Electromagnetic duality in axionic superconductors

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Dualities

Earlier examples in condensed matter:

- Kramers-Wannier duality in the 2D Ising model (1941)
- BKT duality in 2D superfluids (particle-vortex duality)
- Dasgupta-Halperin duality in 3D Ginzburg-Landau superconductors (3=3+0 or 3=2+1)

Earlier examples in high-energy physics:

- Dualities in lattice gauge theories
- S-duality
- Seiberg-Witten duality (SuSy needed)
- AdS-CFT duality (gravity dual)

Outline

- Electromagnetic duality and particle-vortex duality
- Axion terms: the Witten effect
- S-duality and topological states of matter
- Application: the Josephson-Witten effect

Electromagnetic Duality

- Maxwell equations are self-dual in vacuum: electric and magnetic fields can be exchanged

- In the presence of matter, self-duality holds only if magnetic charges are included (Dirac, Schwinger)

Electromagnetic Duality

Maxwell equations:

$$\nabla \cdot (\mathbf{E} + i\mathbf{B}) = q$$

$$\partial_t(\mathbf{E} + i\mathbf{B}) + i\nabla \times (\mathbf{E} + i\mathbf{B}) = \mathbf{j}$$

$$q = 0 \text{ and } \mathbf{j} = 0 \implies \text{invariance under } \mathbf{E} \to -\mathbf{B},\ \mathbf{B} \to \mathbf{E}$$

Electromagnetic Duality

Maxwell equations:

$$\nabla \cdot (\mathbf{E} + i\mathbf{B}) = q + iq_m$$

$$\partial_t(\mathbf{E} + i\mathbf{B}) + i\nabla \times (\mathbf{E} + i\mathbf{B}) = \mathbf{j} + i\mathbf{j}_m$$

$$q_m \neq 0 \text{ and } \mathbf{j}_m \neq 0 \Longrightarrow \text{invariance under } \mathbf{E} \to -\mathbf{B},\, \mathbf{B} \to \mathbf{E}$$

More generally (Schwinger):

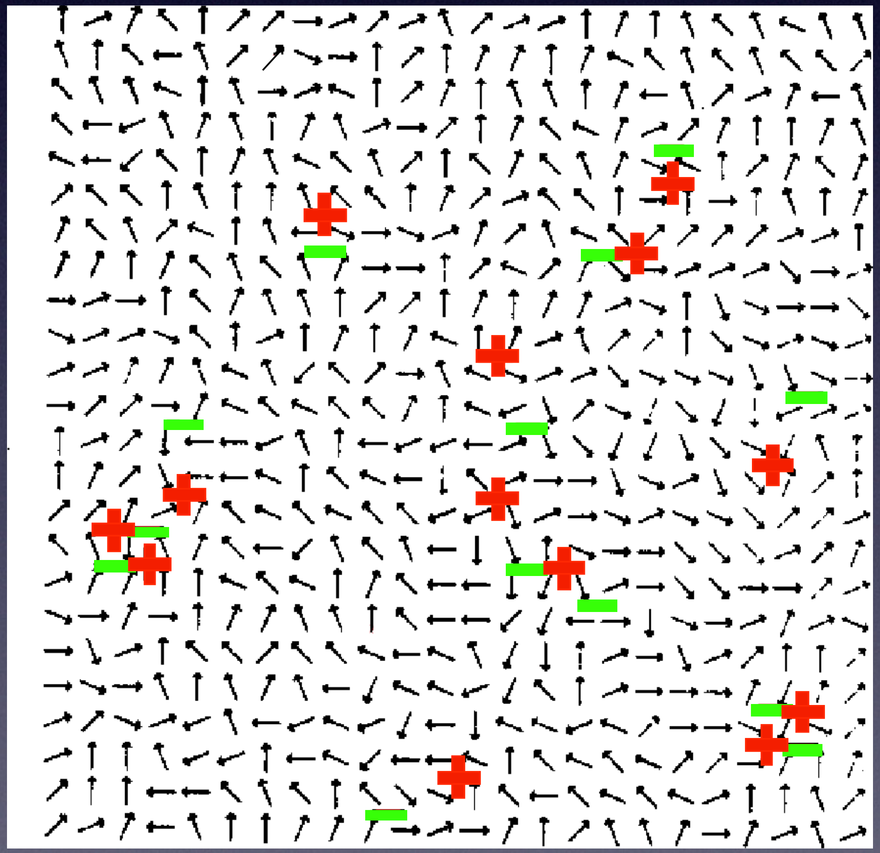
$$(\mathbf{E} + i\mathbf{B}) \to e^{i\phi}(\mathbf{E} + i\mathbf{B})$$

