Random Models *vs* Non Random Models

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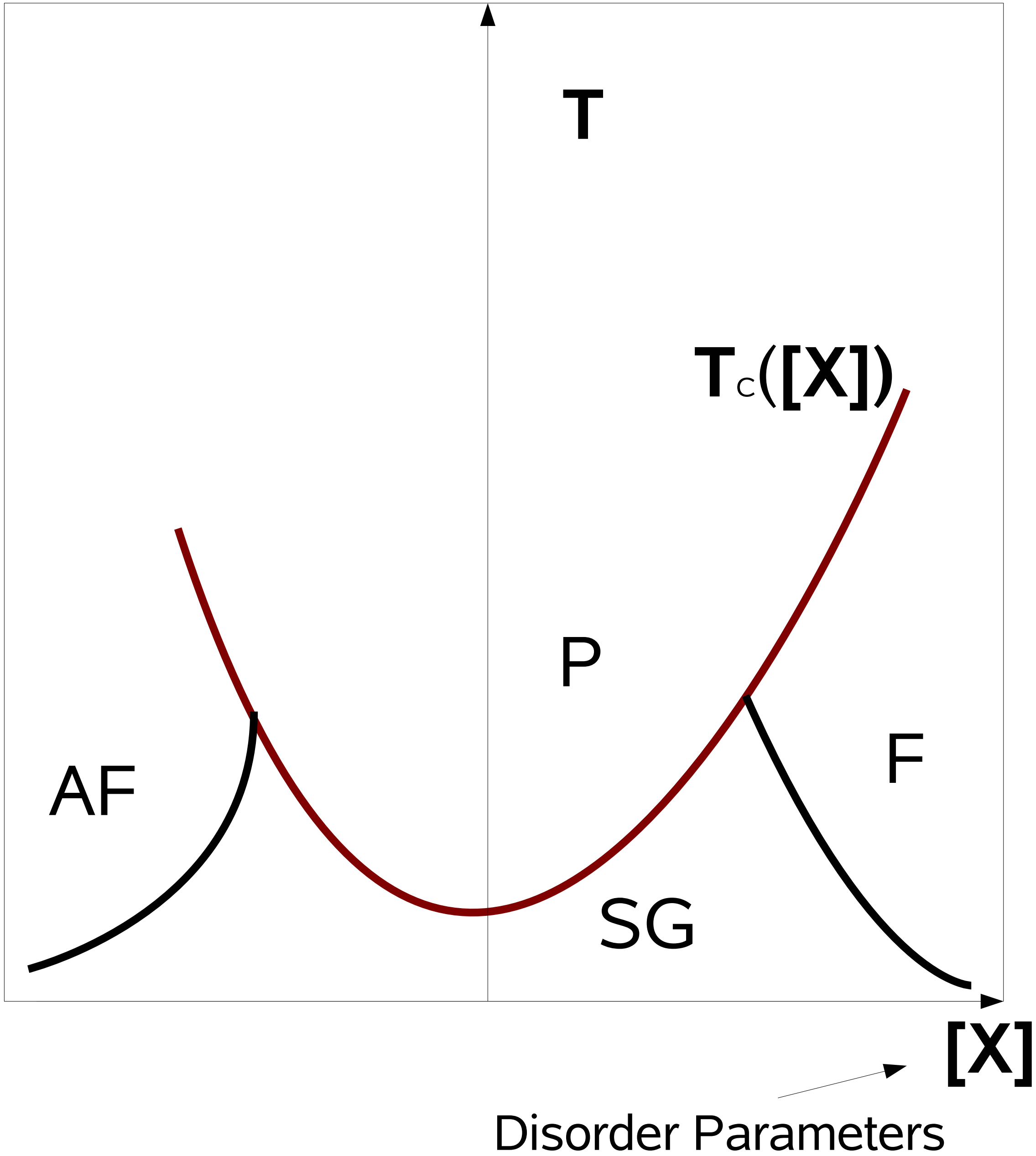
To start with

In a Random Ising model, typically one has a phase diagram of this kind:


T
$T_c([X])$
P
F
AF
SG
[X]
Disorder Parameters

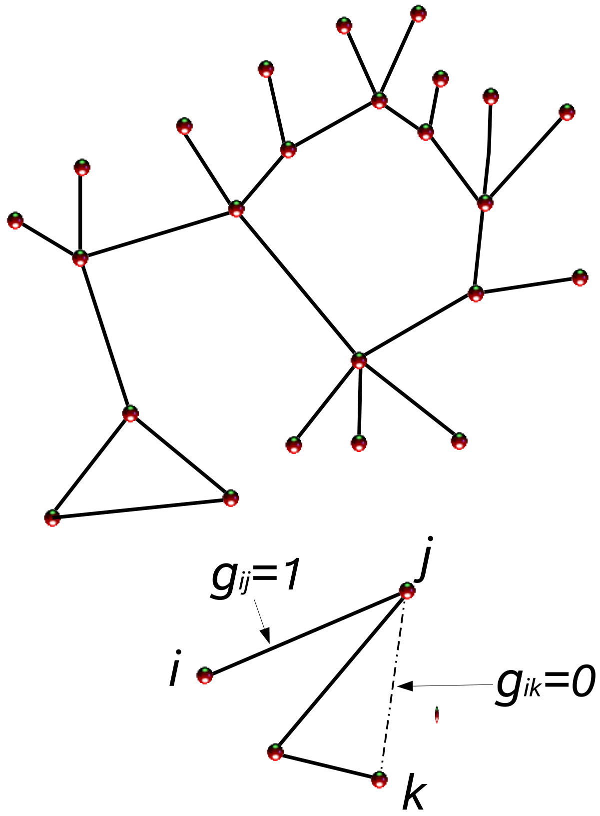
Outline

- Graphs - Infinite Dimensionality
- Random Models
- Mapping to a Non Random Model: The Related Ising Model
- Equations of the mapping (very simple!)
- Applications to Random Graphs
- Conclusions and Outlooks

Graphs

Let be given a graph $\boldsymbol{g}$ of N vertices. The set of links Γ can be defined through the adjacency matrix of the graph, $g_{i,j} = 0, 1$:

$$\Gamma_{\boldsymbol{g}} \equiv \{b = (i_b, j_b) \in \Gamma_f : \ g_{i_b, j_b} = 1, \ i_b < j_b\}, \qquad \Gamma_f = \text{Fully Connected Graph}$$



Example with N=28 and |Γ| =27

Overlap versus Dimensionality

In the thermodynamic limit, the mapping becomes exact if Γ becomes infinite dimensional in a strict or in a weak sense:

- $D(\Gamma) \to \infty$ in the strict sense if two paths of any length overlap with probability 0.

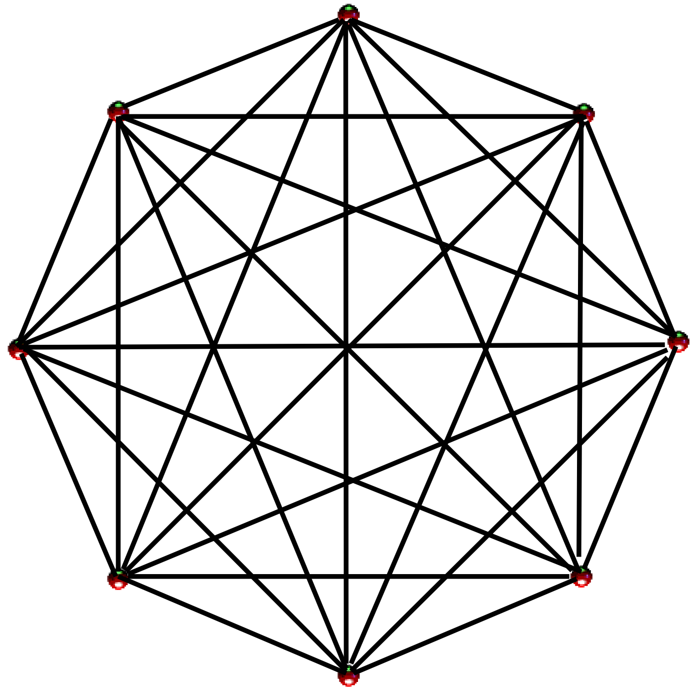
  Typical example: the Fully Connected Graph $\Gamma = \Gamma_f$

- $D(\Gamma) \to \infty$ in the weak sense if two long paths overlap l times with a probability $p(l) \leq C\,\mathrm{exp}(-\alpha l)$.

