Colouring Random Regular Graphs

Combinatorial Approach

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(Joint work with Díaz, Kaporis, Kemkes, Kirousis and Wormald)

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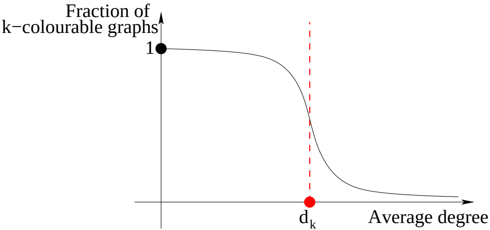
Threshold of k-colourability

$\mathcal{G}(n,p)$ *(Erdős & Rényi)*

Average degree: d

$p = d/(n-1) \sim d/n$



Theorem (Achlioptas and Friedgut '99)

For $k \geq 3$, there is $d_k(n)$ s.t. $\forall \epsilon > 0$:

- $\mathbf{P}\left(\mathcal{G}(n, \frac{d_k(n)-\epsilon}{n}) \text{ is } k\text{-col.}\right) \to 1.$
- $\mathbf{P}\left(\mathcal{G}(n, \frac{d_k(n)+\epsilon}{n}) \text{ is } k\text{-col.}\right) \to 0.$

Value of d_k

Rigorous bounds on d_k

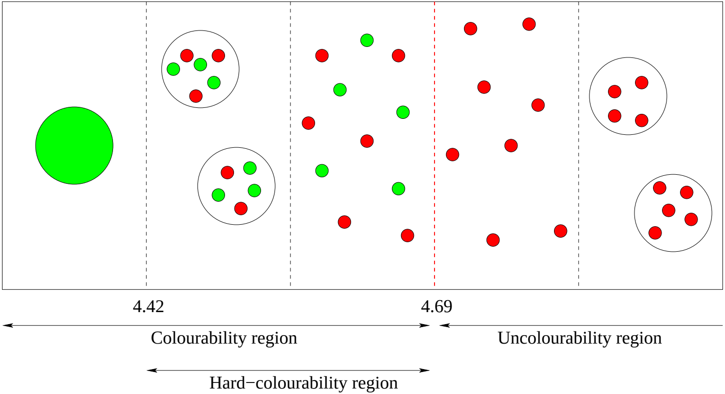
- $d_k(n) \leq 2k \log k - \log k - 1 + o(1)$ (Łuczak '91)
- $d_k(n) \geq 2k \log k - \log k - 2 + o(1)$ (Achlioptas & Naor '05)

- $d_3 \approx 4.7$ *(Experimental)* (Culberson & Gent '91)
- $d_3 \approx 4.69$ *(Replica method)* (Braunstein, Mulet, Pagnani, Weigt & Zecchina '03)

Rigorous bounds on d_3

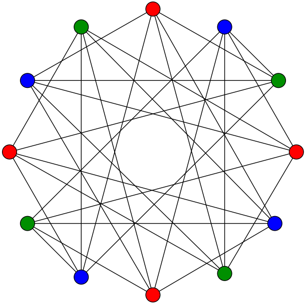
- $d_3 \geq 3.35$ *(k-core)* (Pittel, Spencer & Wormald '96)
- $d_3 \geq 4.03$ *(Algorithmic)* (Achlioptas & Moore '03)
- $d_3 \leq 4.99$ *(1MM)* (Kaporis, Kirousis & Stamatiou '00)

Hard-colourability region



(Braunstein, Mulet, Pagnani, Weigt & Zecchina '03)
(Krząkała, Pagnani & Weigt '04)

Random regular graphs



$G \in \mathcal{G}_{5,n}$

$$\chi(G) = 3$$

Colouring random regular graphs

Theorem (Molloy and Reed '92)

For $d \geq 6$, $\chi(\mathcal{G}_{d,n}) \geq 4$ a.a.s.

Theorem (Achlioptas and Moore '04)

Let k be the smallest integer such that $d < 2k \log k$.

- $\chi(\mathcal{G}_{d,n}) = k$ or $k + 1$ or $k + 2$ a.a.s.
- If $d > (2k - 1) \log k$, then $\chi(\mathcal{G}_{d,n}) = k + 1$ or $k + 2$ a.a.s.

Theorem (Shi and Wormald '07)

- $\chi(\mathcal{G}_{4,n}) = 3$ a.a.s.
- $\chi(\mathcal{G}_{6,n}) = 4$ a.a.s.
- $\chi(\mathcal{G}_{5,n}) = 3$ or 4 a.a.s.