$$(q + iq_m) \to e^{i\phi}(q + iq_m)$$

Dirac quantization: $qq_m = 2\pi n$

Particle-vortex duality

In 2D: superfluids are equivalent to a Coulomb gas of point vortices



Superfluid Hamiltonian:

$$H_{\mathrm{SF}} = \frac{\rho_s}{2T}(\nabla\theta)^2$$

Field theory of a Coulomb gas:

$$H_{\mathrm{SG}} = \frac{T}{8\pi\rho_s}(\nabla\varphi)^2 - z\cos\varphi$$

In 2D the SG theory undergoes a phase transition
BKT transition: no spontaneous symmetry breaking

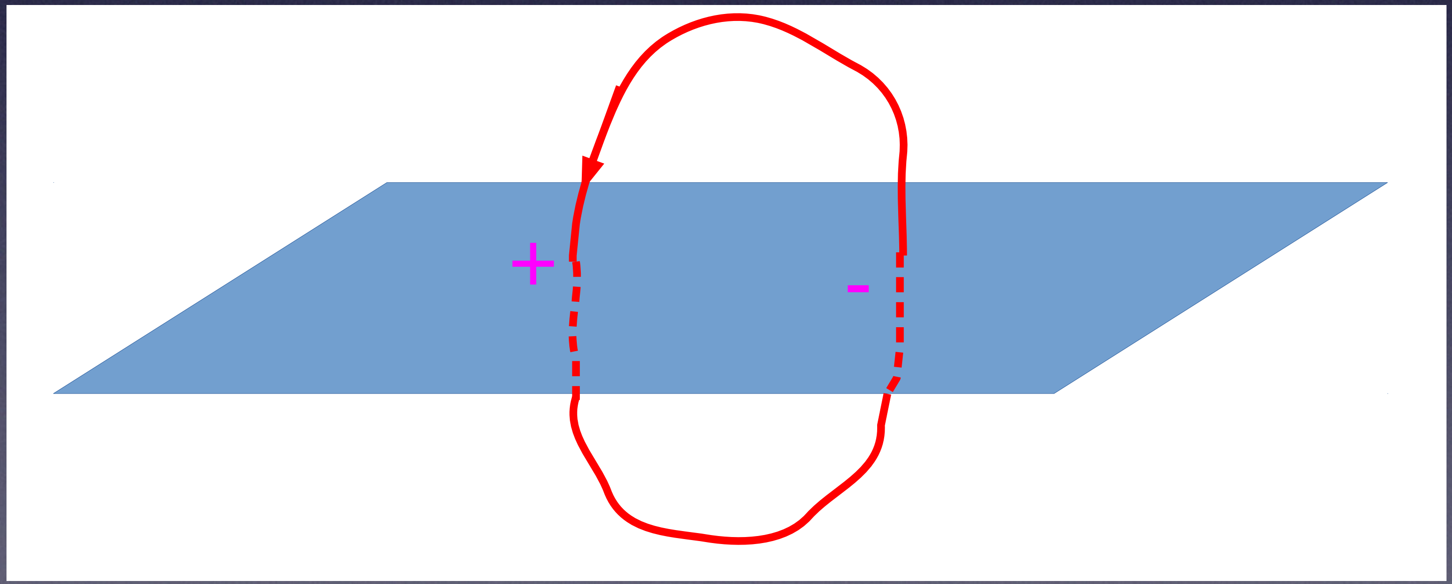
Particle-vortex duality

3D (2+1 or 3+0): Dasgupta-Halperin duality

Vortex superfluidity $\Longleftrightarrow$ Particle superconductivity

XY model $\Longleftrightarrow$ Higgs model

Vortex loop:



Phase transition: 1st order in type I regime
2nd order in type II regime

(Mo, Hove, Sudbo, PRB 2002)

Particle-vortex duality

Duality in abelian Higgs model

4D: Vortices $\Longleftrightarrow$ Worldline of magnetic monopoles

$$\mathcal{L} = \frac{1}{4e^2}\mathcal{F}_{\mu\nu}^2 \quad \text{[compact U(1)]}$$

$$\Big\Updownarrow$$

(Peskin, 1978)

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu - ieA_\mu)\phi|^2 + V(\phi)$$

[U(1) non-compact]