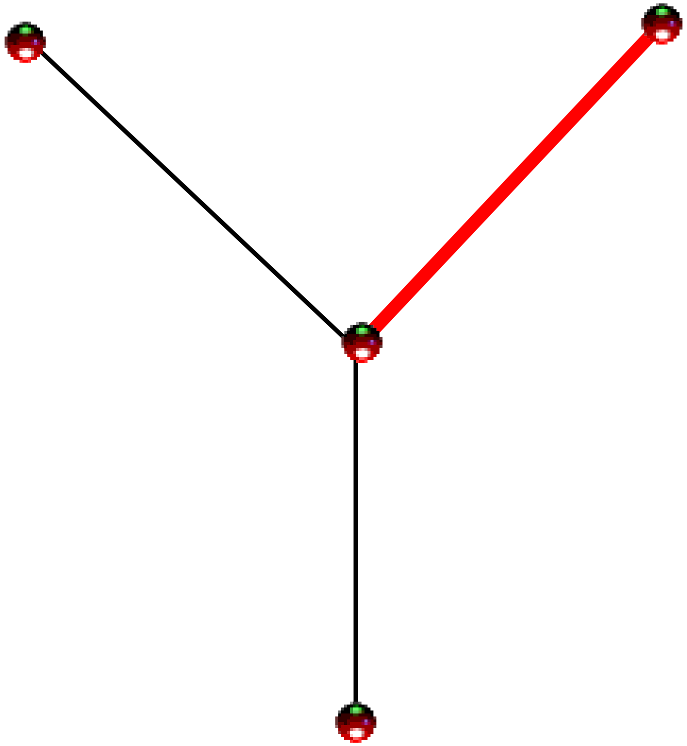
  Typical example: the Bethe Lattice

Γ_f is infinite dimensional in the strict sense

$$\Gamma_f \equiv \{b = (i_b, j_b) : i_b, j_b = 1, \ldots, N,\ i_b < j_b\}, \quad |\Gamma_f| = \frac{N(N-1)}{2}$$

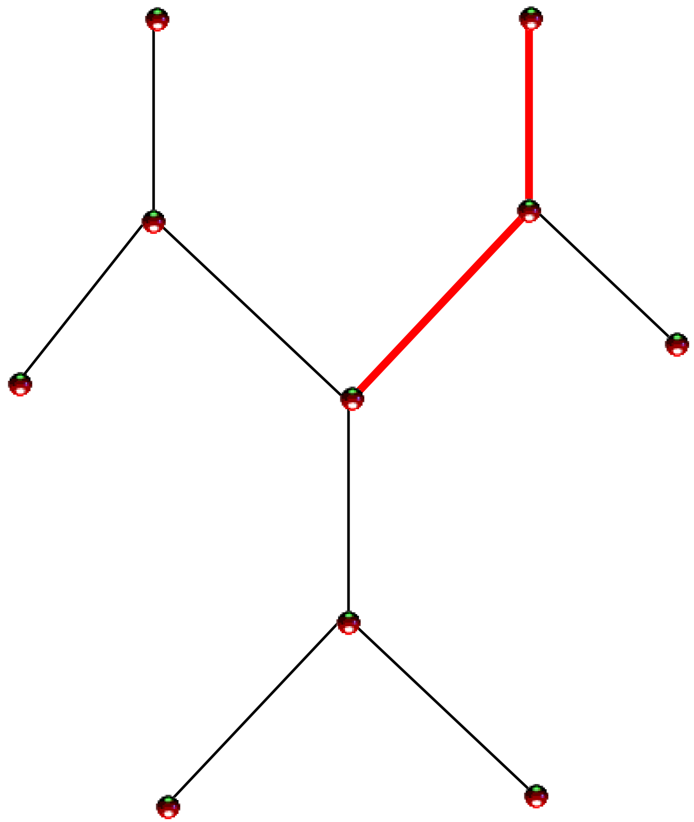


The Bethe Lattice is infinite dimensional in the weak sense



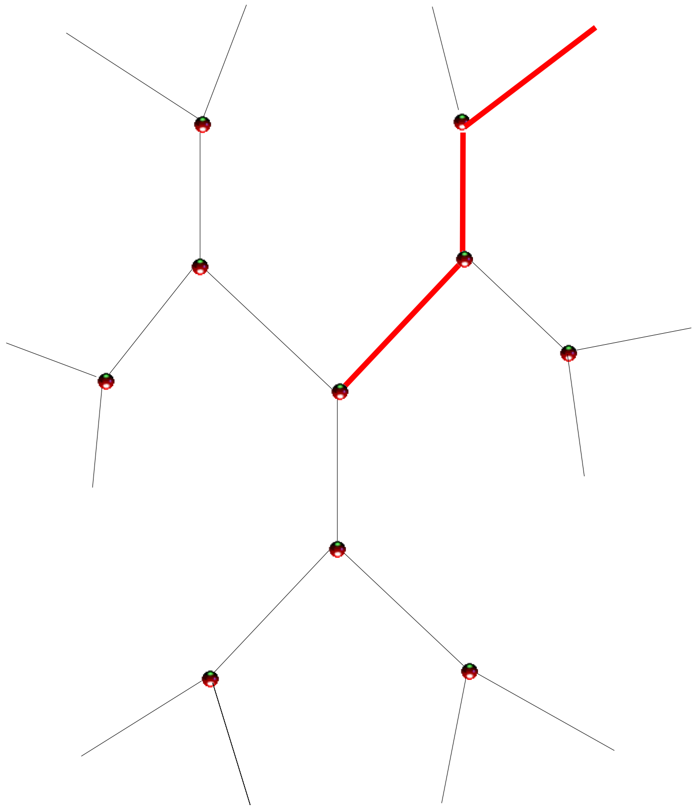
$$p(l = 1) \propto \frac{1}{3}$$

The Bethe Lattice is infinite dimensional in the weak sense



$$p(l = 2) \propto \frac{1}{3} \cdot \frac{1}{2}$$

The Bethe Lattice is infinite dimensional in the weak sense



$$p(l = 3) \propto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \qquad \Rightarrow p(l) \propto \frac{1}{3 \cdot 2^{l-1}}$$

Random Models onto Quenched Graphs

Given Γ we define

$$H\left(\{\sigma_i\};\{J_b\}\right) \equiv -\sum_{b\in\Gamma} J_b \sigma_{i_b}\sigma_{j_b}$$

the J_b's are quenched couplings, σ_i Ising variable at the site i. The free energy F and the physics are defined by

$$-\beta F \equiv \int d\mathcal{P}\left(\{J_b\}\right) \log\left(Z\left(\{J_b\}\right)\right), \quad \overline{\langle O\rangle} = \int d\mathcal{P}\left(\{J_b\}\right)\langle O\rangle$$