Colouring random regular graphs

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5-regular graphs

Non-rigorous Results (*Survey Propagation*), Krząkała et al. '04

- Are 5-regular graphs 3-colourable? **YES**
- Hard colourability region (Greedy algorithms will fail)

$\Downarrow$

Combinatorial approach

Main Result

- F: continuous function defined over $\mathcal{N}$.
- $\mathcal{N}$: 4-dim bounded polytope.
- F has local maximum at $(1/9, 1/9, 1/9, 1/9)$.

Maximum Hypothesis (MH)

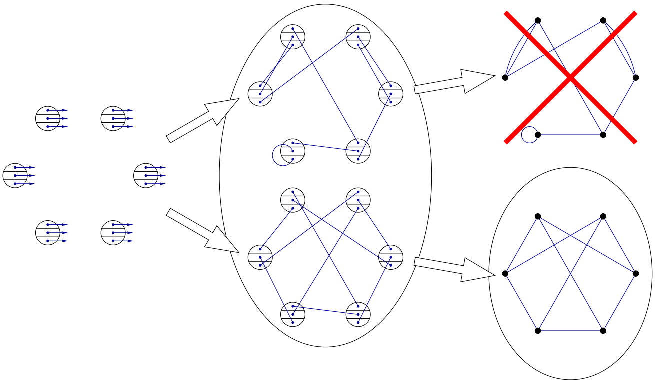
F has a global maximum at $(1/9, 1/9, 1/9, 1/9)$.

(Experimentally verified) *Rigorous proof in progress!*

Theorem

Under the MH, a random 5-regular graph can be 3-coloured with probability bounded away from 0.

Pairing Model



$\mathcal{P}_{d,n}$ $\mathcal{G}_{d,n}$

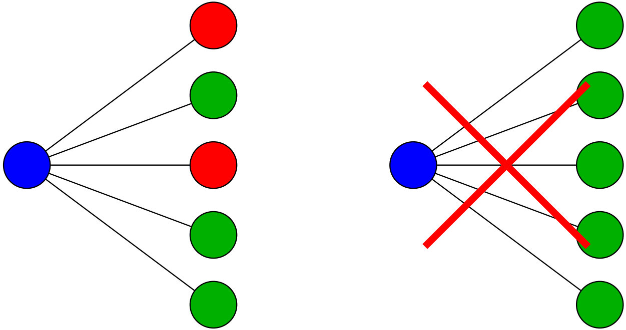
The second moment method

Let $X \geq 0$ be a discrete random variable.

- $\mathbf{P}(X > 0) \geq \dfrac{(\mathbf{E}X)^2}{\mathbf{E}X^2}$
- Thus, if $\mathbf{E}X^2 = \Theta\big((\mathbf{E}X)^2\big)$ then $X > 0$ with probability bounded away from 0.
- This approach fails for ordinary colourings!

Balanced Rainbow 3-Colourings

- **Balanced:** $n/3$ vertices of each colour.
- **Rainbow:** No single vertex can legally change colour. (No vertex has mono-chromatic neighbors).



The moments

- First moment:

$$\mathbf{E}X = \frac{\left|\{(G,C) \mid G \in \mathcal{P}_{5,n}, C \models G\}\right|}{|\mathcal{P}_{5,n}|}$$

- Second moment:

$$\mathbf{E}X^2 = \frac{\left|\{(G,C_1,C_2) \mid G \in \mathcal{P}_{5,n}, C_1, C_2 \models G\}\right|}{|\mathcal{P}_{5,n}|}$$

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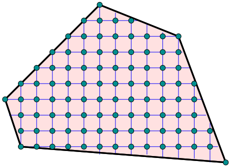
Asymptotics

$k = 1, 2$

Polytope $\mathcal{D}_k$



$\dim(\mathcal{D}_k) = r$

$\mathcal{I} = \mathcal{D}_k \cap \left(\frac{1}{n}\mathbb{Z}\right)^r$

- Exact expression:
$$\mathbf{E}X^k = \sum_{i \in \mathcal{I}} T_i$$
- Asymptotic expression:
$$\mathbf{E}X^k \sim \sum_{i \in \mathcal{I}} \mathrm{poly}_i(n)(\widehat{F}_i)^n$$
- Exponential behaviour:

  If $\widehat{F}_{i_0} \geq \widehat{F}_i$ ($\forall i \in \mathcal{I}$) then $\mathbf{E}X^k \asymp (\widehat{F}_{i_0})^n$.
- Polynomial factors:
$$\mathbf{E}X^k X^k = \Theta\left(n^{r/2}\right) \mathrm{poly}_{i_0}(n)(\widehat{F}_{i_0})^n$$
(Laplace integration technique)

