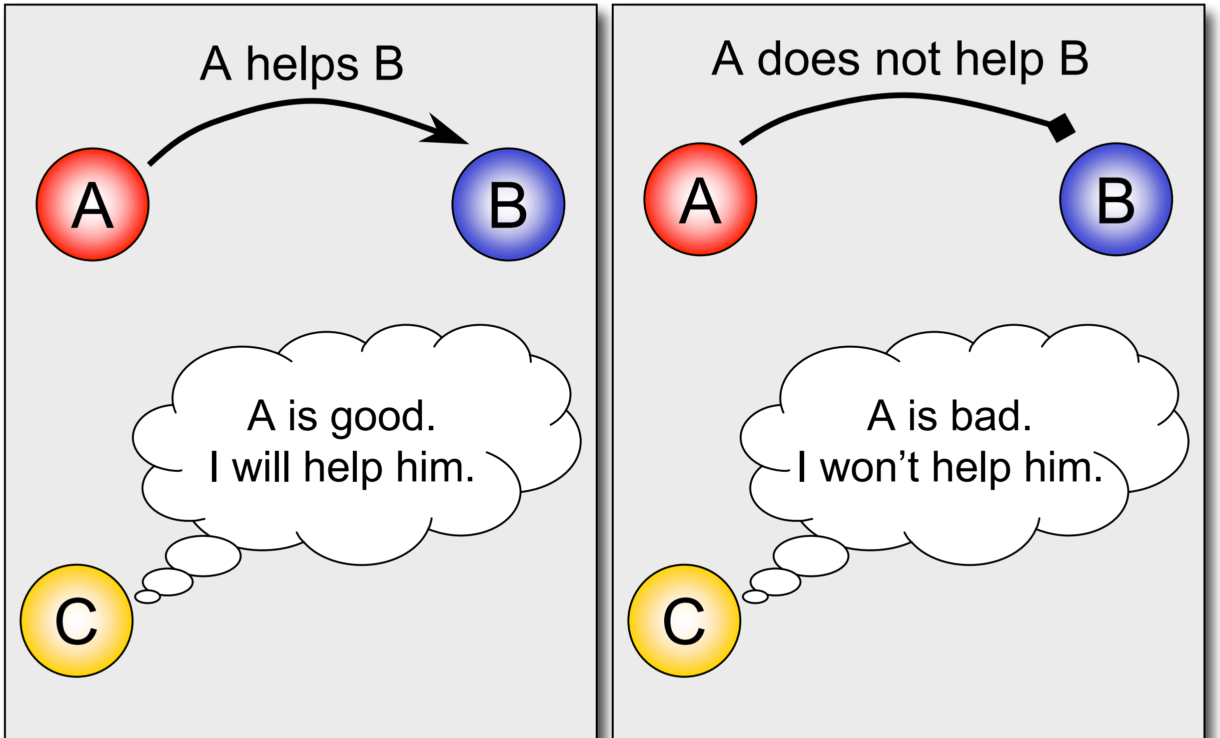

“I scratch your back and you scratch mine.”

- Tit for Tat
- Pavlov
- ...

$$\frac{b}{c} > \frac{1}{w}$$

probability for further interaction

# Indirect reciprocity (reputation)



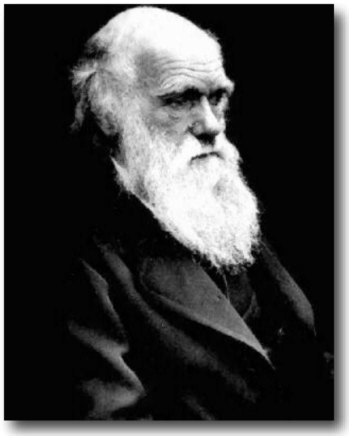

"I scratch your back and you scratch some-one else's."