$Z\left(\{J_b\}\right)$ is the partition function of the quenched system

$$Z\left(\{J_b\}\right) = \sum_{\{\sigma_b\}} e^{-\beta H(\{\sigma_i\};\{J_b\})}, \quad \langle O\rangle = \frac{\sum_{\{\sigma_b\}} e^{-\beta H(\{\sigma_i\};\{J_b\})}\, O}{\sum_{\{\sigma_b\}} e^{-\beta H(\{\sigma_i\};\{J_b\})}}$$

and $d\mathcal{P}\left(\{J_b\}\right)$ is a product measure over all the possible bonds b given in terms of a normalized measure $d\mu \geq 0$

$$d\mathcal{P}\left(\{J_b\}\right) \equiv \prod_{b\in\Gamma_f} d\mu\left(J_b\right), \quad \int d\mu\left(J_b\right) = 1.$$

Universal Critical Parameter

Note that, given an ordinary Ising model with some uniform coupling J, its critical temperature depends on the value of J:

$$\beta_c = \beta_c(J).$$

But in terms of the product βJ there is just one critical parameter. The critical point of the model can be encoded in a single universal critical parameter $w = \tanh(\beta_c(J)J)$.
More precisely we have:

for any $J > 0$ there exists just a positive u. c. p., $w_F > 0$,

for any $J < 0$ there exists just a negative u. c. p., $w_{AF} < 0$

Example:
for $d = 2$, one has (Onsager) $w_F = \sqrt{2} - 1$ and $w_{AF} = -w_F$.

Mapping

Random Model at β

Related Ising Model at $\beta^{(I)}$

$$H\left(\{\sigma_i\};\{J_b\}\right) \equiv -\sum_{b\in\Gamma} J_b \sigma_{i_b}\sigma_{j_b} \quad \Longrightarrow \quad H^{(I)}\left(\{\sigma_i\};J^{(I)}\right) \equiv -J^{(I)}\sum_{b\in\Gamma}\sigma_{i_b}\sigma_{j_b}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\overline{\langle O\rangle} \quad \Longleftarrow \quad \langle O\rangle^{(I)} = f_I\left(\tanh\left(\beta^{(I)}J^{(I)}\right)\right)$$

where, if $D(\Gamma)=\infty$:

$$\overline{\langle O\rangle} = \begin{cases} f_I\left(\int d\mu(J_b)\tanh\left(\beta J_b\right)\right), & \text{for the P} - \text{F transition,} \\ f_I\left(\int d\mu(J_b)\tanh^2\left(\beta J_b\right)\right), & \text{for the P} - \text{SG transition} \end{cases}$$

in other words, the Mapping Trasformations are:

$$\tanh\left(\beta^{(I)}J^{(I)}\right) \to \int d\mu(J_b)\tanh\left(\beta J_b\right), \quad \text{for the P} - \text{F transition,}$$

$$\tanh\left(\beta^{(I)}J^{(I)}\right) \to \int d\mu(J_b)\tanh^2\left(\beta J_b\right), \quad \text{for the P} - \text{SG transition}$$

Equations of the Mapping for the Upper Critical Surface

In particular, if the Related Ising Model is critical at $w_F^{(I)} = \mathrm{tanh}(\beta_c^{(I)} J^{(I)})$, for the Random Model we have:

$$\int d\mu(J_b)\mathrm{tanh}(\beta_c^{(\mathrm{F})} J_b) = w_F^{(I)}, \quad \text{for the P} - \text{F transition},$$

$$\int d\mu(J_b)\mathrm{tanh}^2(\beta_c^{(\mathrm{SG})} J_b) = w_F^{(I)}, \quad \text{for the P} - \text{SG transition},$$

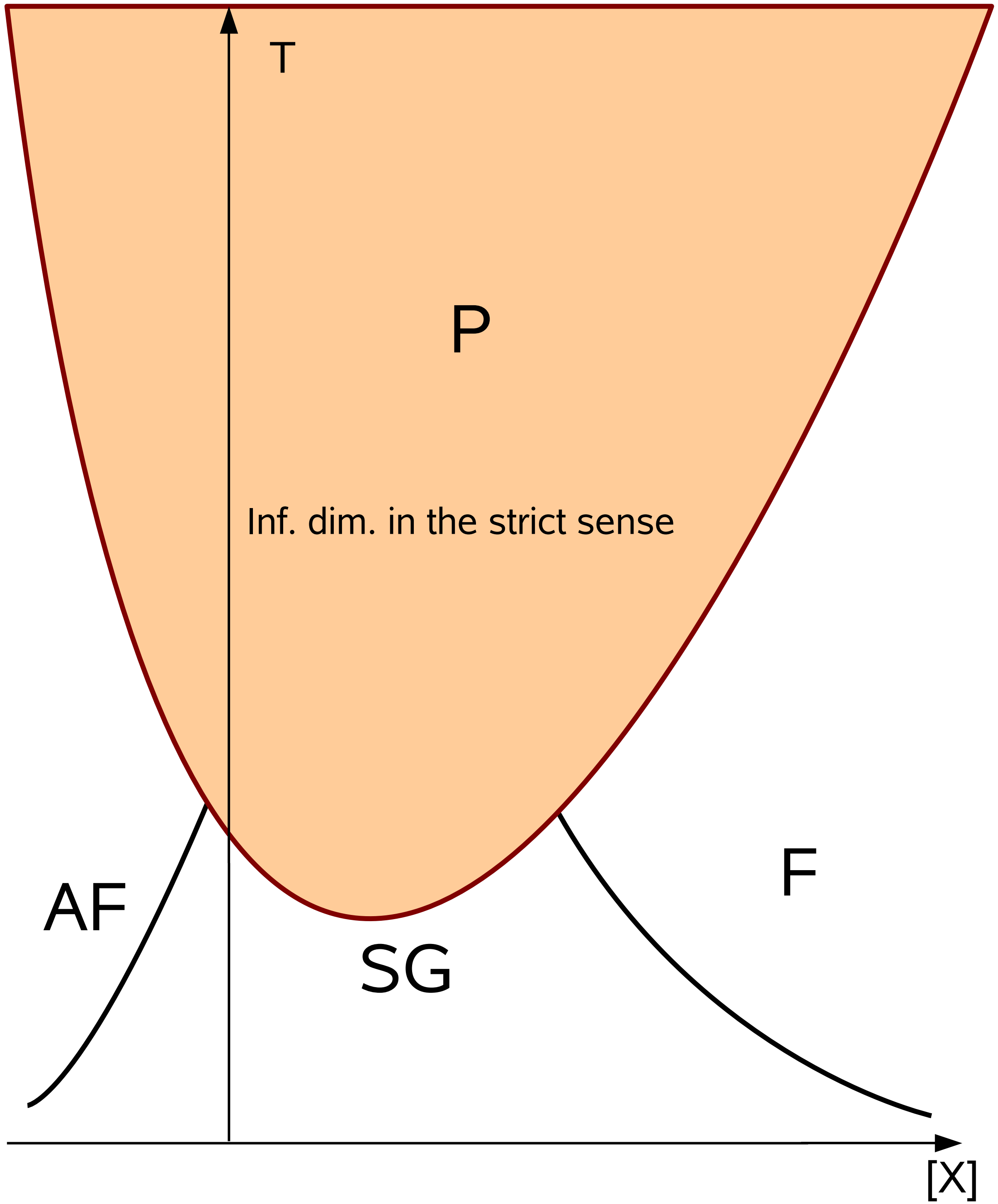
with the stable solution being given by:

$$\beta_c = \mathrm{min}\{\beta_c^{(\mathrm{SG})}, \beta_c^{(\mathrm{F})}\}.$$

If we solve The Related Ising Model we obtain exactly:

- if $D(\Gamma) \rightarrow \infty$ strictly: everything in the P region

- if $D(\Gamma) \rightarrow \infty$ weakly: everything on and infinitely near (above) the upper critical surface


T
P
Inf. dim. in the strict sense
AF
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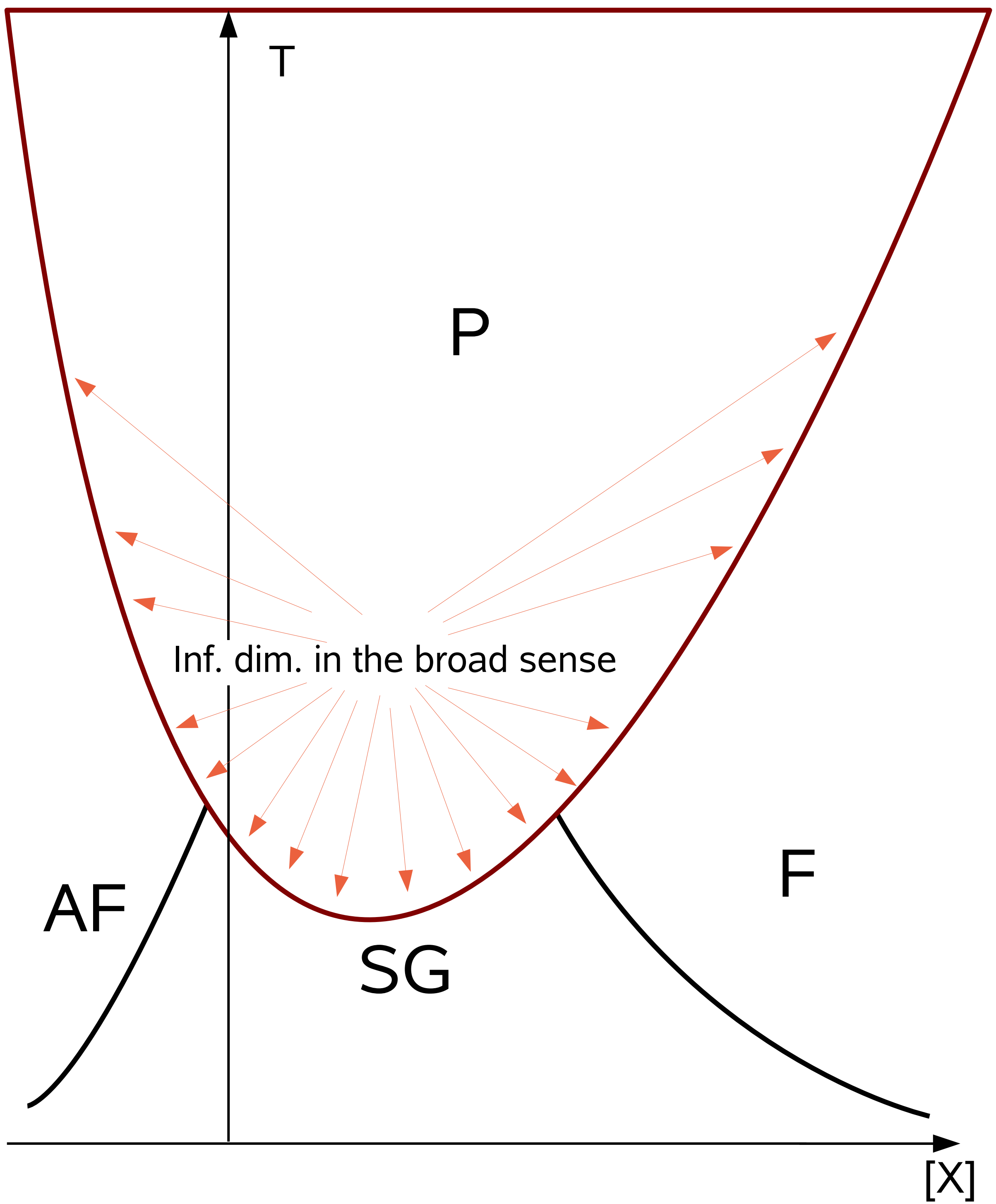
T
P
Inf. dim. in the broad sense
AF
SG
F
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Ex. with $D(\Gamma) \to \infty$ in the strict sense : Sherrington-Kirkpatrick model

The SK model corresponds to the spin glass over the fully connected graph and random couplings $\{J_b\}$ s. t.:

$$\int d\mu(J_b) J_b = J_0/N,$$

$$\int d\mu(J_b)(J_b - J_0/N)^2 = \tilde{J}^2/N.$$

The Hamiltonian of the related Ising model is

$$H_I = -\sum_{b \in \Gamma_f} J^{(I)} \sigma_{i_b} \sigma_{j_b} = -\sum_{(i,j)} J^{(I)} \sigma_i \sigma_j,$$

As is well known, for this model, depending on the sign of the coupling $J^{(I)}$, a ferromagnetic-paramagnetic or an antiferromagnetic-paramagnetic phase transition takes place at the same critical temperature given by

$$\beta_c^{(I)} |J^{(I)}| N = 1,$$

for N large, in terms of the universal quantities $w^{(I)}_{F/AF} = \pm\,\tanh(\beta_c^{(I)}|J^{(I)}|)$ gives

$$w^{(I)}_{F/AF} = \pm\frac{1}{N} + O\left(\frac{1}{N^3}\right)$$

On the other hand for N large the two moments give

$$\int d\mu_b(J_b)\tanh\left(\beta J_b\right) = \frac{(\beta J_0)}{N} + O\left(\frac{1}{N^3}\right).$$

and

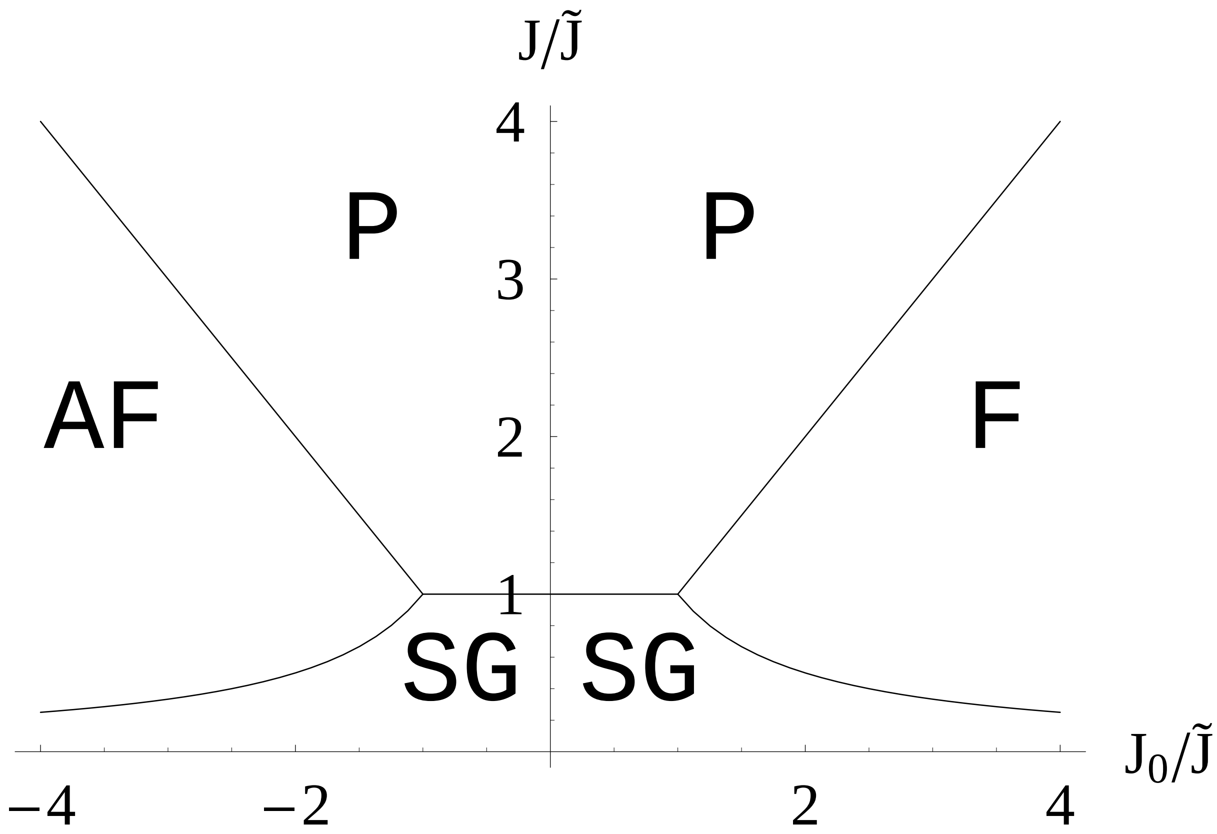
$$\int d\mu_b(J_b)\tanh^2\left(\beta J_b\right) = \frac{\left(\beta\tilde{J}\right)^2}{N} + O\left(\frac{1}{N^3}\right),$$

