Physics and Analytical Approaches in Turbulent Transfer

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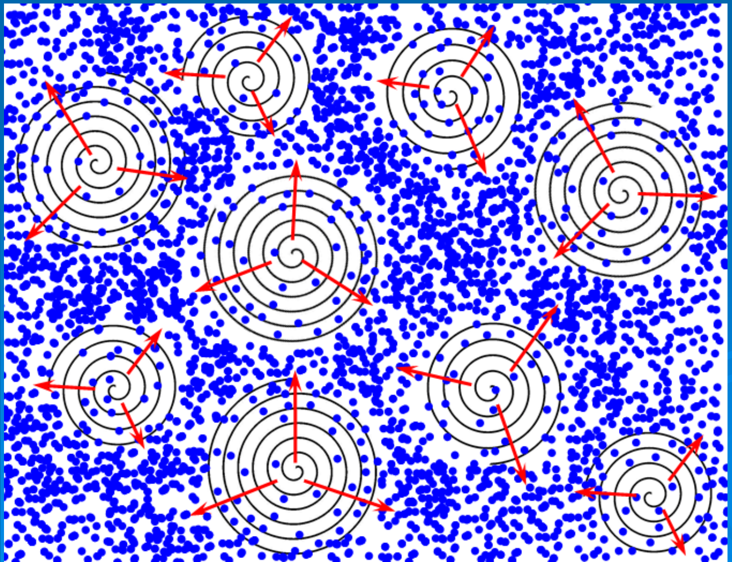


Governing Equations

- **Particles in a turbulent flow:**

$$\frac{\partial n}{\partial t} + \nabla\cdot (n\,\mathbf{v}) - D_m\,\Delta n = 0$$

- **Chemical reactions or Phase transitions:**

$$\frac{\partial n_\beta}{\partial t} + \nabla \cdot (n_\beta \mathbf{v}_\beta) = \alpha_\beta I(n, T_f) + \nabla\cdot (D_\beta \nabla n_\beta)$$

- **Magnetic field in a turbulent flow:**

$$\partial_t \mathrm{B} = (\mathrm{B}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathrm{B} + \eta\,\Delta \mathrm{B},$$

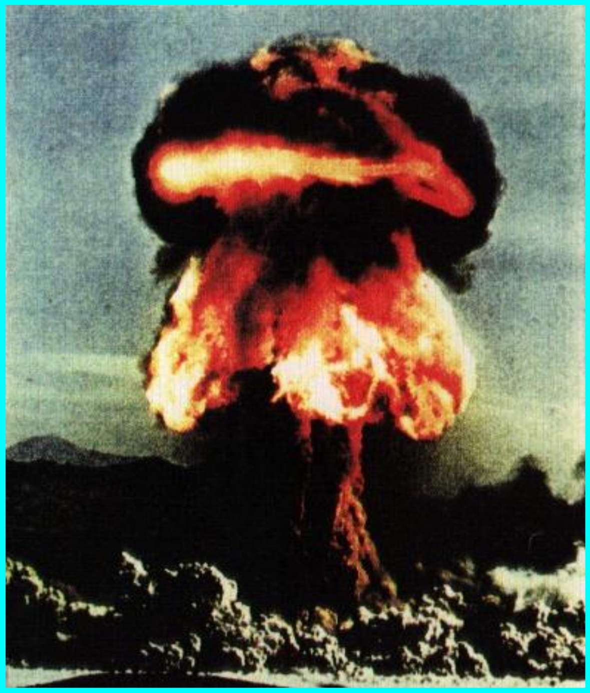
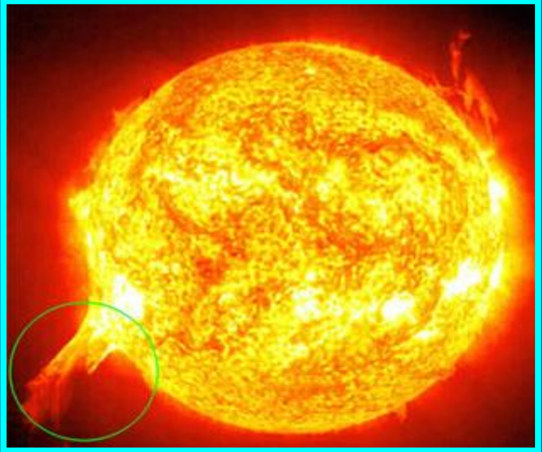
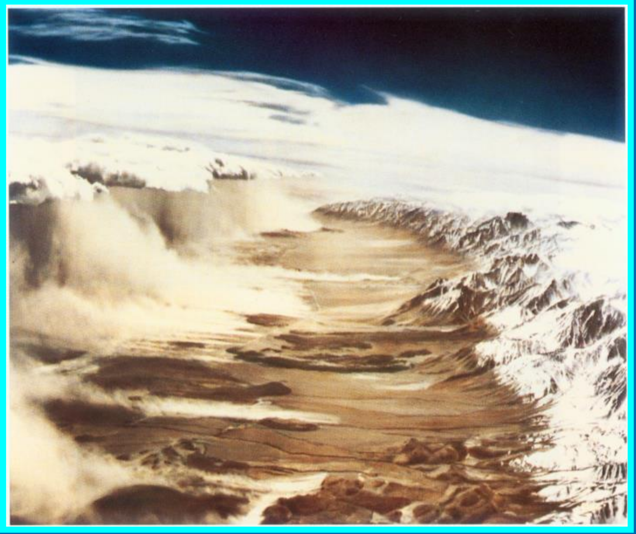
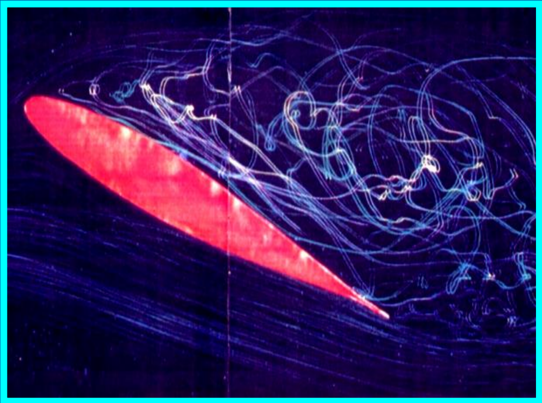
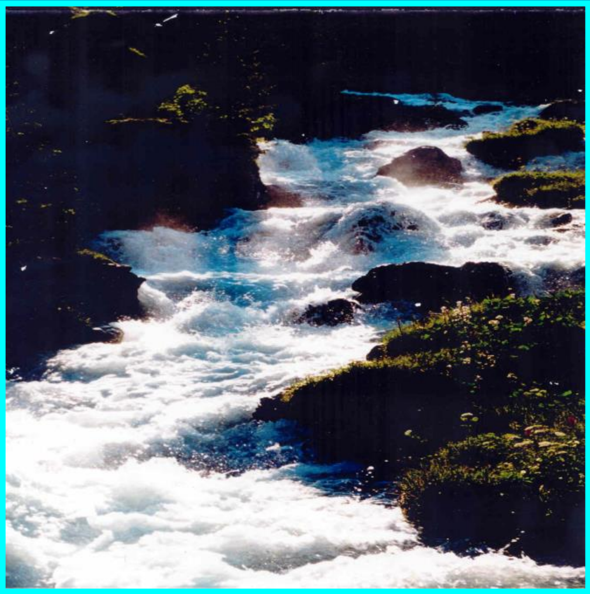
Turbulent Transfer

- What is long-term large-scale evolution

(mean-field theory that allow to determine dynamics in the time and spatial scales which are much larger than the turbulent scales)

- What is the short-term evolution in small-scales

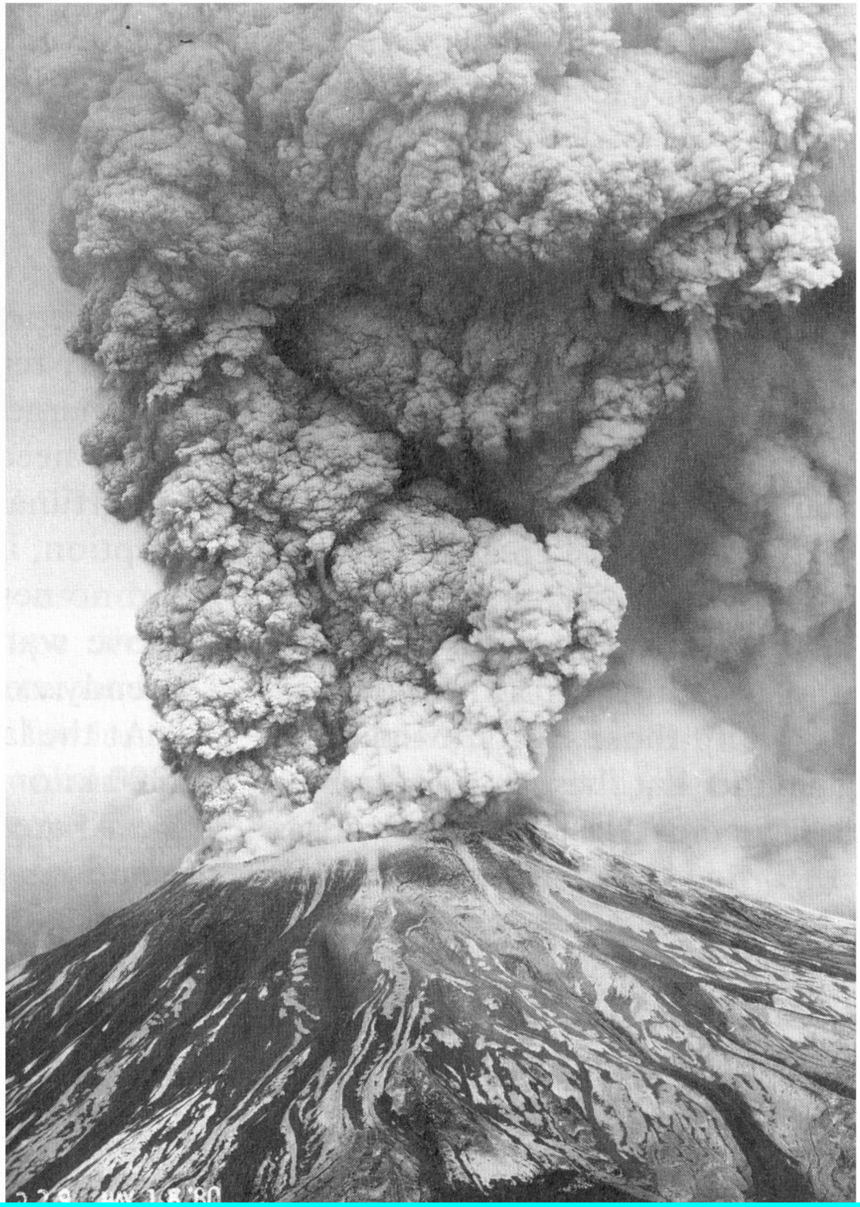
(theory of fluctuations that allow to determine dynamics in the time and spatial scales which are much smaller than the maximum turbulent scales)