$$\frac{b}{c} > \frac{1}{q}$$

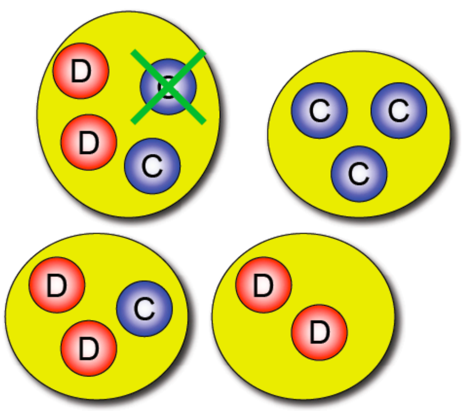
probability to know the reputation

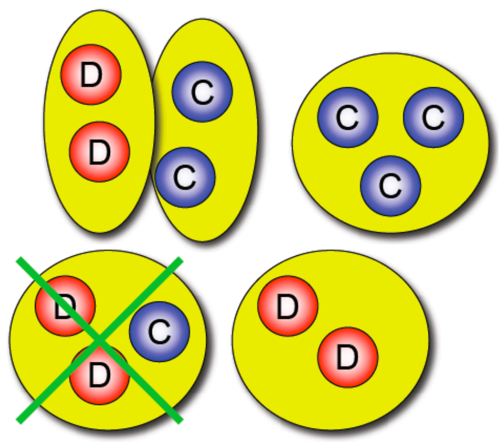
Nowak & Sigmund, Nature 393:573 (1998); Leimar & Hammerstein PRSB 268:745 (2001); Milinski, Semmann, Bakker & Krambeck PRSB 268:2495 (2001)

# Group selection



*"[…] A tribe including many members who […] were always ready to give aid to each other […], would be victorious over other tribes; and this would be natural selection"*



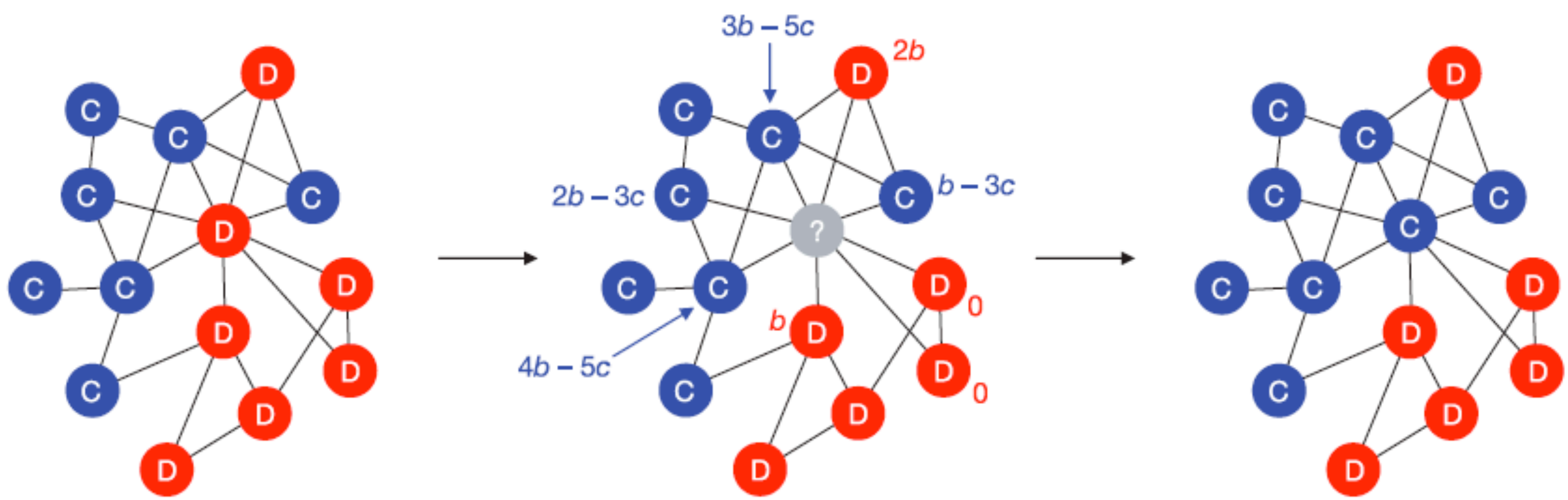

Size of groups

$$\frac{b}{c} > 1 + \frac{n}{m-2}$$

Number of groups

Traulsen & Nowak, PNAS 103:10952 (2006)