Hence, from the equations of the mapping, in the limit $N \to \infty$, we get the following spin glass (SG) and, depending on the sign of J_0, ferromagnetic (F) or antiferromagnetic (AF) boundaries

$$\begin{aligned}\beta_c^{(\mathrm{SG})}\tilde{J} &= 1\\ \beta_c^{(F/AF)}J_0 &= \pm 1.\end{aligned}$$

By taking the envelope of these curves we get the upper phase boundaries.

S-K phase diagram



Ex. with $D(\Gamma) \to \infty$ in the weak sense: Bethe lattice

Spin glass model defined over a Bethe lattice of coordination number q.
The related Ising model is the homogeneous Ising model over a regular Bethe lattice with coordination number q for which the exact solution is known (e.g. Baxter)

$$w_F^{(I)} = \tanh\left(\beta_c^{(I)} J^{(I)}\right) = \frac{1}{q-1},$$

Therefore, by using the equation of the mapping we find that a spin glass (SG) and a disordered ferromagnetic (F) transitions take place at $\beta_c^{(\mathrm{SG})}$ and $\beta_c^{(\mathrm{F})}$ solutions of:

$$\begin{aligned}
\int d\mu(J_b)\tanh\left(\beta_c^{(\mathrm{F})} J_b\right) &= \frac{1}{q-1}, && \mathrm{P-F}, \\
\int d\mu(J_b)\tanh^2\left(\beta_c^{(\mathrm{SG})} J_b\right) &= \frac{1}{q-1}, && \mathrm{P-SG}
\end{aligned}$$

Generalization: $d\mu(J_b) \to \{d\mu_b(J_b)\}$

Let define as many different $J_b^{(I)}$ as many are the different $d\mu_b$. Let be $z_b^{(I)} = \tanh(\beta J_b^{(I)})$ and let

$$G_I(\{z_b^{(I)}\}) = 0$$

represents the equation (possibly vectorial) for the critical surface of the related Ising model. It describes a transition between the P phase and an ordered F/AF phase. The upper critical temperature of the spin glass model is given by

$$\beta_c = \min\{\beta_c^{(\mathrm{SG})}, \beta_c^{(F/AF)}\}$$

where $\beta_c^{(\mathrm{SG})}$ and $\beta_c^{(F/AF)}$ satisfy

$$\begin{aligned} G_I\left(\{\int d\mu_b(J_b)\tanh^2(\beta_c^{(\mathrm{SG})} J_b)\}\right) &= 0, \\ G_I\left(\{\int d\mu_b(J_b)\tanh(\beta_c^{(F/AF)} J_b)\}\right) &= 0. \end{aligned}$$

Note that the last two equations describe completely the upper critical surface.

So, for example, in a case with two families of bonds b_1 and b_2, the equation

$$G_I\left(\int d\mu_{b_1}(J_{b_1})\,\mathrm{tanh}(\beta J_1), \int d\mu_{b_2}(J_{b_2})\,\mathrm{tanh}^2(\beta J_2)\right) = 0$$

does not describe any upper critical surface; there are no intermediate situations between Eqs. (1) and (1).

Equations (1) - (1) give the exact critical P-SG and P-F/AF temperatures. In the case of a homogeneous measure, the suffix F and AF stay for disordered ferromagnetic and antiferromagnetic phases, respectively. In the general case, such a distinction is possible only in the positive and negative sectors where one has respectively $\int d\mu_b(J_b)\mathrm{tanh}(\beta_c^{(F/AF)} J_b) > 0$ or $\int d\mu_b(J_b)\mathrm{tanh}(\beta_c^{(F/AF)} J_b) < 0$, for any bond, whereas, for the other sectors, we use the symbol F/AF only to stress that the transition is not P-SG.

Near the upper critical surface, the mapping allows also to determine coexistence surfaces and correlation functions.

Random Models onto Random Graphs

Given an ensemble of graphs $\boldsymbol{g} \in \mathcal{G}$ distributed with $P(\boldsymbol{g})$, define

$$H_{\boldsymbol{g}}\left(\{\sigma_i\};\{J_b\}\right) \equiv - \sum_{b \in \Gamma_{\boldsymbol{g}}} J_b \sigma_{i_b} \sigma_{j_b}$$

The free energy F and the physcis are now given by

$$-\beta F \equiv \sum_{\boldsymbol{g} \in \mathcal{G}} P(\boldsymbol{g}) \int d\mathcal{P}\left(\{J_b\}\right) \log\left(Z_{\boldsymbol{g}}\left(\{J_b\}\right)\right),$$

$Z_g\left(\{J_b\}\right)$ is the partition function of the quenched system onto the graph realization $\Gamma_{\boldsymbol{g}}$

$$Z_g\left(\{J_b\}\right) = \sum_{\{\sigma_b\}} e^{-\beta H_g(\{\sigma_i\};\{J_b\})},$$

and $d\mathcal{P}\left(\{J_b\}\right)$ is again a product measure over all the possible bonds b given in terms of a normalized measure $d\mu \geq 0$

$$d\mathcal{P}\left(\{J_b\}\right) \equiv \prod_{b \in \Gamma_f} d\mu\left(J_b\right), \qquad \int d\mu\left(J_b\right) = 1.$$

Application to Unconstrained Random Graphs

For unconstrained random graphs we have

$$P(\boldsymbol{g}) = \prod_{b\in\Gamma_f} P(g_b).$$

In this case it is useful to define the effective coupling $\tilde{J}_b$:

$$\tilde{J}_b \equiv J_b \cdot g_b, \qquad J_b \in \mathcal{R}, \qquad g_b = 0, 1,$$

correspondingly:

$$d\tilde{\mu}(\tilde{J}_b) = d\mu(J_b) \cdot P(g_b), \qquad d\tilde{\mathcal{P}}\left(\{\tilde{J}_b\}\right) = P(\boldsymbol{g}) \cdot d\mathcal{P}\left(\{J_b\}\right)$$

so that

$$d\tilde{\mathcal{P}}\left(\{\tilde{J}_b\}\right) = \prod_{b\in\Gamma_f} d\tilde{\mu}(\tilde{J}_b), \qquad \int d\tilde{\mu}(\tilde{J}_b) f(\cdot) = \sum_{g_b=0,1} \int d\mu(J_b) f(\cdot),$$

and the mapping can be applied as we had a single effective graph Γ_P

$$\Gamma_P \equiv \{b \in \Gamma_f :\ P(g_b = 1) \neq 0\}$$

Application to Unconstrained Random Graphs: Erd$\ddot{o}$s-Reny

Random model defined over Poissonian graphs with cN bonds.