SAND STORM IN IRAQ, APRIL 26, 2005

SAND STORM APPROACHING ISRAEL IN MAY 2007

Mount GALUNGGUNG ,west Java in August 1982, M.-A. del Marmol



Mount St. Helen volcano on 18 May 1980, US Geological Survey

Laminar to turbulent flow transition

“Turbulence” in art

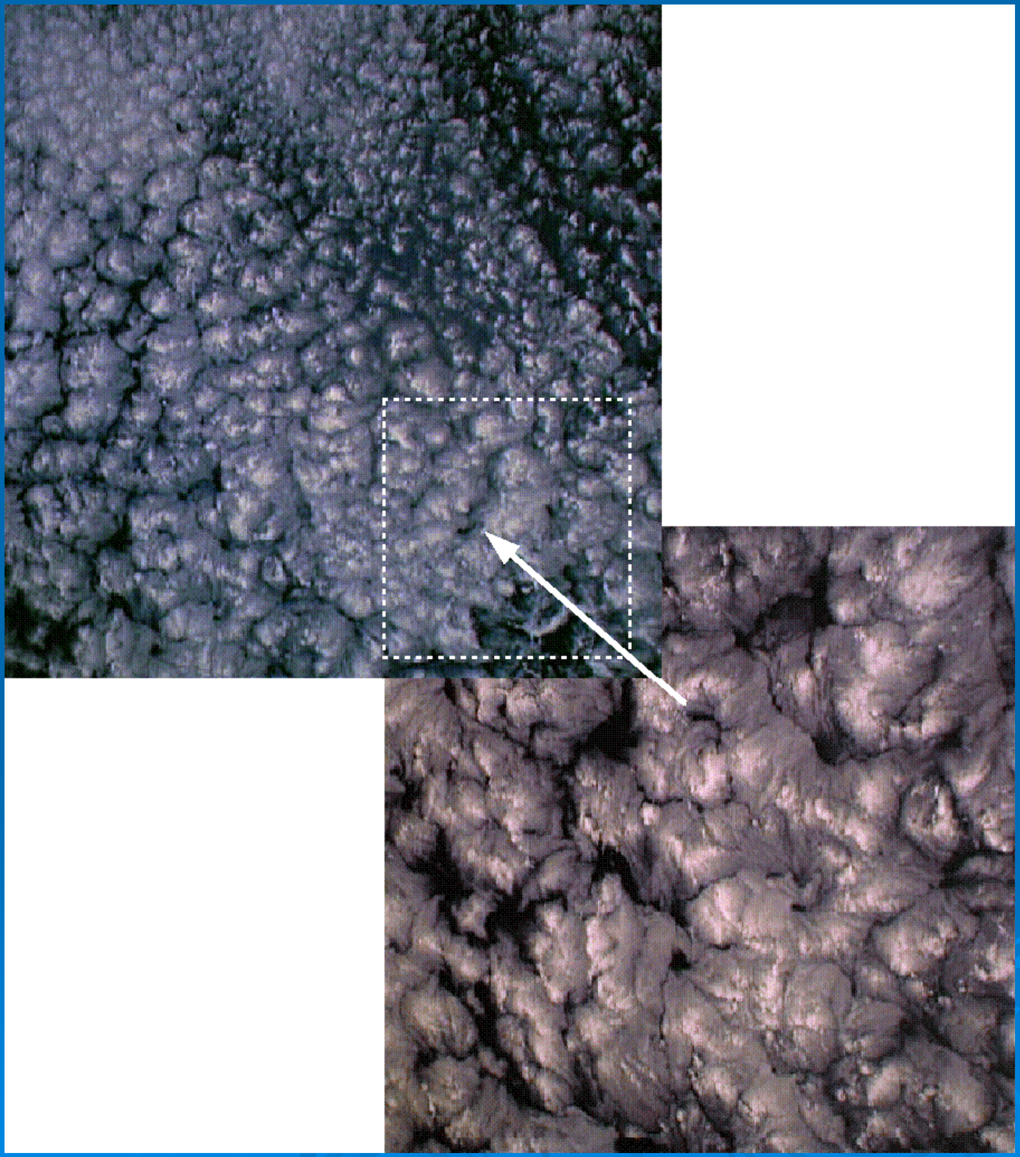
Vincent van Gogh *The Starry Night*, June 1889, The Museum of Modern Art, New York

Turbulent convection in atmosphere

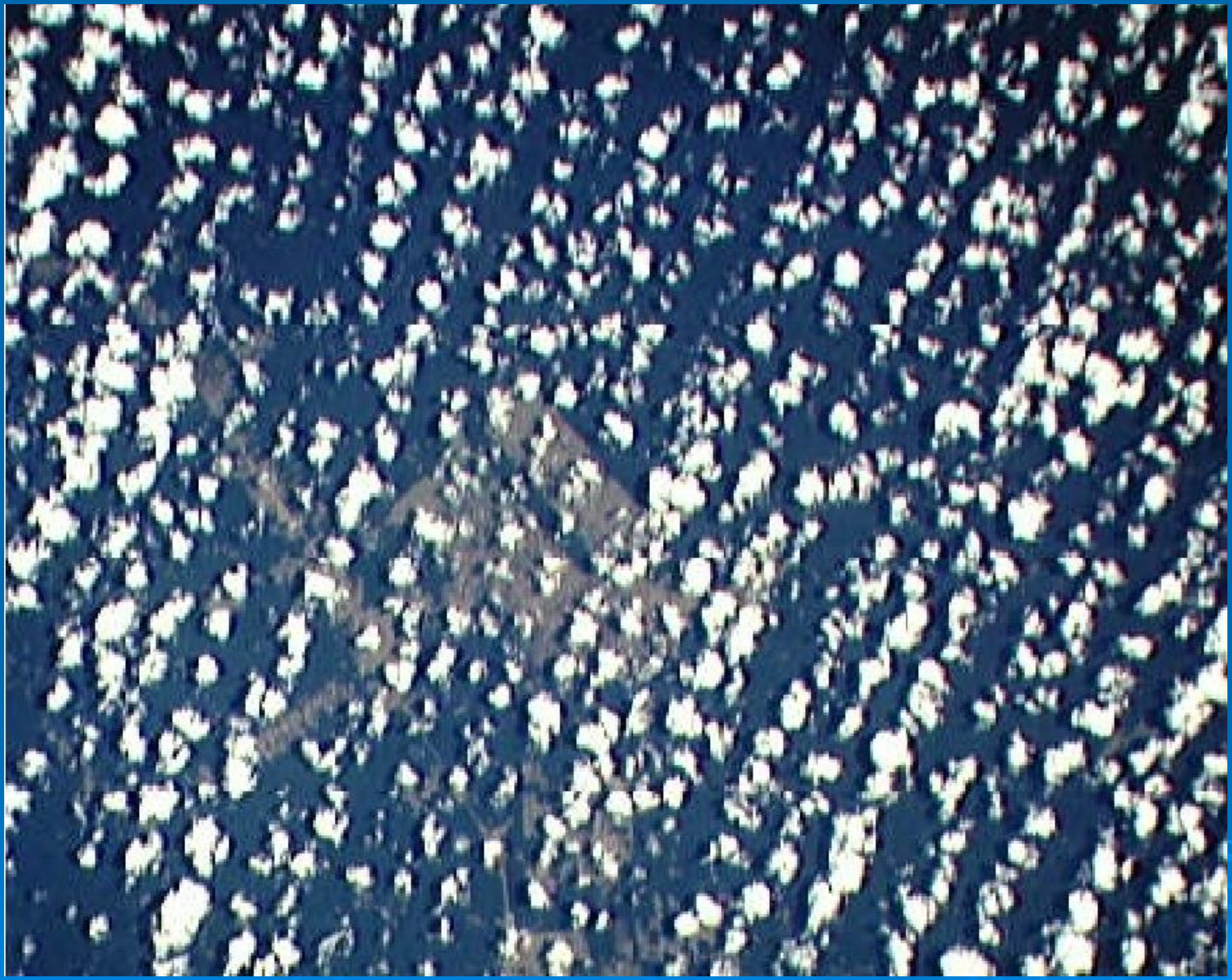
Convective plumes are seen due to condensation of water vapor in updraughts




Closed cloud cells over the Atlantic Ocean



Cloud "streets" over the Amazon River



Mean-Field Approach

- **Instantaneous particle number density:**

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\,\mathbf{v}) - D_m\,\Delta n = 0$$

- **Mean particle number density:** $n = N + n', \qquad N = \langle n \rangle$

$$\frac{\partial N}{\partial t} + \nabla \cdot \langle n\,\mathbf{v} \rangle - D_m\,\Delta N = 0$$

- **Fluctuations of particle number density:**

$$\frac{\partial n'}{\partial t} + \nabla \cdot (n'\,\mathbf{v} - \langle n'\,\mathbf{v} \rangle) - D_m\,\Delta n' = -N\,(\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)N$$

- **Source of fluctuations:**

$$I_{n'} = -N\,(\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)N$$

Methods and Approximations

- **Quasi-Linear Approach** or Second-Order Correlation Approximation (SOCA) or First-Order Smoothing Approximation (FOSA)

  Pe << 1 , Re << 1 (weak nonlinearity) **Plasma Physics**

  Steenbeck, Krause, Rädler (1966); Roberts, Soward (1975); Moffatt (1978)

- **Path-Integral Approach** (delta-correlated in time random velocity field or short yet finite correlation time); **R. Feynman**

  Kraichnan-Kazantsev model of random velocity field

  Zeldovich, Molchanov, Ruzmaikin, Sokoloff (1988)