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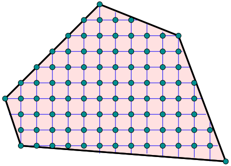

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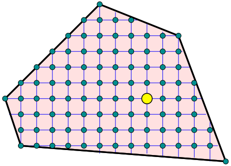

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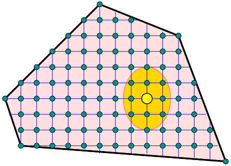

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First Moment

Lemma

$$\mathbf{E}X \sim \sqrt{\frac{2^2 3^6 5^3}{11^3} \frac{1}{(2\pi n)^2} \left(\frac{25}{24}\right)^n}$$

Second Moment

Lemma

Under the MH,

$$\mathbf{E}X^2 \sim \frac{2^2 3^{19} 5^{16}}{7^6 11^7 79^2 \sqrt{13\ 17}} \frac{1}{(2\pi n)^2} \left(\frac{25}{24}\right)^n$$

Proof.

- Location of $\max_{\mathcal{D}_2}(\widehat{F})$ (Requires MH).
  Dificult! $\dim(\mathcal{D}_2) = 301$.
- Application of Laplace integration technique.

Maximisation of $\widehat{F}$

- We transform the problem *(Lagrange multipliers)*
  - $\max_{\mathcal{D}_2}(\widehat{F}) \Longleftrightarrow \max_{\mathcal{N}}(F)$
  - **Good:** $\dim \mathcal{N} = 4$.
  - **Bad:** F is implicit.
- We find $\max_{\mathcal{N}}(F)$ numerically
  - Fine sweep of the domain (step $= 10^{-4}$).
  - IBM-Mare Nostrum (Barcelona Supercomputing Center): 2.268 dual 64-bit processor blade nodes, 4.536 2.2 GHz PPC970FX processors.

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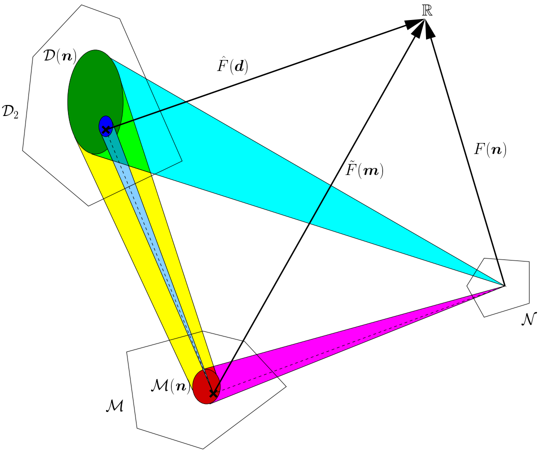

- $\mathcal{D}_2 \subset \mathbb{R}^{324}$, dim $\mathcal{D}_2 = 301$
- $\mathcal{M} \subset \mathbb{R}^{36}$, dim $\mathcal{M} = 13$
- $\mathcal{N} \subset \mathbb{R}^{9}$, dim $\mathcal{M} = 4$
- $\mathcal{M}(\boldsymbol{n}) \subset \mathbb{R}^{36}$, dim $\mathcal{M}(\boldsymbol{n}) = 9$
- $\log \widetilde{F}$ concave in $\mathcal{M}(\boldsymbol{n})$

First Result

Theorem

Under the MH, random 5-regular configurations can be 3-coloured with probability bounded away from 0.

Proof.

$$\frac{(\mathbf{E}[X])^2}{\mathbf{E}\left[X^2\right]} \sim \frac{7^6 11^4 79^2 \sqrt{13\,17}}{3^{13} 5^{13}} \approx 0.08211508230\ldots$$

From configurations to simple graphs

Theorem

Under the MH, random 5-regular graphs can be 3-coloured with probability bounded away from 0.

Proof.

$$\frac{\left(\mathbf{E}_{\mathcal{G}_{5,n}}[X]\right)^2}{\mathbf{E}_{\mathcal{G}_{5,n}}\left[X^2\right]} = \Theta\left(\frac{(\mathbf{E}[X])^2}{\mathbf{E}\left[X^2\right]}\right) = \Theta(1).$$

Further work

- Formal proof of the MH. *Work in progress*
- Extend result to a.a.s. *Done! (Kemkes and Wormald '05)*
- Is there an algorithm which provably 3-colours 5-regular graphs?
- May this approach be extendable for computing the exact chromatic number for general d-regular graphs?

The End

Thank you!