$$H\left(\{\sigma_i\};\{J_b\}\right) \equiv -\sum_{i<j} J_{i,j} g_{i,j} \sigma_i \sigma_j = -\sum_{i<j} \tilde{J}_{i,j} \sigma_i \sigma_j, \quad \text{where}$$

$$P(g_{i,j}) = \frac{c}{N}\delta_{g_{i,j},1} + (1-\frac{c}{N})\delta_{g_{i,j},0},$$

where

$$c = \langle k \rangle, \quad \text{the average degree.}$$

What is the Related Ising Model of this random model?

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where

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What is the Related Ising Model of this random model?

$$H^{(I)}\left(\{\sigma_i\}; J^{(I)}\right) \equiv -J^{(I)} \sum_{i<j} \sigma_i \sigma_j,$$

The Fully Connected Model ($\Gamma_P = \Gamma_f$)!

We can solve exactly the uniform and fully connected model:

$$m_I = \tanh\left[\beta^{(I)} J^{(I)} N m_I + \beta^{(I)} h\right],$$

from which in particular we get its critical temperature

$$N\beta_c^{(I)} J^{(I)} = 1 \qquad \Rightarrow \qquad w_F^{(I)} = \tanh(\beta_c^{(I)} J^{(I)}) = \tanh(\frac{1}{N}) = \frac{1}{N}$$

By applying the equations of the mapping

$$\int d\tilde{\mu}(\tilde{J}_b)\tanh(\beta_c^{(\mathrm{F})}\tilde{J}_b) = w_F^{(I)}, \qquad \mathrm{P-F}$$

$$\int d\tilde{\mu}(\tilde{J}_b)\tanh^2(\beta_c^{(\mathrm{SG})}\tilde{J}_b) = w_F^{(I)}, \qquad \mathrm{P-SG}$$

with $d\tilde{\mu}(\tilde{J}_b) = d\mu(J_b) \cdot P(g_b)$, $d\mu(J_b)$ arbitrary and $P(g_b = 1) = \frac{c}{N}$,
we find:

$$c \int d\mu(J_b) \tanh(\beta_c^{(\mathrm{F})} J_b) = 1, \qquad \mathrm{P-F}$$
$$c \int d\mu(J_b) \tanh^2(\beta_c^{(\mathrm{SG})} J_b) = 1, \qquad \mathrm{P-SG}$$

Viana and Bray 1985

Critical Behavior

The mapping is rigorously defined only for $\beta \leq \beta_c = \min(\beta_c^{(\mathsf{F})}, \beta^{(AF)})$. However, except for the free energy, by analytic continuation below β_c we get very easily good estimations of the effective fields

$$
\begin{aligned}
m^{(\mathsf{F})} &= \tanh\left[c \int d\mu(J_b) \tanh(\beta J_b) m^{(\mathsf{F})}\right], \\
m^{(\mathsf{SG})} &= \tanh\left[c \int d\mu(J_b) \tanh^2(\beta J_b) m^{(\mathsf{SG})}\right],
\end{aligned}
$$

where $m^{(\mathsf{SG})} = \sqrt{q_{EA}}$.

As expected, the critical behavior of the Ising model over the E.- R. random graph is Mean-Field-like.

Application to Unconstrained Random Graphs: Small World

Random model defined over a Small World Graph. Poissonian graphs with cN bonds superimposed onto a one dimensional ring:

$$H(\{\sigma_i\};\{J_b\}) \equiv -J_0 \sum_i \sigma_i \sigma_{i+1} - \sum_{i<j} J_{i,j} c_{i,j} \sigma_i \sigma_j, \quad \text{where}$$

$$P(c_{i,j}) = \frac{c}{N}\delta_{c_{i,j},1} + (1-\frac{c}{N})\delta_{c_{i,j},0}.$$

What is the related Ising model of this random model?

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$$P(c_{i,j}) = \frac{c}{N}\delta_{c_{i,j},1} + (1-\frac{c}{N})\delta_{c_{i,j},0}.$$

What is the related Ising model of this random model?

$$H^{(I)}\left(\{\sigma_i\}; J_0^{(I)}, J^{(I)}\right) \equiv -J_0^{(I)} \sum_i \sigma_i \sigma_{i+1} - J^{(I)} \sum_{i<j} \sigma_i \sigma_j,$$

Fully Connected superimposed onto a one dimensional ring!

We can solve exactly this model. Its critical surface is given by

$$N\beta_c^{(I)} J^{(I)} e^{2\beta_c^{(I)} J_0^{(I)}} = 1$$

By applying the Mapping Substitutions to $J^{(I)}$ and $J_0^{(I)}$:

$$\begin{cases} \tanh(\beta^{(I)} J^{(I)}) \to \int d\mu(J_{i,j}) \tanh(\beta J_{i,j}), \\ \tanh(\beta^{(I)} J_0^{(I)}) \to \int d\mu(J_0) \tanh(\beta J_0), \end{cases} \qquad \mathsf{P-F}$$

and

$$\begin{cases} \tanh(\beta^{(I)} J^{(I)}) \to \int d\mu(J_{i,j}) \tanh^2(\beta J_{i,j}), \\ \tanh(\beta^{(I)} J_0^{(I)}) \to \int d\mu(J_0) \tanh^2(\beta J_0), \end{cases} \qquad \mathsf{P-SG}$$

with $d\mu(J_0') = \delta(J_0' - J_0)$, we find

$$c \int d\mu(J_b) \tanh(\beta_c^{(\mathsf{F})} J_b) e^{2\beta_c^{(\mathsf{F})} J_0} = 1, \quad \mathsf{P-F}$$

Nikoletopoulus *et al* 2004

$$c \int dJ_b f(J_b) \tanh^2(\beta_c^{(\mathsf{SG})} J_b) \cosh(2\beta_c^{(\mathsf{SG})} J_0) = 1, \quad \mathsf{P-SG}$$

And similarly to the previous case we can write readily the equations for the effective fields $m^{(\mathsf{F})}$ and $m^{(\mathsf{SG})} = \sqrt{q_{EA}}$.