  Rogachevskii, Kleeorin (1997)

  $$\mathrm{St} = \frac{\tau}{\ell/u} \ll 1$$

- **Tau-approaches** (spectral tau-approximation, minimal tau-approximation) – third-order or high-order closure

  Re >> 1 and Pe >> 1 (strong nonlinearity); **Kinetic Theory**

  Orszag (1970); Pouquet, Frisch, Leorat (1976);

  Kleeorin, Rogachevskii, Ruzmaikin (1990); Blackman, Field (2002)

- **Renormalization Procedure** (renormalization of viscosity, diffusion, electromotive force and other turbulent transport coefficients) –

  (strong nonlinearity) **Quantum Theory**

  Moffatt (1981; 1983); Kleeorin, Rogachevskii (1994)

Particle Flux in Turbulent Flow for Pe << 1 (quasi-linear approach, non-inertial particles)

$$Pe = \frac{u_0 \ell_0}{D_m} \ll 1$$

$$\frac{\partial n'}{\partial t} + \nabla \cdot (n' \mathbf{u} - \langle n' \mathbf{u} \rangle) - D_m \Delta n' = I_{n'}$$

$$I_{n'} = -N (\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla) N$$

Solution:

$$n'(\omega, \mathbf{k}) = (D_m k^2 + i\omega)^{-1} I_{n'}(\omega, \mathbf{k})$$

Turbulent Flux of Particles:

$$\langle u_i n' \rangle = \int \langle u_i(\omega, \mathbf{k}) I_{n'}(-\omega, -\mathbf{k}) \rangle (D_m k^2 - i\omega)^{-1} d\omega \, d\mathbf{k}$$

Model of Background Turbulence: $\nabla \cdot \mathbf{u} = u_i \lambda_i$ $\qquad \lambda_i = -\frac{\nabla_i \rho}{\rho}$

$$\langle u_i(\omega, \mathbf{k}) u_j(-\omega, -\mathbf{k}) \rangle = \frac{\langle \mathbf{u}^2 \rangle E(k)}{8\pi^2 k^2 \tau_c (\omega^2 + \tau_c^{-2})} \left[\delta_{ij} - \frac{k_i k_j}{k^2} + \frac{i}{k^2} (\lambda_i k_j - \lambda_j k_i) \right]$$

Turbulent Flux of Particles: $\langle \mathbf{u} \, n' \rangle = N \, \mathbf{V}^{\mathrm{eff}} - D_T \nabla N$

Particle Flux in Turbulent Flow (Quasi-Linear Approach)

Turbulent Flux of Particles:

$$\langle \mathbf{u}\, n' \rangle = N\, \mathbf{V}^{\text{eff}} - D_T \boldsymbol{\nabla} N$$

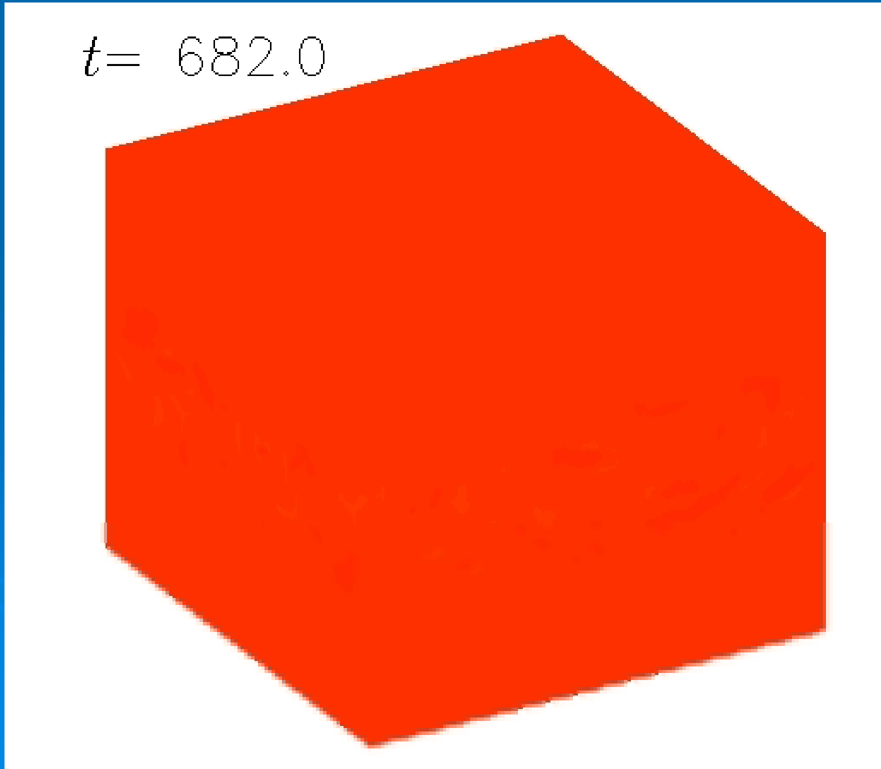
Effective Pumping Velocity:

$$\mathbf{V}^{\text{eff}} = D_T \frac{\boldsymbol{\nabla}\rho}{\rho} = -D_T \frac{\boldsymbol{\nabla} T}{T}$$

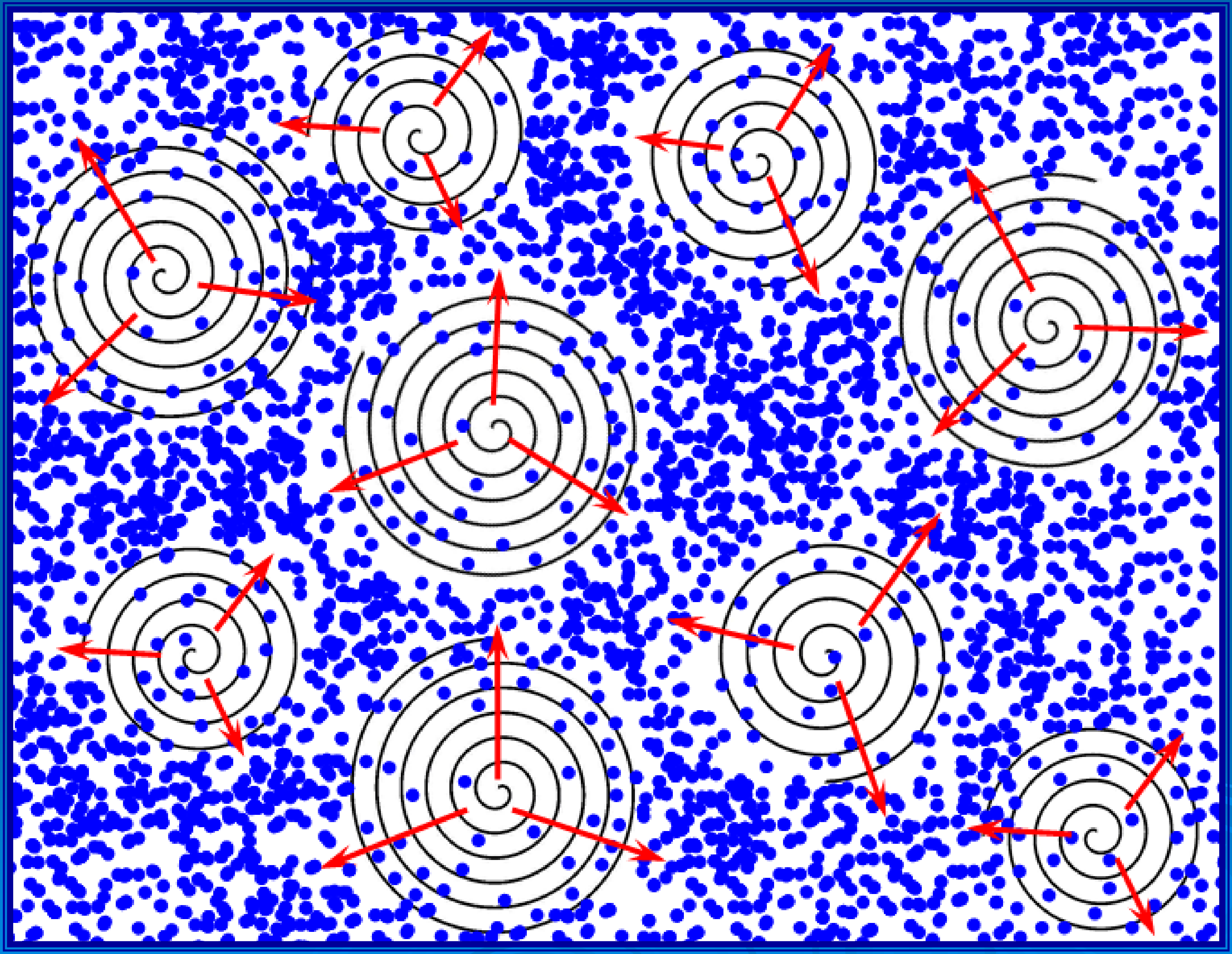
Turbulent Diffusion Coefficient:

$$Pe = \frac{u_0 \ell_0}{D_m} \ll 1$$

$$D_T = \frac{q-1}{q+1}\, \frac{u_0\, \ell_0}{3}\, \text{Pe}$$



Particle Inertia Effect



Turbulent Thermal Diffusion

$T_2 > T_1$

$\mathbf{u}'$ → a $\qquad$ ← $\mathbf{u}'$ b

$T_2 \qquad\qquad \langle \mathbf{u}' \theta' \rangle \longrightarrow \qquad\qquad T_1$

$$u' > 0 \qquad\qquad u' < 0$$

$$u'\theta' > 0 \;\Rightarrow\; \theta' > 0 \qquad\qquad \theta' < 0 \;\Leftarrow\; u'\theta' > 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$p' > 0 \qquad\qquad p' < 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$u' n' > 0 \;\Leftarrow\; n' > 0 \qquad\qquad n' < 0 \;\Rightarrow\; u' n' > 0$$

$\langle \mathbf{u}' n' \rangle \longrightarrow$

Non-diffusive mean flux of particles is in the direction of the mean heat flux (i.e., in the direction of minimum fluid temperature).