# Network reciprocity



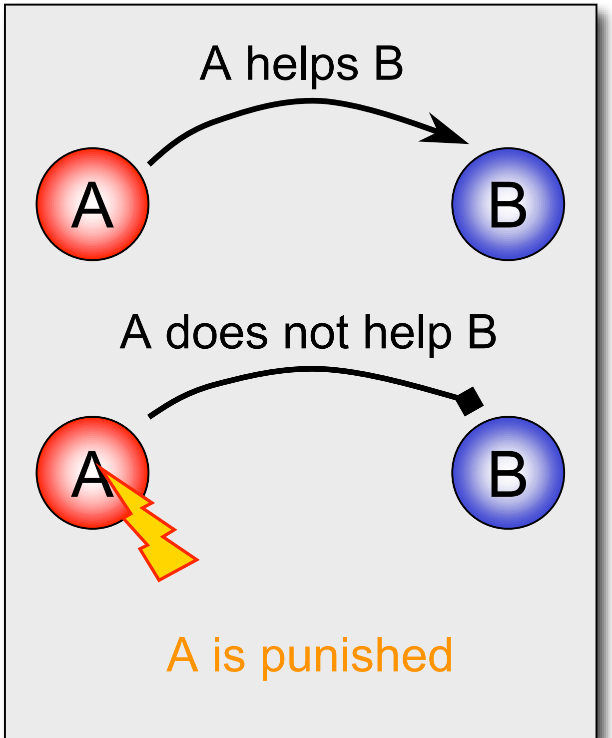

“Help your neighbors!”

$$\frac{b}{c} > k$$

Degree of the graph

Ohtsuki, Hauert, Lieberman & Nowak, Nature 441:502 (2006); Ohtsuki, Pacheco & Nowak, PRL 98:108106 (2007)

# Punishment ?




"If you don't cooperate, I will punish you!"

- Punishment can stabilize cooperation
- Punishment cannot lead to the evolution of cooperation
- How can punishment emerge?

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# The emergence of costly punishment

- Why are humans willing to punish others at a personal cost to themselves?

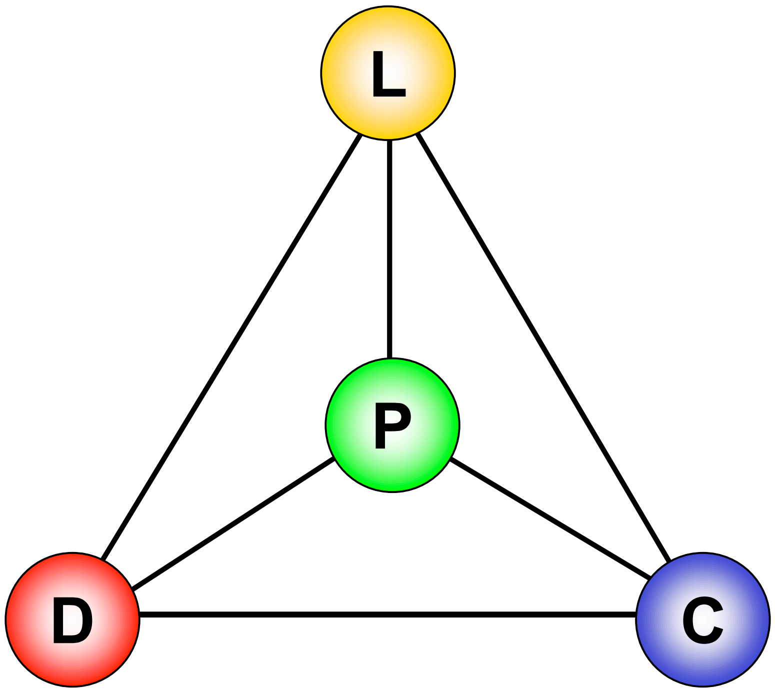
**C** Play the game. Always cooperate with others.