Application to Unconstrained Random Graphs: Small World in general

We can even approximately solve the related Ising model directly by the mean-field approach. In general, given a Poissonian random graph superimposed onto a d-dimensional hypercube lattice with couplings J_0, we have:

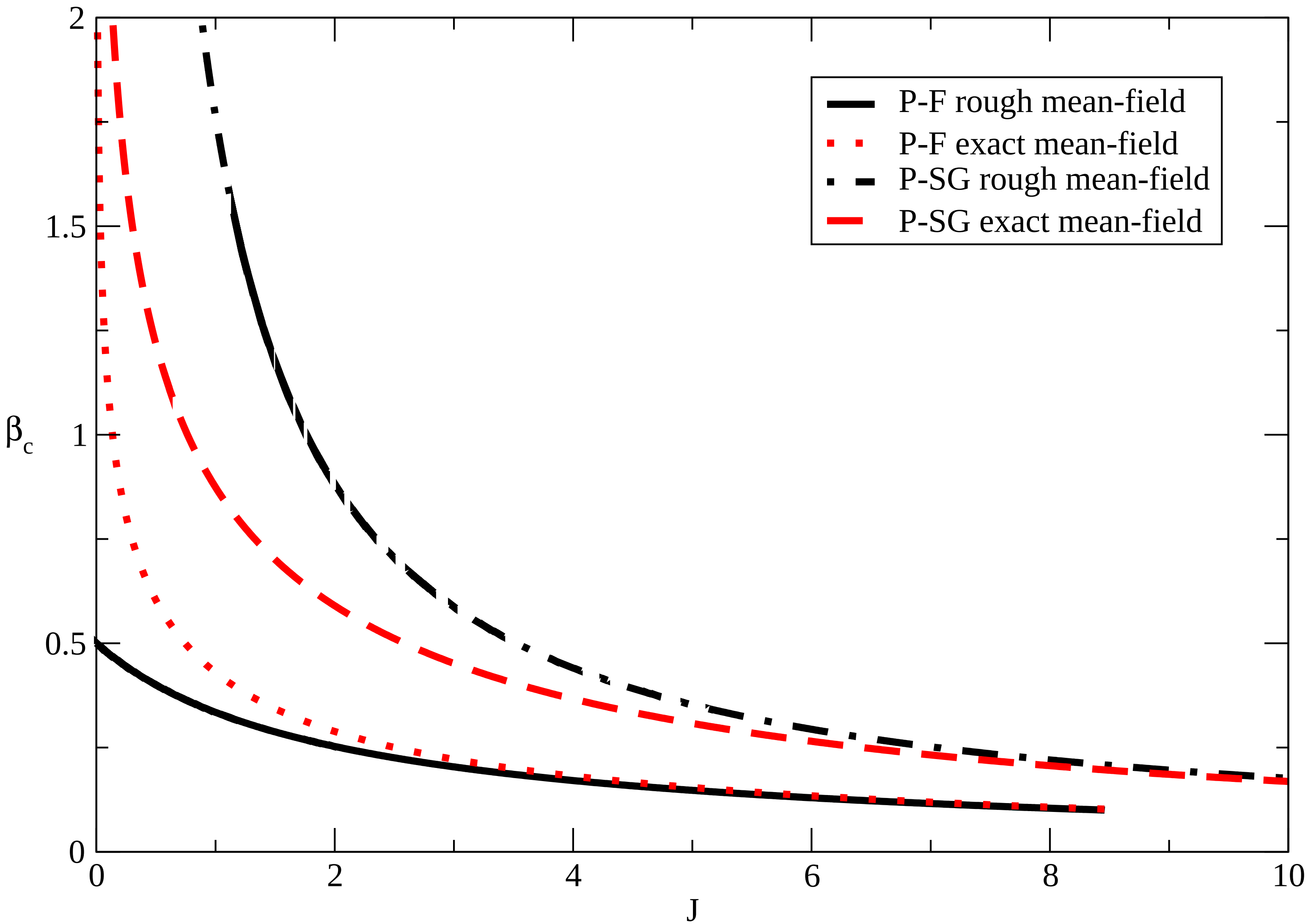
$$m_I \simeq \tanh\left[\beta^{(I)} J^{(I)} N m_I + 2d\beta^{(I)} J_0^{(I)} m_I\right],$$

from which by applying the mapping we get

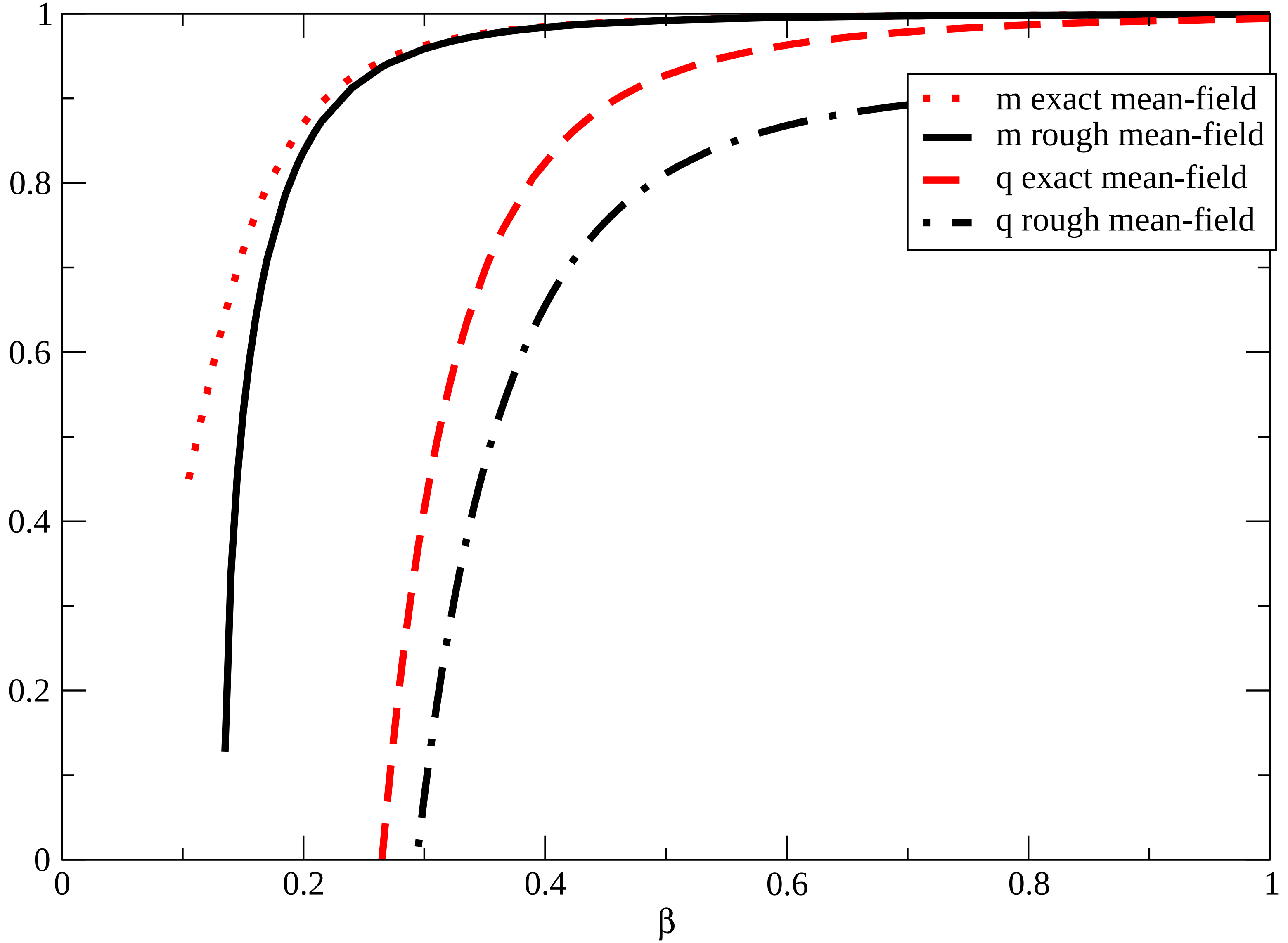
$$\begin{aligned} m^{(\mathrm{F})} &\simeq \tanh\left[c\int d\mu(J_b)\tanh(\beta J_b) m^{(\mathrm{F})} + 2d\beta J_0 m^{(\mathrm{F})}\right], \\ m^{(\mathrm{SG})} &\simeq \tanh\left[c\int d\mu(J_b)\tanh(\beta J_b) m^{(\mathrm{SG})} + 2d\tanh^{-1}(\tanh^2(\beta J_0)) m^{(\mathrm{SG})}\right]. \end{aligned}$$

The mapping establishs clearly that on unconstrained random graphs, the Univeral Mean-Field-like critical behavior takes place!