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Path-Integral Approach

$$\frac{\partial n}{\partial t} + \nabla\cdot(\mathbf{v}\, n) = D^{(n)}\,\Delta n,$$

$$n(t = t_0, x) = n_0(x)$$

$$\mathrm{St} = \frac{\tau}{\ell/u} \ll 1$$

Solution:

$$n(t, x) = \langle G(t, t_0)\, n_0[\xi(t, x|t_0)]\rangle_{\mathbf{w}}$$

Wiener trajectory:

$$\xi(t, x|s) = x - \int_s^t \mathbf{v}[\tau, \xi(t, x|\tau)]\, d\tau + \sqrt{2D^{(n)}}\,\mathbf{w}(t-s),$$

$$\langle w_i(t)\rangle_{\mathbf{w}} = 0,$$

$$\langle w_i(t+\tau) w_j(t)\rangle_{\mathbf{w}} = \tau\delta_{ij},$$

Green function:

$$b = \nabla\cdot\mathbf{v}.$$

$$G(t, t_0) = \exp\{-\int_{t_0}^t b[\sigma, \xi(t, x|\sigma)]\, d\sigma\},$$

Kraichnan-Kazantsev model of random velocity field:

$$\langle v_i(t_1, \mathbf{x}) v_j(t_2, \mathbf{y})\rangle \propto \delta\left(\frac{t_1 - t_2}{\tau_c}\right)$$

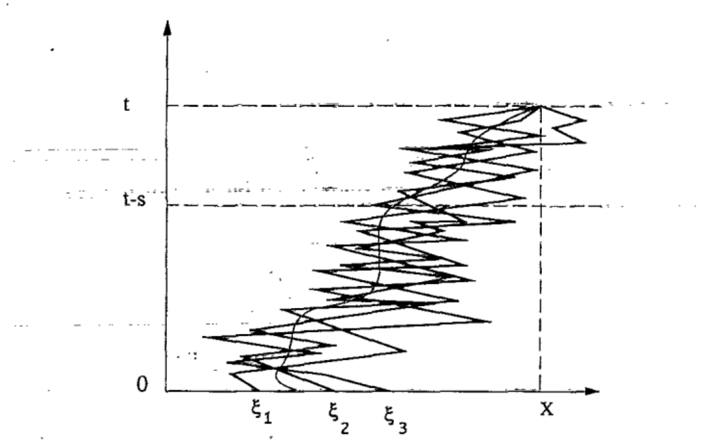


FIG. 1. Random Wiener trajectories. The random trajectories pass through the point **x** at time t.

Path-Integral Approach

$$n(t+\Delta t, \boldsymbol{x}) = \langle G(t+\Delta t, t)\, n[t, \boldsymbol{\xi}(t+\Delta t, \boldsymbol{x}|t)]\rangle_{\mathbf{W}},$$

$$n[t, \xi_{\Delta t}] = n(t, \boldsymbol{x}) + \frac{\partial n(t, \boldsymbol{x})}{\partial x_m}(\xi_{\Delta t} - \boldsymbol{x})_m + \frac{1}{2}\frac{\partial^2 n(t, \boldsymbol{x})}{\partial x_m \partial x_l}(\xi_{\Delta t} - \boldsymbol{x})_m \times (\xi_{\Delta t} - \boldsymbol{x})_l + \ldots.$$

$$\begin{aligned}[\xi_{t_2-t_1} - \boldsymbol{x}]_m &= -\int_0^{t_2-t_1}\left[\mathrm{v}_m(t,\boldsymbol{x}) + \frac{\partial \mathrm{v}_m(t,\boldsymbol{x})}{\partial x_l}\left(-\mathrm{v}_l(t,\boldsymbol{x})s + \sqrt{2D^{(n)}}\,\mathrm{w}_l(s)\right)\right] ds \\ &\quad + \sqrt{2D^{(n)}}\,\mathrm{w}_m(t_2 - t_1) \\ &= -\mathrm{v}_m\,(t_2 - t_1) + \frac{1}{2}\mathrm{v}_l\,\frac{\partial \mathrm{v}_m}{\partial x_l}(t_2 - t_1)^2 - \sqrt{2D^{(n)}}\,\frac{\partial \mathrm{v}_m(t,\boldsymbol{x})}{\partial x_l}\end{aligned}$$

$$\begin{aligned}n[t+\Delta t, \xi_{\Delta t}] &= n(t,\boldsymbol{x}) + \frac{\partial n(t,\boldsymbol{x})}{\partial x_m}\left[-\mathrm{v}_m\,\Delta t + \frac{1}{2}\mathrm{v}_l\,\frac{\partial \mathrm{v}_m}{\partial x_l}(\Delta t)^2\right] \\ &\quad + \frac{1}{2}\frac{\partial^2 n(t,\boldsymbol{x})}{\partial x_m \partial x_l}\left[\mathrm{v}_m \mathrm{v}_l\,(\Delta t)^2 + 2D^{(n)}\mathrm{w}_m \mathrm{w}_l\right] + O[(\Delta t)^3]\end{aligned}$$

$$\begin{aligned}b(\sigma, \xi_\sigma) &= b(t,\boldsymbol{x}) + \frac{\partial b(t,\boldsymbol{x})}{\partial x_m}(\xi_\sigma - \boldsymbol{x})_m + \ldots \\ &= b(t,\boldsymbol{x}) + \frac{\partial b(t,\boldsymbol{x})}{\partial x_m}\left[-\mathrm{v}_m \sigma + \sqrt{2D^{(n)}}\mathrm{w}_m(\sigma)\right] + \ldots.\end{aligned}$$

$$\begin{aligned}G(t+\Delta t, t) &= 1 - b\,\Delta t + \frac{1}{2}\mathrm{v}_m \frac{\partial b}{\partial x_m}(\Delta t)^2 + \frac{1}{2}b^2(\Delta t)^2 \\ &\quad - \sqrt{2D^{(n)}}\frac{\partial b}{\partial x_m}\int_t^{t+\Delta t}\mathrm{w}_m\, d\sigma + O[(\Delta t)^3].\end{aligned}$$

Mean-Field Equation
Path-Integral Approach

Mean-Field Equation:

$$\frac{\partial \overline{n}}{\partial t} + \boldsymbol{\nabla} \cdot \left[\left(\overline{V} + V^{\rm eff} \right) \overline{n} \right] = \nabla_m \nabla_p \left[\left(D^{(n)} \delta_{mp} + D_{mp}^{(T)} \right) \overline{n} \right],$$

Turbulent Diffusion Tensor:

$$D_{mp}^{(T)} = \langle \tau u_m \, u_p \rangle$$

Turbulent Flux of Particles:

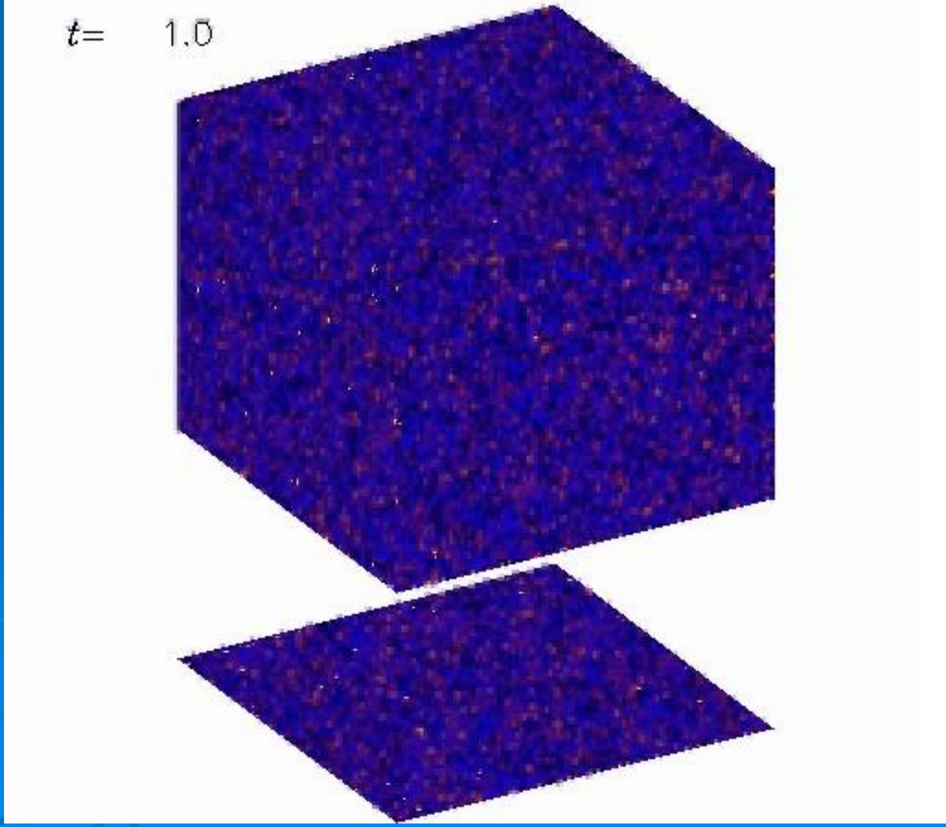
$$\langle \mathbf{u} \, n' \rangle = N \, \mathbf{V}^{\rm eff} - D_T \boldsymbol{\nabla} N$$

Effective Pumping Velocity:

$$\mathbf{V}^{\rm eff} = -\tau \, \langle \mathbf{u} \, (\boldsymbol{\nabla} \cdot \mathbf{u}) \rangle = -\alpha \, D_T \frac{\boldsymbol{\nabla} T}{T}$$

T. Elperin, N. Kleeorin, I. Rogachevskii,
Phys. Rev. Lett. **76**, 224 (1996)

N.E.L. Haugen, N. Kleeorin, I. Rogachevskii ,
A. Brandenburg, Phys. Fluids 24, 075106 (2012).