Electromagnetic duality and the Witten effect

Original derivation for a non-Abelian Higgs model
[E. Witten, Phys. Lett. B 86,283 (1979)]

Axion electrodynamics:

$$\mathcal{L} = \frac{1}{8\pi}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{e^2\theta}{4\pi^2}\mathbf{E}\cdot\mathbf{B} - \rho\phi - \mathbf{j}\cdot\mathbf{A}$$

$$\mathbf{D} = \frac{\partial\mathcal{L}}{\partial\mathbf{E}} \qquad \mathbf{H} = -\frac{\partial\mathcal{L}}{\partial\mathbf{B}}$$

$$\nabla\cdot\mathbf{D} = \rho \qquad \nabla\cdot\mathbf{B} = \rho_m$$

$$\implies \nabla\cdot\mathbf{E} = 4\pi\left(\rho - \frac{e^2\theta}{4\pi^2}\rho_m\right)$$

$$\implies Q = \underbrace{q}_{q=ne} - \frac{e^2\theta}{4\pi^2}\underbrace{e_m}_{e_m=2\pi/e} = e\left(n - \frac{\theta}{2\pi}\right)$$

Electrodynamics of topological insulators

Classical electrodynamics of conventional insulators:

$$\mathcal{L} = \frac{1}{8\pi}\left(\epsilon \mathbf{E}^2 - \frac{1}{\mu}\mathbf{B}^2\right)$$

Axion electrodynamics of 3D TIs [Qi, Hughes, Zhang (2008)]:

$$\mathcal{L} = \frac{1}{8\pi}\left(\epsilon \mathbf{E}^2 - \frac{1}{\mu}\mathbf{B}^2\right) + \frac{\alpha\theta}{4\pi^2}\mathbf{E}\cdot\mathbf{B}$$

$\Longrightarrow \theta = \pi$ for TIs with TRI

Meaning of θ:

$$\theta = \frac{1}{8\pi}\int \mathrm{d}^3k \ \mathrm{tr}\left[\mathbf{a}(\mathbf{k})\wedge\mathbf{f}(\mathbf{k}) - \frac{2}{3}\mathbf{a}(\mathbf{k})\wedge\mathbf{a}(\mathbf{k})\wedge\mathbf{a}(\mathbf{k})\right]$$

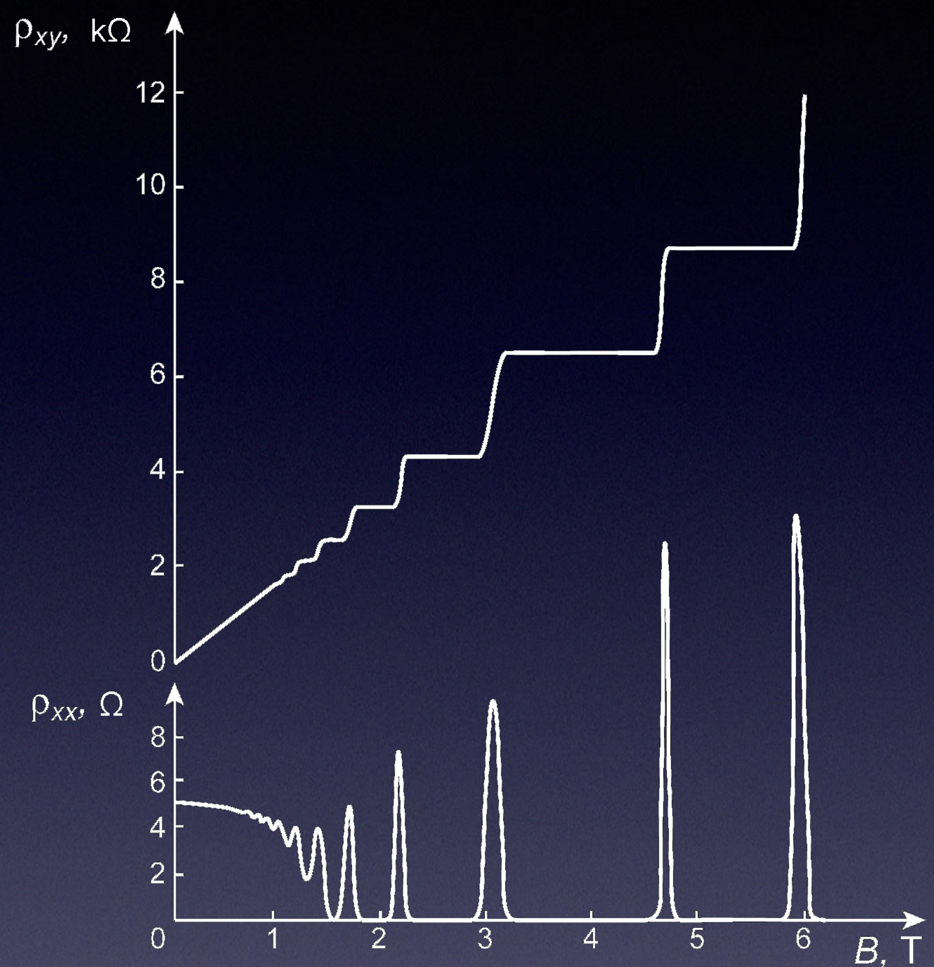
$$\mathbf{f}(\mathbf{k}) = \mathrm{d}\mathbf{a}(\mathbf{k}) + i\mathbf{a}(\mathbf{k})\wedge\mathbf{a}(\mathbf{k})$$