**D** Play the game. Exploit others that cooperate.

**P** Play the game. Always cooperate and punish those who do not cooperate.

**L** Do not join the game.

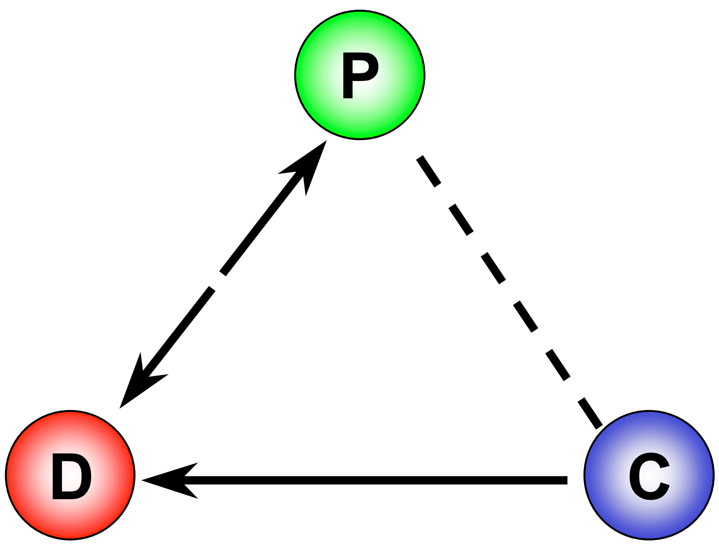
# Interaction of four strategies




- 2 Types:
  $(C,D)=(i,N-i)$
- 3 Types:
  $(C,D,L)=(i,j,N-i-j)$
- 4 Types:
  $(C,D,L,P)=(i,j,k,N-i-j-k)$

# Punishers in the Prisoner’s Dilemma

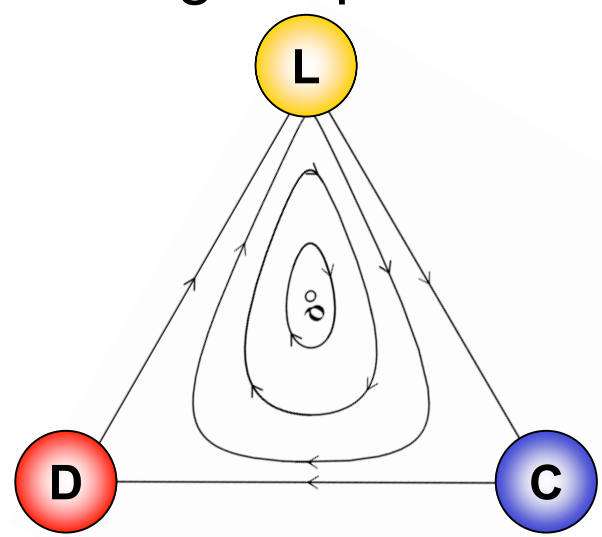
- Adding a punishment option does not lead to the emergence of punishment




Only D is stable!

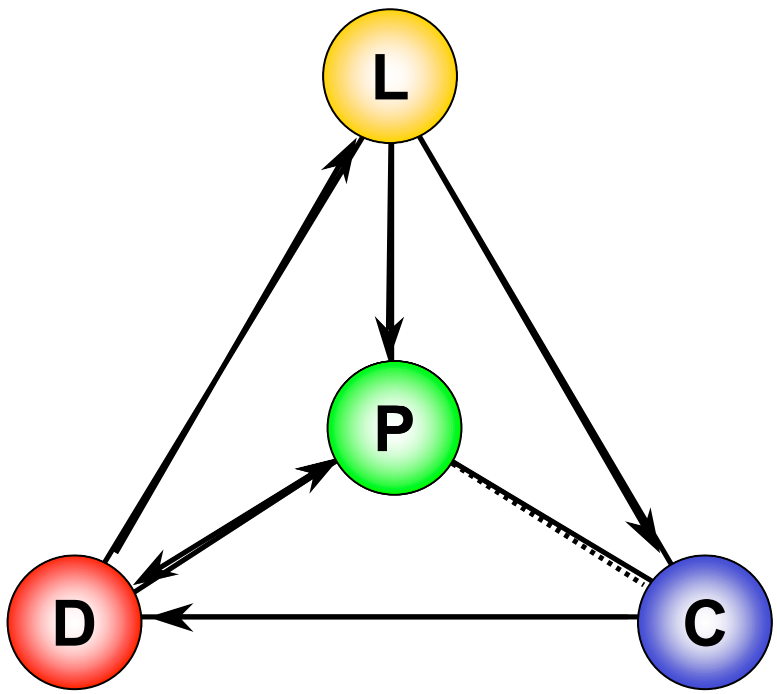
# Voluntary participation

- "Loners" avoid the cost of joining the game σ
  - Loners are advantageous among defectors
  - Cooperation is advantageous if there are no defectors
  - Defection is advantageous among cooperators