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Tau Approach

$$\frac{\partial n'}{\partial t} = -Q^{(n)} + D^{(n)}\Delta n' - (\boldsymbol{u}\cdot\boldsymbol{\nabla})\overline{n} - \overline{n}(\boldsymbol{\nabla}\cdot\boldsymbol{u}).$$

$$Q^{(n)} = \boldsymbol{\nabla}\cdot(\boldsymbol{u}n' - \langle \boldsymbol{u}n'\rangle) - D^{(n)}\Delta n'.$$

$$F_i^{(n)}(\boldsymbol{k}) = \langle u_i(t,\boldsymbol{k})\, n'(t,-\boldsymbol{k})\rangle$$

$$E^{(n)}(\boldsymbol{k}) = \langle n'(t,\boldsymbol{k})\, n'(t,-\boldsymbol{k})\rangle$$

$$f_{ij}(\boldsymbol{k}) = \langle u_i(t,\boldsymbol{k})\, u_j(t,-\boldsymbol{k})\rangle$$

$$\frac{dF_j^{(n)}(\boldsymbol{k})}{dt} = -\left[(\nabla_i\overline{n}) - ik_i\,\overline{n}\right] f_{ij}(-\boldsymbol{k}) + \hat{\mathcal{M}}^{(n)}F_i^{(III)}(\boldsymbol{k}),$$

$$\frac{dE^{(n)}(\boldsymbol{k})}{dt} = -\left[F_j^{(n)}(\boldsymbol{k}) + F_j^{(n)}(-\boldsymbol{k})\right](\nabla_j\overline{n}) - \left[F_j^{(n)}(\boldsymbol{k}) - F_j^{(n)}(-\boldsymbol{k})\right](ik_j\,\overline{n})$$
$$+\hat{\mathcal{M}}^{(n)}E_n^{(III)}(\boldsymbol{k}),$$

Tau Approximation:

$$\hat{\mathcal{M}}F^{(III)}(\boldsymbol{k}) - \hat{\mathcal{M}}F^{(III,0)}(\boldsymbol{k}) = -\frac{1}{\tau_r(k)}\left[F^{(II)}(\boldsymbol{k}) - F^{(II,0)}(\boldsymbol{k})\right],$$

Turbulence model:

$$f_{ij}(\boldsymbol{k}) \equiv \langle u_i(\boldsymbol{k})\, u_j(-\boldsymbol{k})\rangle = \frac{\langle \boldsymbol{u}^2\rangle\, E(k)}{8\pi k^2}\left[\delta_{ij} - \frac{k_i\,k_j}{k^2} + \frac{i}{k^2}\,(\lambda_i\,k_j - \lambda_j\,k_i)\right],$$

Tau Approach

Equations for the correlation functions for:

- The particle flux $\left(M_j^{(II)}(\mathbf{k})\right)_F = \langle n' u_j \rangle$
- The number density fluctuations $\left(M^{(II)}(\mathbf{k})\right)_n = \langle n' n' \rangle$

The spectral τ-approximation (the third-order closure procedure)

$$\hat{D}M^{(III)}(\mathbf{k}) - \hat{D}M_0^{(III)}(\mathbf{k}) = -\frac{M^{(II)}(\mathbf{k}) - M_0^{(II)}(\mathbf{k})}{\tau_c(\mathbf{k})}$$

$$\hat{D}M_j^{III} = \langle u_j \boldsymbol{\nabla} \cdot (n' \mathbf{u}) \rangle$$

Tau Approach: Results

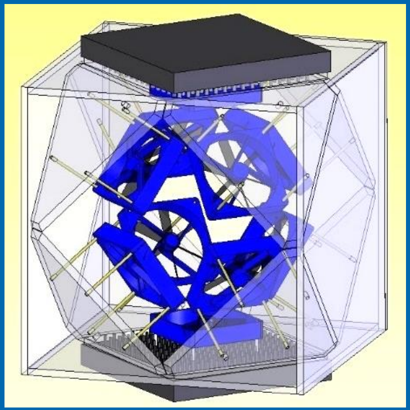
$$\langle \boldsymbol{u}\, n' \rangle = \boldsymbol{V}^{\mathrm{eff}}\, \overline{n} - D_T\, \boldsymbol{\nabla} \overline{n},$$

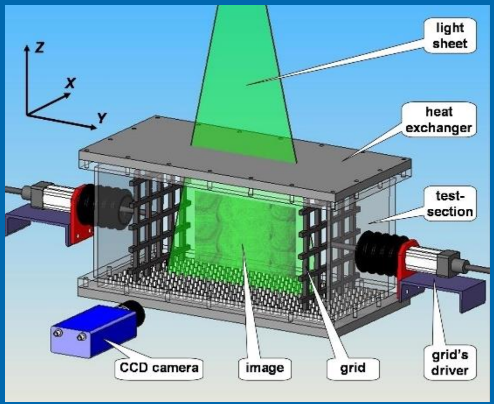
$$D_T = \frac{1}{3}\, u_0\, \ell_0, \qquad \boldsymbol{V}^{\mathrm{eff}} = -D_T \boldsymbol{\lambda} = D_T\, \frac{\boldsymbol{\nabla} \overline{\rho}}{\overline{\rho}},$$

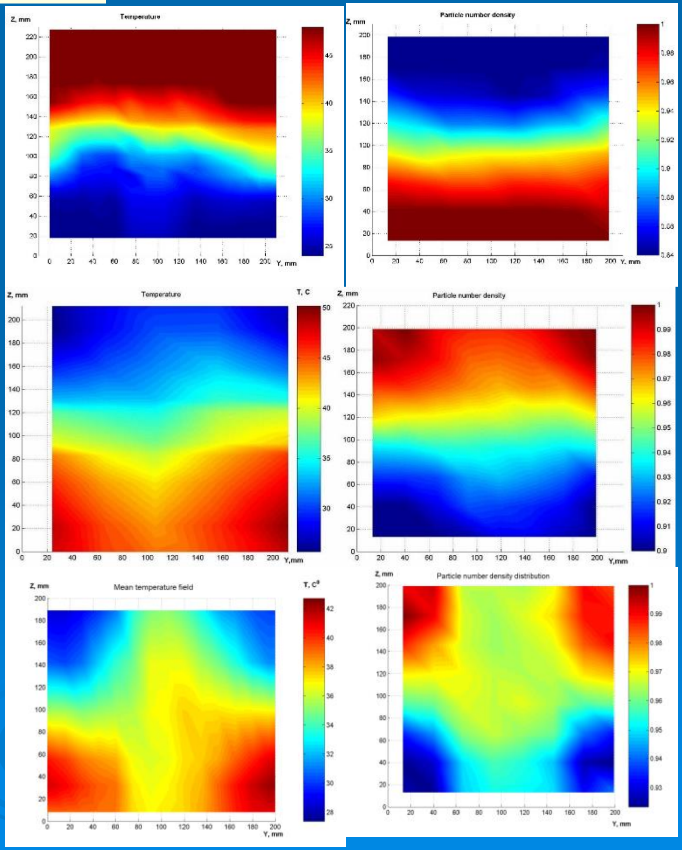
$$\langle n'^2 \rangle = \frac{8}{9}\, \ell_0^2\, \left[(\boldsymbol{\nabla} \overline{n})^2 + (\boldsymbol{\lambda} \cdot \boldsymbol{\nabla}) \overline{n}^2 \right].$$

$$\frac{\partial \overline{n}}{\partial t} + \boldsymbol{\nabla} \cdot \left[\overline{n}\, (\boldsymbol{V}_{\mathrm{g}} + \boldsymbol{V}^{\mathrm{eff}}) - (D^{(n)} + D_T)\, \boldsymbol{\nabla} \overline{n} \right] = 0,$$

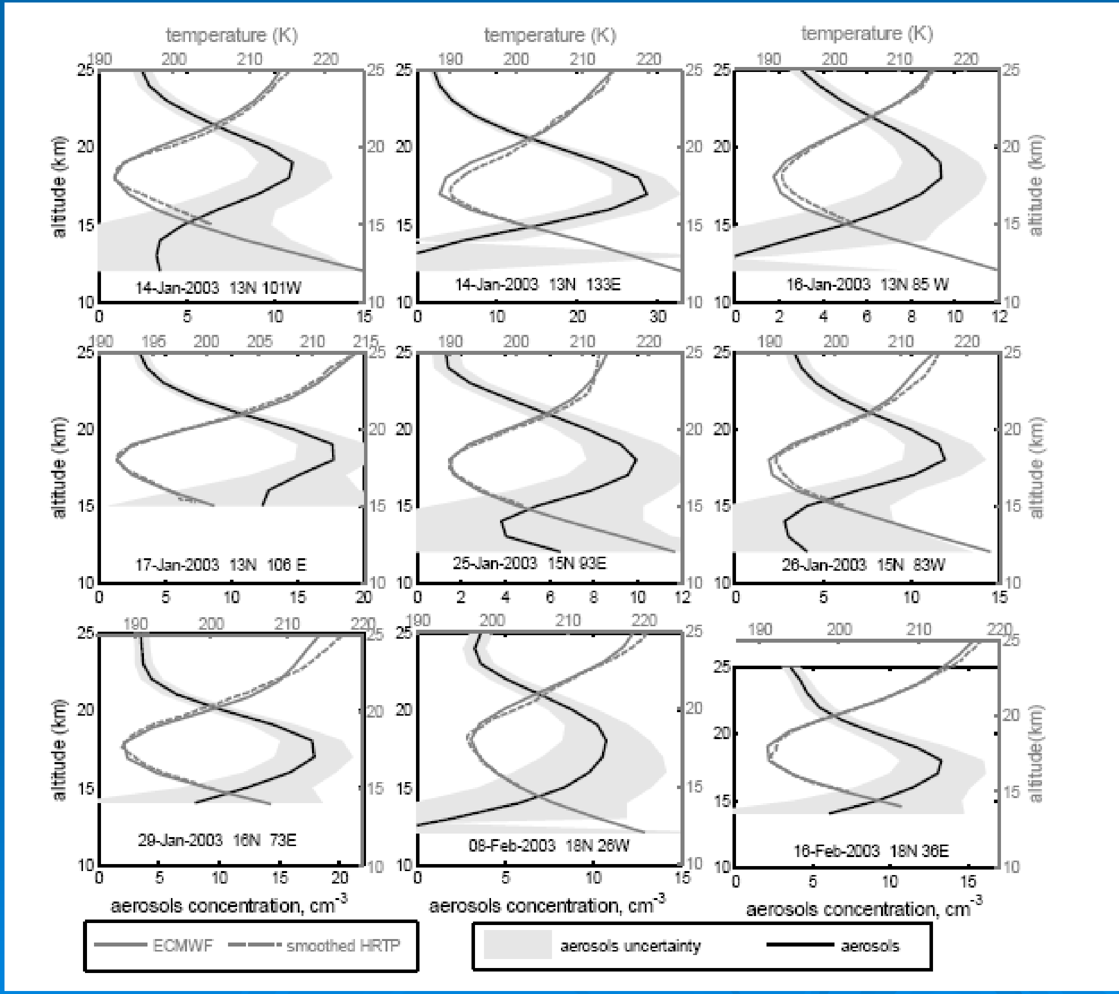
$$\frac{\overline{n}}{\overline{n}_*} = \left(\frac{\overline{T}}{\overline{T}_*} \right)^{-\frac{\alpha D_T}{D^{(n)} + D_T}} \exp\left[- \int_{z_0}^{z} \frac{V_{\mathrm{g}}}{D^{(n)} + D_T}\, dz' \right],$$







Distribution of Number Density of Aerosols (black) and Mean Temperature Distribution (gray) (Satellite Gomos Data)



M. Sofiev, V. F. Sofieva, T. Elperin, N. Kleeorin, I. Rogachevskii and S. Zilitinkevich, J. Geophys. Res. 114, D18209 (2009).