$$\mathbf{a}_{\alpha\beta}(\mathbf{k}) = -i\langle\alpha,\mathbf{k}|\nabla_{\mathbf{k}}|\beta,\mathbf{k}\rangle$$

Non-abelian Berry vector potential for the Bloch state $|\alpha,\mathbf{k}\rangle$

Topological insulators

The first TI: quantum Hall effect



TKNN invariant
(Thouless, Kohmoto,Nightingale, den Nijs):

$$\sigma_{xy} = m\frac{e^2}{h} \qquad m \in \mathbb{Z}$$

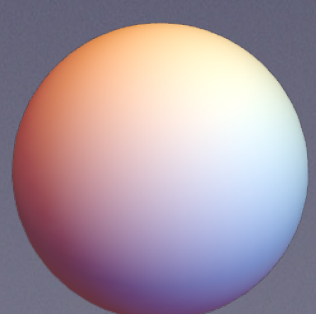
$$m = \sum_n \int \frac{d^2k}{2\pi}\left(\frac{\partial a_{ny}}{\partial k_x} - \frac{\partial a_{nx}}{\partial k_y}\right)$$

$$a_{nk} = i\langle u_{nk}|\nabla_{\mathbf{k}}|u_{nk}\rangle$$

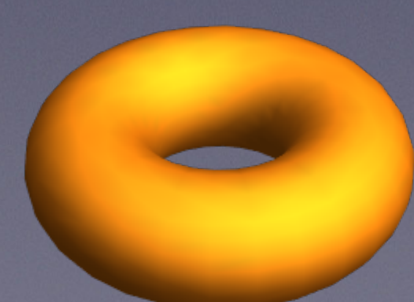
σ_{xy} : physical realization of Gauss-Bonnet theorem