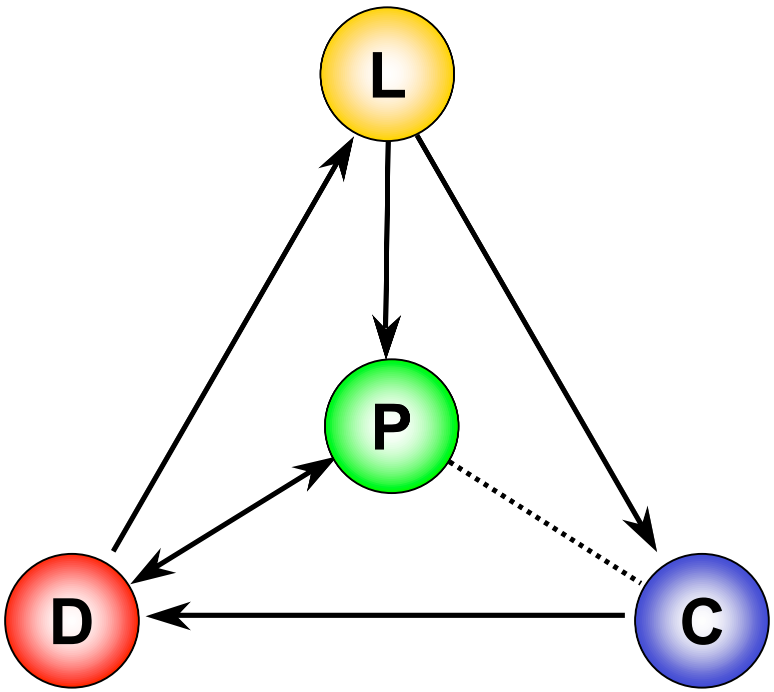



Hauert, de Monte, Hofbauer & Sigmund, Science 296:1129 (2002)

# Adding a fourth strategy



# Stochastic dynamics in finite populations

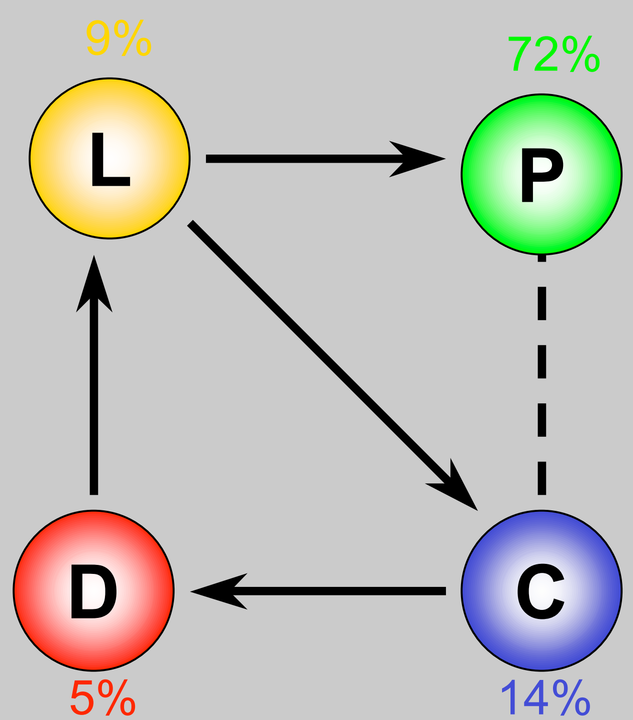



Where will the system spend most of the time?