Effect of Chemical Reactions Turbulent Diffusion

Instantaneous particle number density of admixture:

$$\frac{\partial n_\beta}{\partial t} + \nabla\cdot(n_\beta\, \boldsymbol{v}) = -\nu_\beta \hat{W}(n_\beta, T) + \hat{D}(n_\beta),$$

The source term:

$$-\nu_\beta \hat{W}(n_\beta, T)$$

The Arrhenius law:

$$\hat{W} = A \exp\left(-E_a/RT\right)\, \Pi_{\beta=1}^{m} (n_\beta)^{\nu_\beta},$$

Instantaneous fluid temperature field:

$$\frac{\partial T}{\partial t} + (\boldsymbol{v}\cdot\nabla)T + (\gamma - 1)T(\nabla\cdot\boldsymbol{v}) = q\hat{W}(n_\beta, T) + \hat{D}(T),$$

Q is the reaction energy release; $q = Q/\rho c_p$

ν_β is the stoichiometric coefficient that is the order of the reaction;

Mean-Field Approach

Mean-Field Equations:

$$\frac{\partial \overline{N}_\beta}{\partial t} + \boldsymbol{\nabla}\cdot\langle n'_\beta\, \boldsymbol{u}\rangle = -\nu_\beta \overline{W} + \hat{D}(\overline{N}_\beta),$$

$$\frac{\partial \overline{T}}{\partial t} + \boldsymbol{\nabla}\cdot\langle \theta\, \boldsymbol{u}\rangle + (\gamma - 2)\langle \theta(\boldsymbol{\nabla}\cdot \boldsymbol{u})\rangle = q\overline{W} + \hat{D}(\overline{T}),$$

Equations for fluctuations:

$$n'_\beta = n_\beta - \overline{N}_\beta$$

$$\theta = T - \overline{T}$$

$$\frac{\partial n'_\beta}{\partial t} + \boldsymbol{\nabla}\cdot\left(n'_\beta\, \boldsymbol{u} - \langle n'_\beta\, \boldsymbol{u}\rangle\right) = -\nu_\beta\left(\hat{W} - \overline{W}\right) - \boldsymbol{\nabla}\cdot(\overline{N}_\beta\, \boldsymbol{u}) + \hat{D}(n'_\beta),$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{\nabla}\cdot\left(\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u}\rangle\right) + (\gamma - 2)\left[\theta(\boldsymbol{\nabla}\cdot \boldsymbol{u}) - \langle \theta(\boldsymbol{\nabla}\cdot \boldsymbol{u})\rangle\right]$$

$$= q\left[\hat{W} - \overline{W}\right] - (\boldsymbol{u}\cdot\boldsymbol{\nabla})\overline{T} - (\gamma - 1)\overline{T}(\boldsymbol{\nabla}\cdot \boldsymbol{u}) + \hat{D}(\theta).$$

Assumption

$$\hat{W} - \overline{W} = \sum_{\beta=1}^{m} \left(\frac{\partial \hat{W}}{\partial n_\beta}\right)_{\overline{N}_\beta} n'_\beta + \left(\frac{\partial \hat{W}}{\partial T}\right)_{\overline{T}} \theta + O\left[(n'_\beta)^2; \theta^2; n'_\beta\, \theta\right] \equiv \overline{W}(C'_n + C'_T)$$

As follows from the Arheneus law:

$$\left(\frac{\partial \ln \hat{W}}{\partial n_\beta}\right)_{\overline{N}_\beta} = \frac{\nu_\beta}{\overline{N}_\beta}, \qquad \left(\frac{\partial \ln \hat{W}}{\partial T}\right)_{\overline{T}} = \frac{E_a}{R\overline{T}^2}$$

New variables:

$$C'_n = \sum_{\beta=1}^{m} \frac{\nu_\beta}{\overline{N}_\beta} n'_\beta, \qquad C'_T = \frac{E_a}{R\overline{T}^2}\theta,$$

Equations for fluctuations:

$$\frac{\partial n'_\beta}{\partial t} + \boldsymbol{\nabla}\cdot\Big(n'_\beta\, \boldsymbol{u} - \langle n'_\beta\, \boldsymbol{u}\rangle\Big) = -\nu_\beta \overline{W}(C'_n + C'_T) - \boldsymbol{\nabla}\cdot(\overline{N}_\beta\, \boldsymbol{u}) + \hat{D}(n'_\beta),$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{\nabla}\cdot\Big(\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u}\rangle\Big) + (\gamma - 2)\Big[\theta(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \langle \theta(\boldsymbol{\nabla}\cdot\boldsymbol{u})\rangle\Big]$$

$$= q\overline{W}(C'_n + C'_T) - (\boldsymbol{u}\cdot\boldsymbol{\nabla})\overline{T} - (\gamma - 1)\overline{T}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \hat{D}(\theta).$$

Turbulent Diffusion of Gases

T. Elperin, N. Kleeorin, M. Liberman, I. Rogachevskii, Phys. Rev E 90, 053001 (2014)

$$\langle n'_\beta \, \boldsymbol{u} \rangle = -D_\beta^T \, \boldsymbol{\nabla} \overline{N}_\beta + \sum_{\lambda=1;\lambda\neq\beta}^{m} D_\lambda^{\rm MTD}(\beta) \, \boldsymbol{\nabla} \overline{N}_\lambda + \boldsymbol{V}_{\rm eff} \, \overline{N}_\beta,$$

$$\langle \theta \, \boldsymbol{u} \rangle = -D^T \, \boldsymbol{\nabla} \overline{T} - \sum_{\lambda=1}^{m} D_\lambda^{\rm TDE} \, \boldsymbol{\nabla} \overline{N}_\lambda.$$

the coefficient of turbulent diffusion:

$$D_\beta^T = \frac{D_0^T}{{\rm Da}_{\rm T}} \left(1 - \frac{\ln(1 + 2{\rm Da}_{\rm T})}{2{\rm Da}_{\rm T}}\right)$$

$$D_0^T = \tau_0 u_0^2/3$$

${\rm Da}_{\rm T} = \tau_0/\tau_c$ is the turbulent Damköhler number

the simplest chemical reaction $A \to B$

$$O_2 \to O + O$$

${\rm Da}_{\rm T} \gg 1$

$$D_\beta^T = D_0^T/{\rm Da}_{\rm T} = \tau_c u_0^2/3$$

Concentration of reagent A decreases much faster during the chemical time, so that the usual turbulent diffusion based on the turbulent time does not contribute to the mass flux of a reagent A.

Turbulent Diffusion of Gases

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The coefficient of turbulent diffusion:

$$D_\beta^T = D_0^T \left\{1 - \frac{\nu_\beta^2}{\overline{N}_\beta \left(\alpha_n - \alpha_T\right)} \left[1 - \frac{1}{\mathrm{Da}_\mathrm{T}} \left(1 - \frac{\ln(1 + 2\mathrm{Da}_\mathrm{T})}{2\mathrm{Da}_\mathrm{T}}\right)\right]\right\}$$

$$\alpha_n = \sum_{\beta=1}^{m} \frac{\nu_\beta}{\overline{N}_\beta}, \quad \alpha_T = \frac{qE_a}{R\overline{T}^2}.$$

The second-order chemical reaction: $A + B \to C + D.$

$$H + O_2 \to OH + O.$$

$$\alpha_n = 2/\overline{N} \gg \alpha_T, \text{ where } \overline{N}_A = \overline{N}_B \equiv \overline{N}$$

The coefficient of turbulent diffusion:

$$\mathrm{Da}_\mathrm{T} \gg 1$$

$$D_{A,B}^T = \frac{1}{2} D_0^T \left(1 + \mathrm{Da}_\mathrm{T}^{-1}\right)$$

The stoichiometric third-order reaction with different stoichiometric coefficients of the reagents:

$$2A + B \to 2C$$

$$2C + O_2 \to 2CO$$

$$2H_2 + O_2 \to 2H_2O$$

$$D_A^T = D_0^T / \mathrm{Da}_\mathrm{T}, \quad D_B^T = \frac{1}{2} D_0^T \left(1 + \mathrm{Da}_\mathrm{T}^{-1}\right).$$

Comparison with Numerical Simulations

A. Brandenburg, N. E. L. Haugen and N. Babkovskaia, Phys. Rev. E 83, 016304 (2011)

Kolmogorov-Petrovskii-Piskunov-Fisher Equation (advection-reaction-diffusion equation):

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\, v) = \frac{n}{\tau_c}\left(1 - \frac{n}{n_0}\right) + D\Delta n,$$

$$\frac{D_\beta^{\mathrm{T}}}{D_0^T} = \frac{1}{\mathrm{Da}_{\mathrm{T}}}\left(1 - \frac{\ln(1+2\mathrm{Da}_{\mathrm{T}})}{2\mathrm{Da}_{\mathrm{T}}}\right)$$

Mean-Field KPPF-equation:

$$\frac{\partial \overline{N}}{\partial t} + \tau \frac{\partial^2 \overline{N}}{\partial t^2} = \frac{\overline{N}}{\tau_c}\left(1 - \frac{\overline{N}}{n_0}\right) + D_{\mathrm{T}}\Delta\overline{N},$$

the reaction speed (the front speed):

$$s_{\mathrm{T}} = (d/dt)\int(\overline{N}/n_0)\, dz$$

$$s_{\mathrm{T}} = 2(D_\beta^T/\tau_c)^{1/2}$$

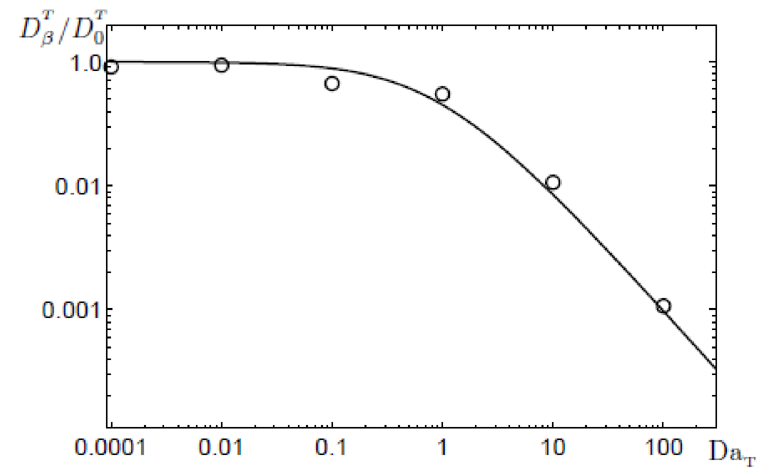

FIG. 1. Comparison of the theoretical dependence of turbulent diffusion coefficient D_β^T/D_0^T versus turbulent Damköhler number Da_{T} with the corresponding results of MFS performed in [49].