$$\int_{\mathcal{M}} \kappa = 2\pi\chi(\mathcal{M})$$

$\kappa \Rightarrow$ Gaussian curvature
$\chi \Rightarrow$ Euler characteristic



$\Longrightarrow \chi = 2$



$\Longrightarrow \chi = 0$

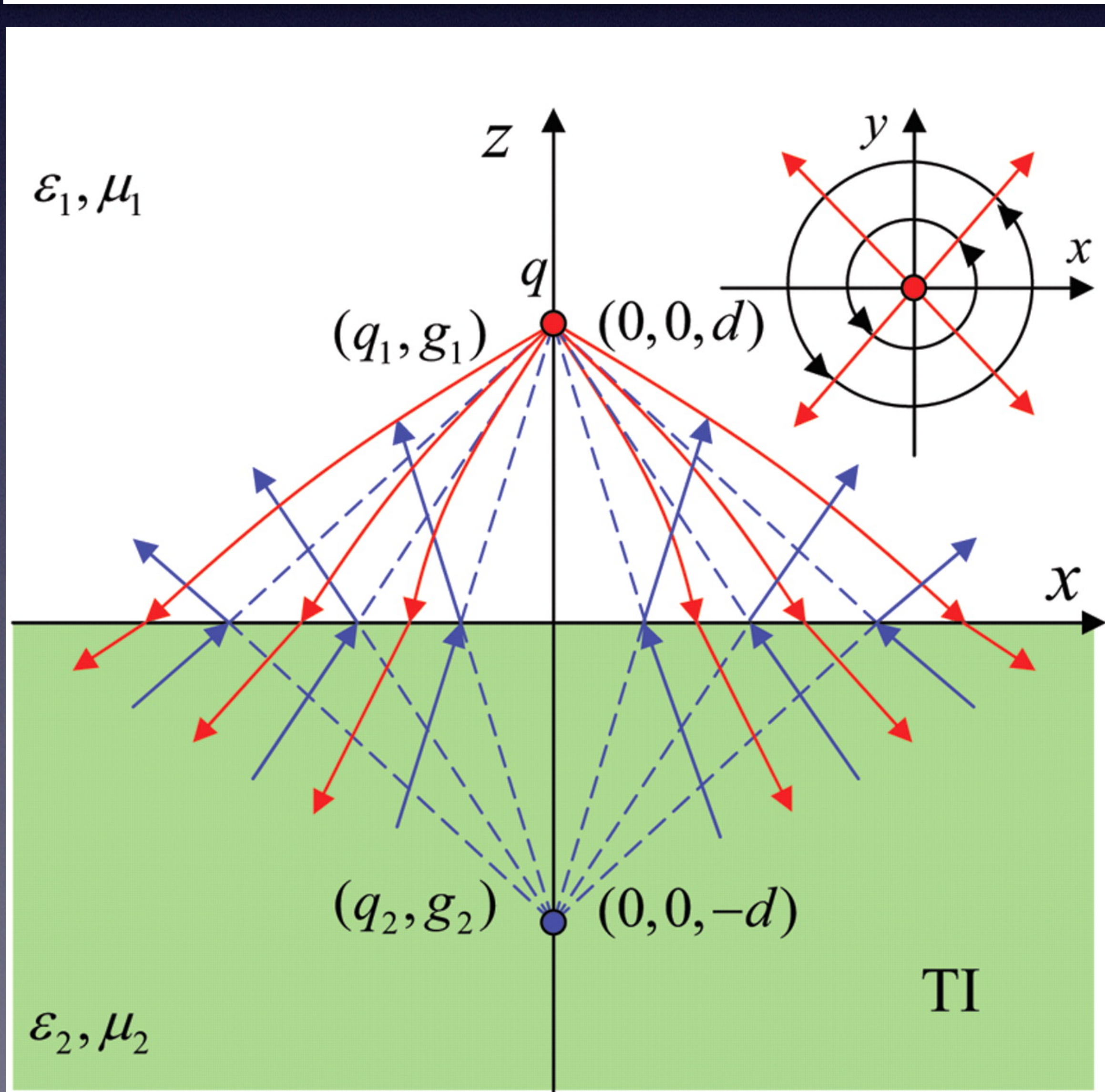
Electrodynamics of topological insulators

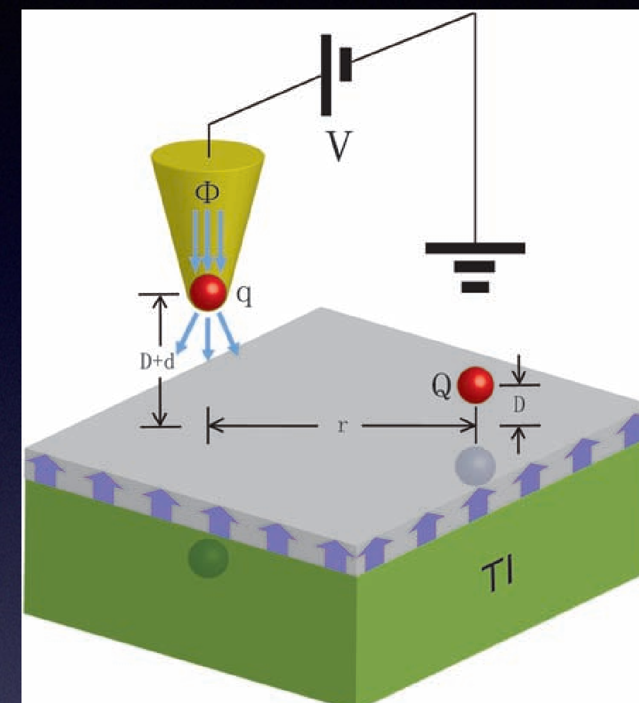
Inducing a Magnetic Monopole with Topological Surface States