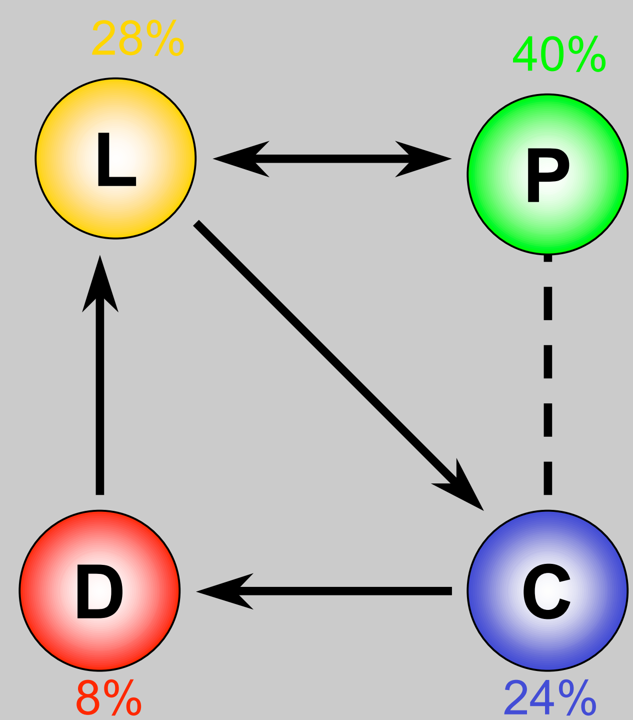
- Small mutation rates
- Embedded Markov chain on pure states
- Solve for stationary distribution

# Punishment can evolve in finite populations

**Punishment of defectors**




**Punishment of defectors and loners**



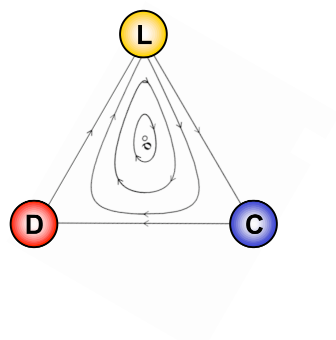

Hauert, Traulsen, Brandt, Nowak & Sigmund, Science, 316:1905 (2007)

# Game theoretic experiments

SCIENCE VOL 296 10 MAY 2002

## Volunteering as Red Queen Mechanism for Cooperation in Public Goods Games

Christoph Hauert,[1,2] Silvia De Monte,[1,3] Josef Hofbauer,[1] Karl Sigmund[1,4*]

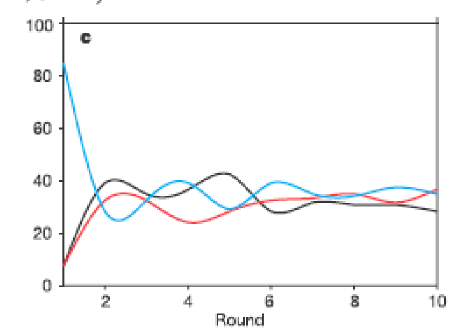



NATURE | VOL 425 | 25 SEPTEMBER 2003 |

## Volunteering leads to rock–paper–scissors dynamics in a public goods game

Dirk Semmann, Hans-Jürgen Krambeck & Manfred Milinski

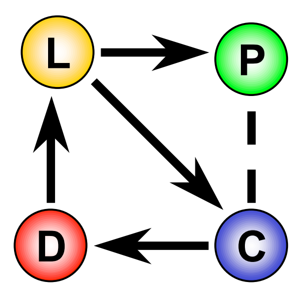
*Department of Evolutionary Ecology, Max Planck Institute of Limnology, 24306 Plön, Germany*




SCIENCE VOL 316 29 JUNE 2007

## Via Freedom to Coercion: The Emergence of Costly Punishment

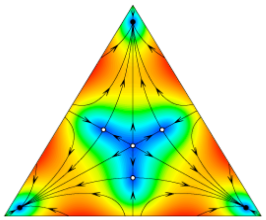
Christoph Hauert,[1] Arne Traulsen,[1] Hannelore Brandt,[2] Martin A. Nowak,[1] Karl Sigmund[3,4*]



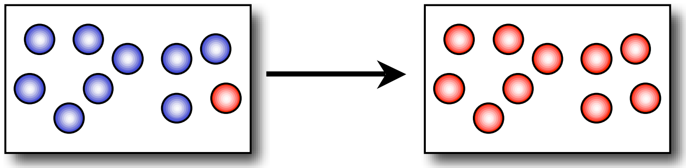

*work in progress*

# Summary

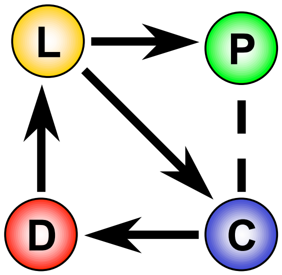
- Evolutionary game dynamics in finite populations
  - Converges to replicator equation for large populations



  - Equivalent to classical population dynamics for constant selection



  - Applications: The evolution of cooperation, punishment,...



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