Methods and Approximations

- **Quasi-Linear Approach** or Second-Order Correlation Approximation (SOCA) or First-Order Smoothing Approximation (FOSA)

  Pe << 1 , Re << 1 (weak nonlinearity) **Plasma Physics**

  Steenbeck, Krause, Rädler (1966); Roberts, Soward (1975); Moffatt (1978)

- **Path-Integral Approach** (delta-correlated in time random velocity field or short yet finite correlation time); **R. Feynman**

  Kraichnan-Kazantsev model of random velocity field

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- **Renormalization Procedure** (renormalization of viscosity, diffusion, electromotive force and other turbulent transport coefficients) –

  (strong nonlinearity) **Quantum Theory**

  Moffatt (1981; 1983); Kleeorin, Rogachevskii (1994)

Renormalization Procedure

- The first step is the averaging over the scale that is inside the inertial range of turbulence.
- The next stage of the renormalization procedure comprises a step-by-step increase of the scale of the averaging up to the maximum scale of turbulent motions.
- This procedure allows the derivation of equations for the turbulent transport coefficients: eddy viscosity, turbulent diffusion, turbulent heat conductivity, electromotive force coefficients, etc.
- To apply this procedure an equation invariant under the renormalization of the turbulent transport coefficients must be determined.

Renormalization Procedure

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + \mathbf{f},$$

$$\frac{\partial n}{\partial t} + (\mathbf{v}\cdot\nabla)n = \eta\Delta n + \epsilon,$$

$$\frac{\partial V_j}{\partial t} + (\mathbf{V}\cdot\nabla)V_j + \frac{1}{\rho}\frac{\partial p}{\partial x_j} - \nu\Delta V_j - f_j = \frac{\partial}{\partial x_j}\sigma_{ij},$$

$$\frac{\partial N}{\partial t} + (\mathbf{V}\cdot\nabla)N - \eta\Delta N = -\frac{\partial}{\partial x_j}\Psi_j,$$

$$\sigma_{ij} = \sigma_{ij}^{(0)} - \langle u_i^{(1)}u_j^{(0)}\rangle - \langle u_i^{(0)}u_j^{(1)}\rangle - \langle u_i^{(1)}u_j^{(1)}\rangle,$$

$$\Psi_j = \langle q^{(0)}u_j^{(1)}\rangle + \langle q^{(1)}u_j^{(0)}\rangle + \langle q^{(1)}u_j^{(1)}\rangle,$$

$$E(k) = (\beta-1)\left(\frac{u_0^2}{k_0}\right)\left(\frac{k}{k_0}\right)^{-\beta}$$

$$\frac{d\nu}{dk} = -\frac{7}{60\nu k^2}E(k),$$

$$\frac{d\eta}{dk} = -\frac{\mathrm{Pr}(k)}{3\nu k^2(1+\mathrm{Pr})}E(k).$$

$$\nu^2(\xi) = \nu_0^2 + \frac{7}{10}[\xi(k) - \xi_d],$$

$$\mathrm{Re}(\mathrm{Pr}) \equiv \frac{\nu(\mathrm{Pr})}{\nu_0} = \frac{\mathrm{Pr}}{\mathrm{Pr}_0}\left|\frac{\mathrm{Pr}_0 - a_1}{\mathrm{Pr} - a_1}\right|^{\alpha_1}\left|\frac{\mathrm{Pr}_0 + a_2}{\mathrm{Pr} + a_2}\right|^{\alpha_2},$$

$$\eta(\mathrm{Pr}) = \frac{\nu(\mathrm{Pr})}{\mathrm{Pr}},$$

$$\xi = \int_k^\infty \frac{E(k')}{3(k')^2}dk' = \frac{E(k)}{3k(\beta+1)}$$

Small-Scale Clusterings in Stratified Turbulent Flows (with imposed mean temperature gradient)

- Large-scale clustering (large-scale inhomogeneous structures)

$$L_c \gg \ell_0 \qquad \frac{L_{\text{box}}}{\ell_{\text{forcing}}} \approx 5 - 20$$

- Small-Scale Tangling Clustering $\frac{\partial \theta}{\partial t} \propto -(\mathbf{v}\cdot\boldsymbol{\nabla})T + \ldots$

$$L_c \ll \ell_0 \qquad \frac{\partial n'}{\partial t} \propto -(\mathbf{v}\cdot\boldsymbol{\nabla})N + \ldots$$

ℓ_0 is the integral (maximum) scale of turbulent motions

L_c is the characteristic size of clusters

Inertial Clustering of Small Solid Particles

- Inertia causes particles inside the turbulent eddies to drift out to the boundary regions between the eddies (i.e. regions with low vorticity or high strain rate and maximum of fluid pressure).

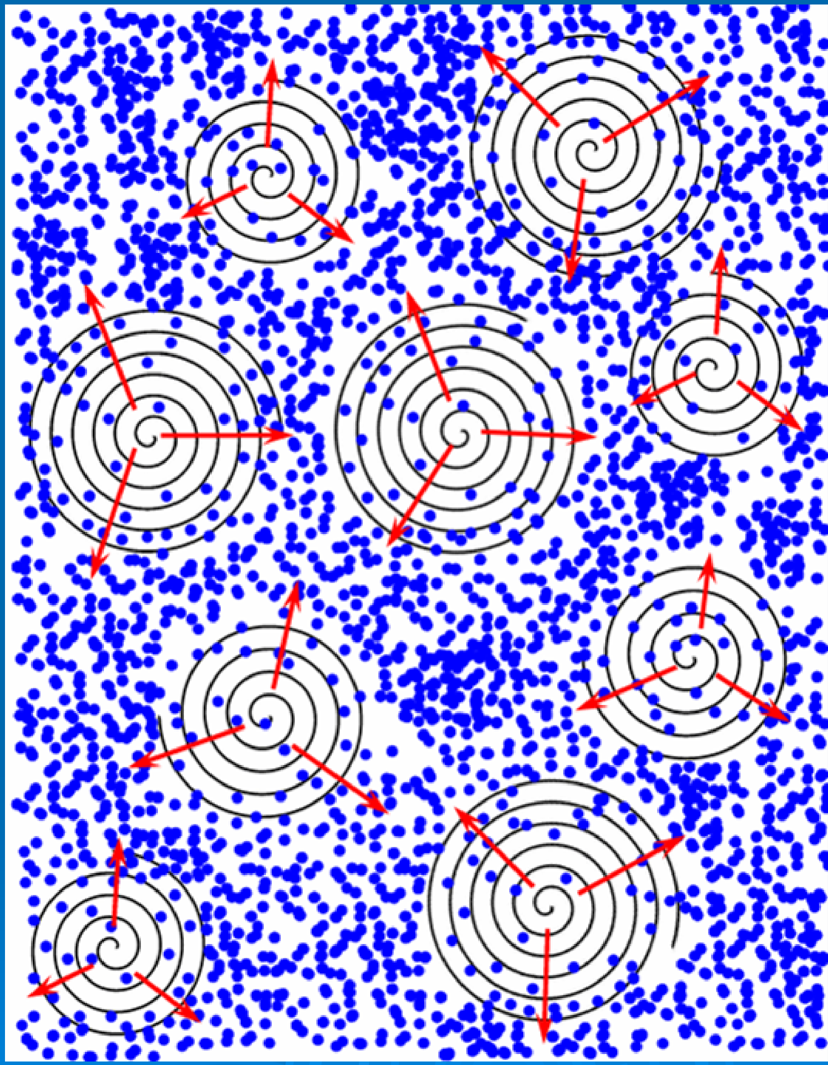
- This mechanism acts in a wide range of scales of turbulence.

- Scale-dependent turbulent diffusion causes relaxation of particle clusters.

- In small scales

$$D_T(\ell) \to D_m$$

- Thus, clusters of particles are localized in small scales.



M. R. Maxey, J. Fluid Mech. 174, 441 (1987).

J.K. Eaton and J.R. Fessler, Int. J. Multiphase Flow 20, 169 (1994).