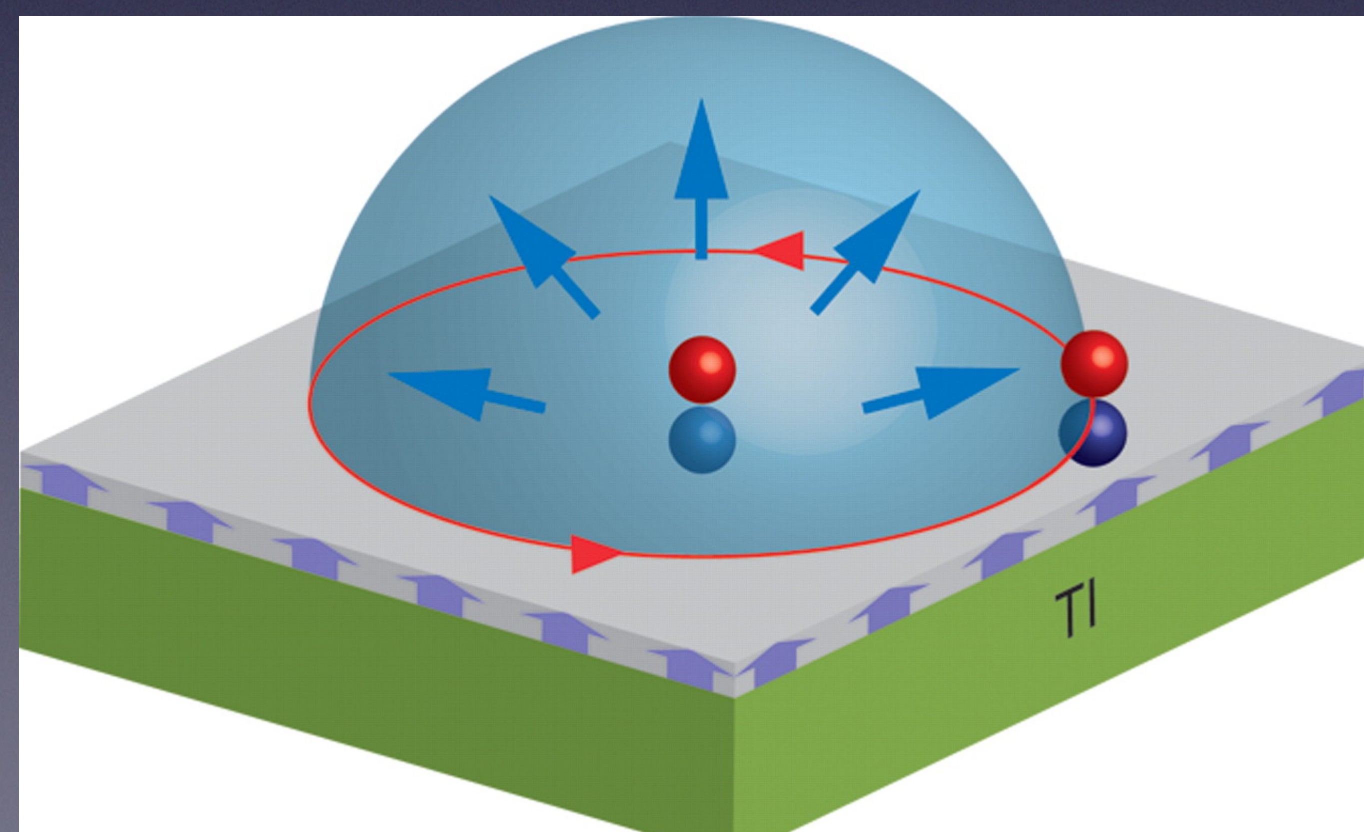
Xiao-Liang Qi,[1] Rundong Li,[1] Jiadong Zang,[2] Shou-Cheng Zhang[1]*

Existence of the magnetic monopole is compatible with the fundamental laws of nature; however, this elusive particle has yet to be detected experimentally. We show theoretically that an electric charge near a topological surface state induces an image magnetic monopole charge due to the topological magneto-electric effect. The magnetic field generated by the image magnetic monopole may be experimentally measured, and the inverse square law of the field dependence can be determined quantitatively. We propose that this effect can be used to experimentally realize a gas of quantum particles carrying fractional statistics, consisting of the bound states of the electric charge and the image magnetic monopole charge.

Science 323, 1184 (2009)







Electrodynamics of topological insulators

For θ uniform, the axion term is a total derivative:

$$S_{axion} = \frac{\alpha\theta}{4\pi^2}\int d^4x \mathbf{E}\cdot\mathbf{B} = \frac{\alpha\theta}{32\pi^2}\int d^4x \epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = \frac{\alpha\theta}{16\pi^2}\int d^4x \partial^\mu(\epsilon^{\mu\nu\lambda\rho}A^\nu F^{\lambda\rho})$$

$\Longrightarrow$ For an infinite system, Maxwell equations do not modify

However, TIs have surfaces and θ changes there!