Small World, D=1, c=2, J_0=1, P-F and P-SG lines



Effective fields m and q_{EA} J_0=1, J=6, c=2



Application to Complex Networks

In this case, the graphs $\Gamma_{\boldsymbol{g}}$ are drawn out from an ensemble $\mathcal{G}$ with some given distribution $P(g)$ characterized by some given constrain $\Rightarrow$ this implies that the $\{g_b\}$ are no longer independent random variables

$$P(\boldsymbol{g}) \neq \prod_{b\in\Gamma_f} P(g_b).$$

$\Rightarrow$ the mapping cannot be applied to this ensemble.
However, in any fixed (quenched) sample $\Gamma_{\boldsymbol{g}}$ the $\{J_b\}$ are again independent, ($g_b \equiv 1$ if $b \in \Gamma_{\boldsymbol{g}}$) so that we have

$$d\mathcal{P}_{\Gamma_{\boldsymbol{g}}}(\{J_b\}) = \prod_{b\in\Gamma_{\boldsymbol{g}}} d\mu\,(J_b)$$

and the mapping can be applied to the single graph if $\Gamma_{\boldsymbol{g}}$ is infinite dimensional in the weak sense *a.s.w.r.* to $P(\boldsymbol{g})$. If now the graphs obey to a self-average property, we can simply exploit the solution known for the complex graph with uniform coupling.

In particular, if we condiser the random graphs where the only constraint is $p(k)$, for a uniform coupling we have

Dorogovtesv *et al* 2002 - Leone *et al* 2002:

$$\tanh(\beta_c^{(I)} J^{(I)}) = \frac{\langle k \rangle_p}{\langle k^2 \rangle_p - \langle k \rangle_p},$$

and by applying the mapping we get the exact P-F and P-SG critical lines for a complex network with coupling disorder:

$$\int d\mu(J_b) \tanh(\beta_c^{(\mathrm{F})} J_b) = \frac{\langle k \rangle_p}{\langle k^2 \rangle_p - \langle k \rangle_p}, \qquad \mathrm{P-F}$$

Kim *et al* 2005 – Ostilli 2006

$$\int d\mu(J_b) \tanh^2(\beta_c^{(\mathrm{SG})} J_b) = \frac{\langle k \rangle_p}{\langle k^2 \rangle_p - \langle k \rangle_p}, \qquad \mathrm{P-SG}$$

As in the previous cases, we can study also the critical behavior of the coupling disordered model by the effective substitutions

$$\tanh(\beta^{(I)} J^{(I)}) \to \int d\mu(J_b) \tanh(\beta J_b), \qquad \mathrm{P-F},$$

$$\tanh(\beta^{(I)} J^{(I)}) \to \int d\mu(J_b) \tanh^2(\beta J_b), \qquad \mathrm{P-SG},$$

to be plugged into the expressions of the related Ising model.

Key ingredients of the "proof"

1. High temperature expansion

2. Replica trick : $z^n, \ n \in \mathsf{R} \rightarrow z^n, \ n \in \mathsf{N}$

3. Or, alternatively, $\log(z) = \log(1 + (z-1)) = \sum_n \frac{(-1)^{n-1}}{(z-1)^n}$

4. $D(\Gamma) \rightarrow \infty$

There is no functional to be extremized $\Rightarrow$ as a consequence:

NO ANSATZ IS REQUIRED!

High Temperature Expansion

$$Z(\{J_b\}) = 2^N \prod_{b\in\Gamma} \cosh(K_b) \sum_{\gamma} \prod_{b\in\gamma} \tanh(K_b)\,,$$

where the sum runs over all the multipolygon (closed paths) γ and

$$K_b \equiv \beta J_b.$$

Averaging over the disorder we have

$$\int d\mathcal{P}\left(\{J_b\}\right)\log\left(Z\left(\{J_b\}\right)\right) \;=\; \int d\mathcal{P}\left(\{J_b\}\right)\log\left(2^N \prod_{b\in\Gamma}\cosh(K_b)\right)$$
$$+\int d\mathcal{P}\left(\{J_b\}\right)\log\left(\sum_{\gamma}\prod_{b\in\gamma}\tanh(K_b)\right),$$

from which we get

$$-\beta F = N\log(2) + \sum_{b\in\Gamma}\int d\mu_b \log\left(\cosh(K_b)\right) + \phi,$$

where the non trivial part ϕ is given by

$$\phi \equiv \int d\mathcal{P}\left(\{J_b\}\right)\log\left(\sum_{\gamma}\prod_{b\in\gamma}\tanh(K_b)\right).$$

Replica Trick

Let

$$P\left(\{\tanh(K_b)\}\right) \equiv \sum_{\gamma} \prod_{b \in \gamma} \tanh(K_b).$$

To evaluate the free energy we need to consider the averages of P^n for $n \in \mathrm{N}$

$$P^{(n)} \equiv \int d\mathcal{P}\left(\{J_b\}\right) P^n \left(\{\tanh(K_b)\}\right).$$

The free energy term ϕ can be obtained in terms of $P^{(n)}$, via the replica method:

$$\phi = \lim_{n \to 0} \frac{P^{(n)} - 1}{n}.$$

Exploiting $D \to \infty$

Let us consider a measure $d\mu$ with zero average. The general evaluation of the term $P^{(2n)}$ is a formidable task, in which one has to deal with $2n$ paths which can overlap in all the possible ways with $0, 2, \dots, 2n$ overlaps. Nevertheless, if $D \to \infty$ strictly, up to terms $O(1/D)$ the only possible overlap correspond to the "two to two" one $\Rightarrow$ in the thermodynamic limit we have

$$P^{(2n)} \propto \left(P\left(\tanh^2(K_b)\right)\right)^n, \forall \beta \leq \beta_c^{(\mathrm{SG})}.$$

If $D(\Gamma) \to \infty$ only weakly, the above equality holds only in the left-limit $\beta \to \beta_c^{(\mathrm{SG})^-}$. For this last statement it is crucial to use the fact that:

The critical behavior of the system is determined by the paths of arbitrarily large length

By using the last equation with the replica formula we have

$$\phi \propto \phi_I(\tanh^2(K_b))$$

and the mapping follows immediately.

Conclusions and Outlooks

- An exact and general method to get the upper critical surface and the upper critical behavior of quenched models
- No ansatz
- In the P phase we can solve as many random models as many non random models we are able to solve, analytically or numerically

- A method particularly suitable for models defined over networks. In particular we have:
  - Great semplification
  - No tree-like ansatz required! (The related Ising model is Fully Connected)
  - For models on Poissonian Graphs: A clear Universal Mean-Field-like behavior
  - For Complex Networks we are able to study exactly the random versions ($d\mu(J_b) \neq \delta(J_b' - J_b)dJ_b'$)

Open problems

- Generalization to other (non Ising) models? (for p-spin models is already clear)
- What happens adding h? Can we formulate the mapping rigorously also when $h \neq 0$?
- What happens when there are important constrains on the $\{g_b\}$? The effective substitution $\langle k \rangle_p \to \frac{\langle k^2 \rangle_p}{\langle k \rangle_p}$ works well for power law distributions not too flat, $\gamma \geq 5$. Can we formulate the mapping more in general?

References

Mainly for the strict infiniteness:
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