Theory of Fluctuations

$$\boldsymbol{\nabla} T \neq 0$$
$$N = \langle n \rangle$$

- **Fluctuations of particle number density:**

$$\frac{\partial n'}{\partial t} + \boldsymbol{\nabla}\cdot\left(n'\,\mathbf{v} - \langle n'\,\mathbf{v}\rangle\right) - D_m\,\Delta n' = -N\left(\boldsymbol{\nabla}\cdot\mathbf{v}\right) - \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)N$$

- **Two-point correlation function :** $\Phi(t,\mathbf{R}) = \langle n'(t,\mathbf{x})\,n'(t,\mathbf{y})\rangle$

$$\frac{\partial \Phi}{\partial t} = \left[B(\mathbf{R}) + 2\mathbf{U}^{(A)}(\mathbf{R})\cdot\boldsymbol{\nabla} + D_{ij}(\mathbf{R})\nabla_i\nabla_j\right]\Phi(t,\mathbf{R}) + I(\mathbf{R})$$

- **Source of tangling clustering:** $\mathbf{U}^{(S,A)}(\mathbf{R}) = (1/2)\left[\mathbf{U}(\mathbf{R}) \pm \mathbf{U}(-\mathbf{R})\right]$

$$I(\mathbf{R}) = B(\mathbf{R})N^2 + \mathbf{U}^{(S)}(\mathbf{R})\cdot\boldsymbol{\nabla}N^2 + D^T_{ij}(\mathbf{R})\left(\nabla_i N\right)\left(\nabla_j N\right)$$

where $D_{ij} = 2D_m\delta_{ij} + D^T_{ij}(0) - D^T_{ij}(\mathbf{R})$

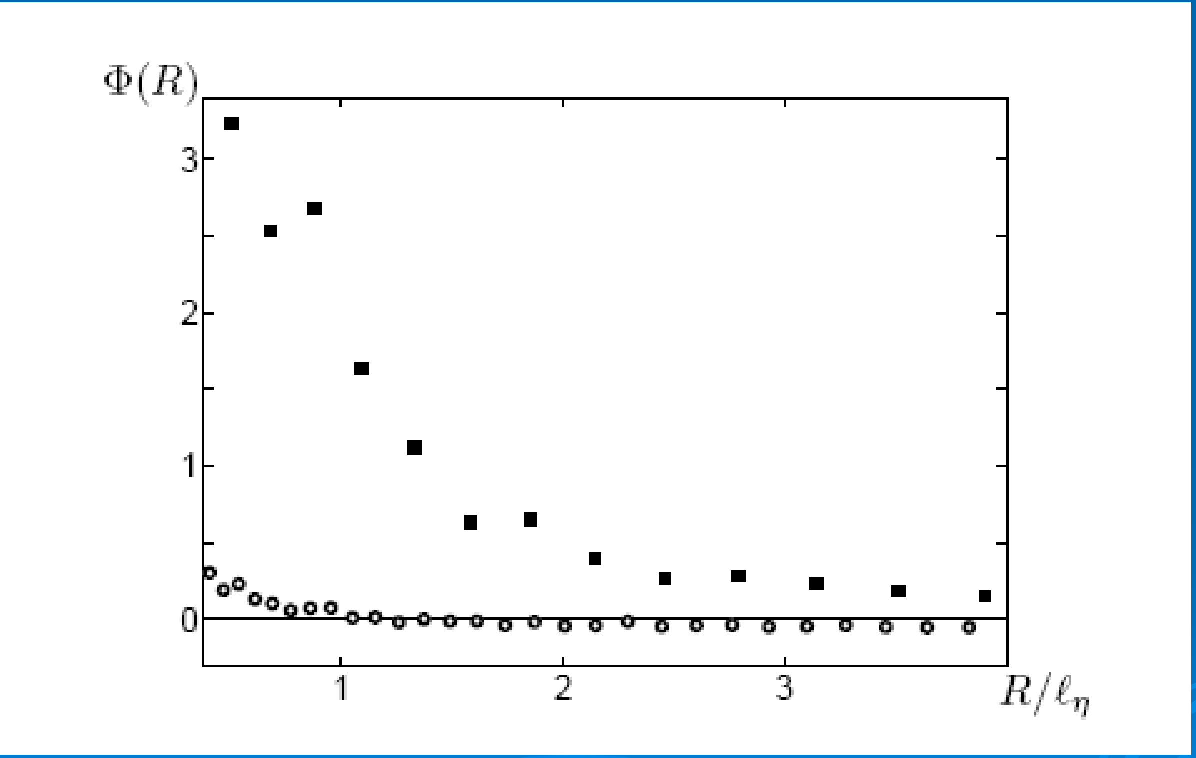
$$B(\mathbf{R}) \approx 2\,\langle\tau\left(\boldsymbol{\nabla}\cdot\mathbf{v}(\mathbf{x})\right)\left(\boldsymbol{\nabla}\cdot\mathbf{v}(\mathbf{y})\right)\rangle$$

$$\mathbf{U}_i(\mathbf{R}) \approx -2\,\langle\tau\,\mathbf{v}_i(\mathbf{x})\left(\boldsymbol{\nabla}\cdot\mathbf{v}(\mathbf{y})\right)\rangle$$

$$D^T_{ij}(\mathbf{R}) \approx 2\,\langle\tau\,\mathbf{v}_i(\mathbf{x})\,\mathbf{v}_j(\mathbf{y})\rangle$$

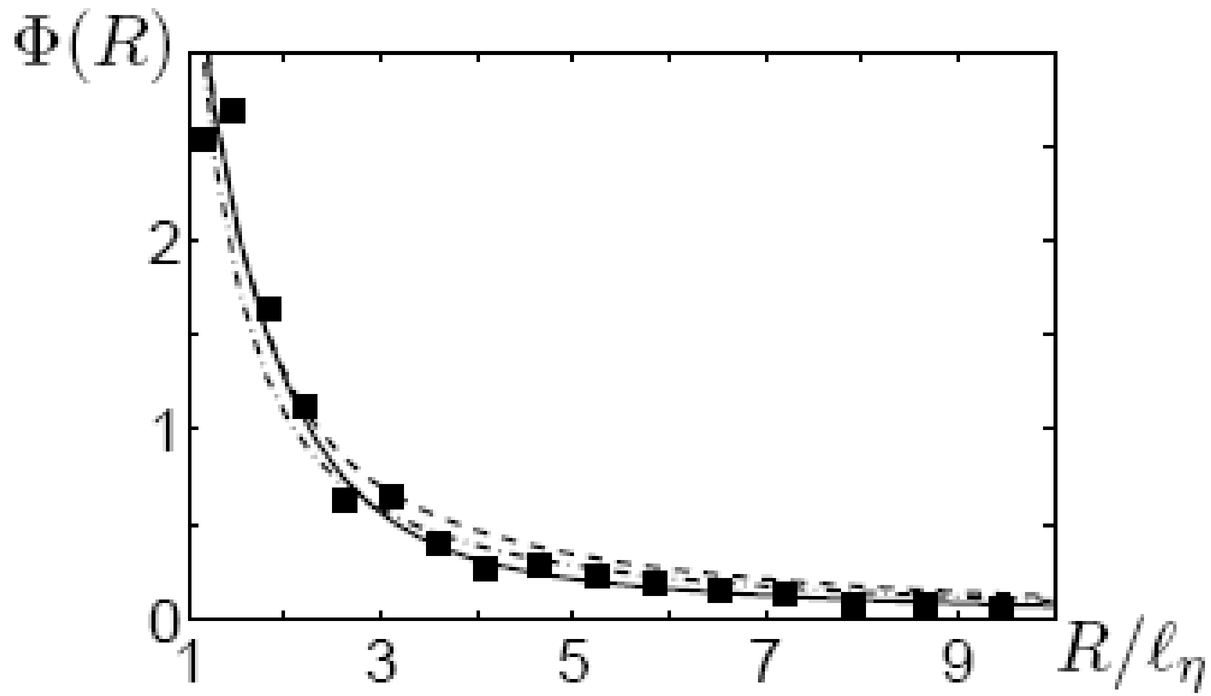
Normalized second-order correlation function determined in our experiments for

(i) inertial clustering (isothermal turbulence, circles)
(ii) tangling clustering (non-isothermal turbulence, squares)

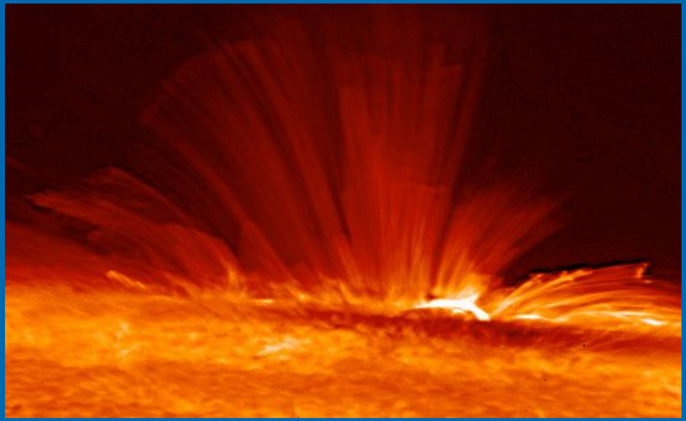


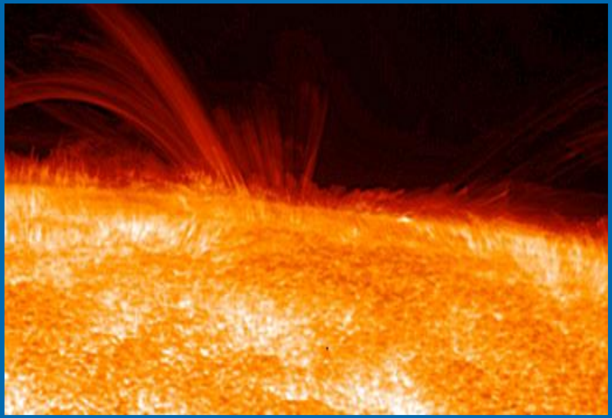
A. Eidelman, T. Elperin, N. Kleeorin, B. Melnik, I. Rogachevskii, Physical Review E **81**, 056313 (2010)

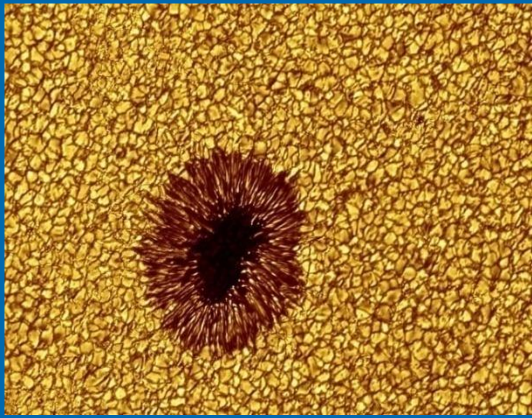
Normalized second-order correlation function determined in our experiments (filled squares) and from our theoretical model (solid line)



Magnetic Fields ---- Particles







$\langle \mathbf{u} \cdot (\nabla \times \mathbf{u}) \rangle$ **is the kinetic helicity**

$\alpha = -\frac{1}{3} \tau \langle \mathbf{u} \cdot (\nabla \times \mathbf{u}) \rangle$ **is the alpha effect**

Particles:

$\mathbf{V}_{\mathrm{eff}} = -\tau \langle \mathbf{u} \, \nabla \cdot \mathbf{u} \rangle$ **is the effective velocity of particles**

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THE END