Dissipationless current: $\mathbf{j} = \frac{e^2}{4\pi^2}\nabla\theta \times \mathbf{E}$

Hall conductivity: $\sigma_{xy} = \frac{e^2}{2\pi}\left(n - \frac{\theta}{2\pi}\right)$

Electrodynamics of topological insulators

Hall conductivity	Witten effect
$\sigma_{xy} = \frac{e^2}{2\pi}\left(n - \frac{\theta}{2\pi}\right)$	$Q = e\left(n - \frac{\theta}{2\pi}\right)$
(no monopoles)	(monopoles)

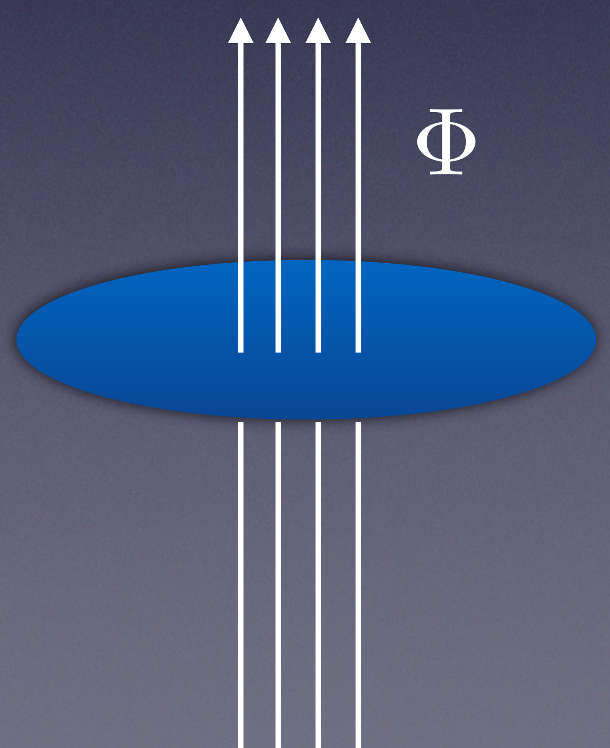
Is there a correspondence between the two?

$\Longrightarrow$ Particle on a ring with magnetic flux

Current: $j = \frac{e^3}{2\pi r^2}\left(n - \frac{e\Phi}{2\pi}\right)$

$$\theta \to e\Phi$$

In 3D: take vortices and a surface



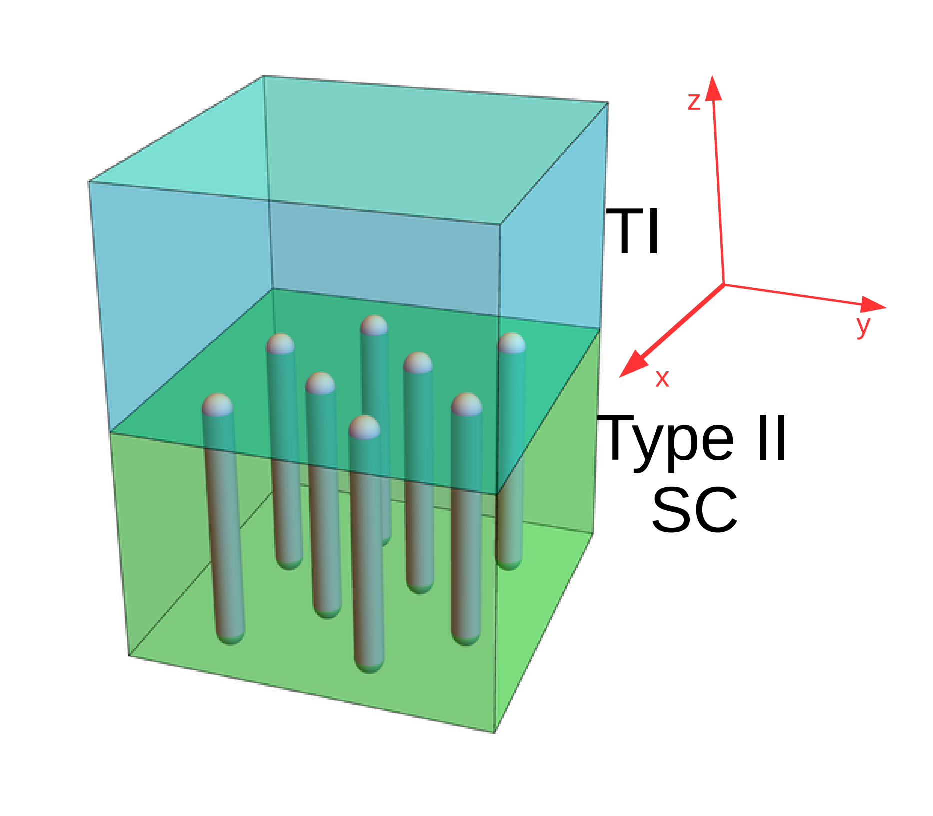
TI-SC structures

Gauss law:

$$\nabla \cdot \mathbf{E} = 4\pi\rho + (e^2/\pi)\nabla\theta \cdot \mathbf{B}$$

$$\Longrightarrow Q = q + \frac{e^2}{4\pi^2}\int d^2r B(r)\int_{-\infty}^{0} dz \frac{d\theta}{dz}$$

$$= q + \frac{e^2\theta}{4\pi^2}\Phi_B$$



$$\Phi_B = N_v \frac{2\pi}{(2e)} = \frac{N_v \pi}{e} \quad \Longrightarrow \quad \text{Charge fractionalization}$$

$\Longrightarrow$ Witten effect without monopoles!

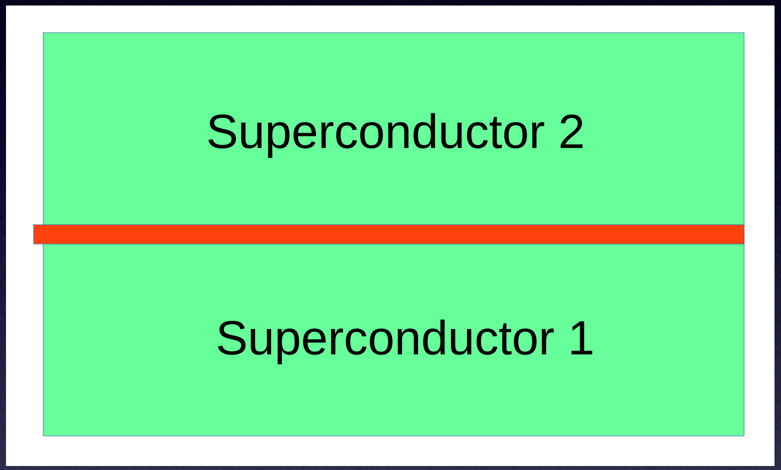
Josephson and Witten effects

- Josephson effect:

$$I_J = I_c \sin(\Delta\varphi)$$

$$\partial_t \Delta\varphi = 2eV$$

$$\Delta\varphi = \varphi_1 - \varphi_2$$

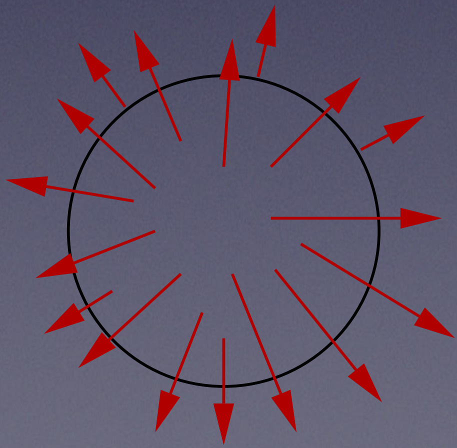


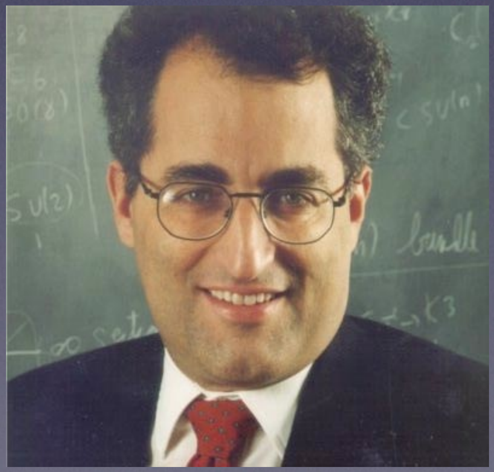
- Witten effect:

charge fractionalization due to magnetic monopoles

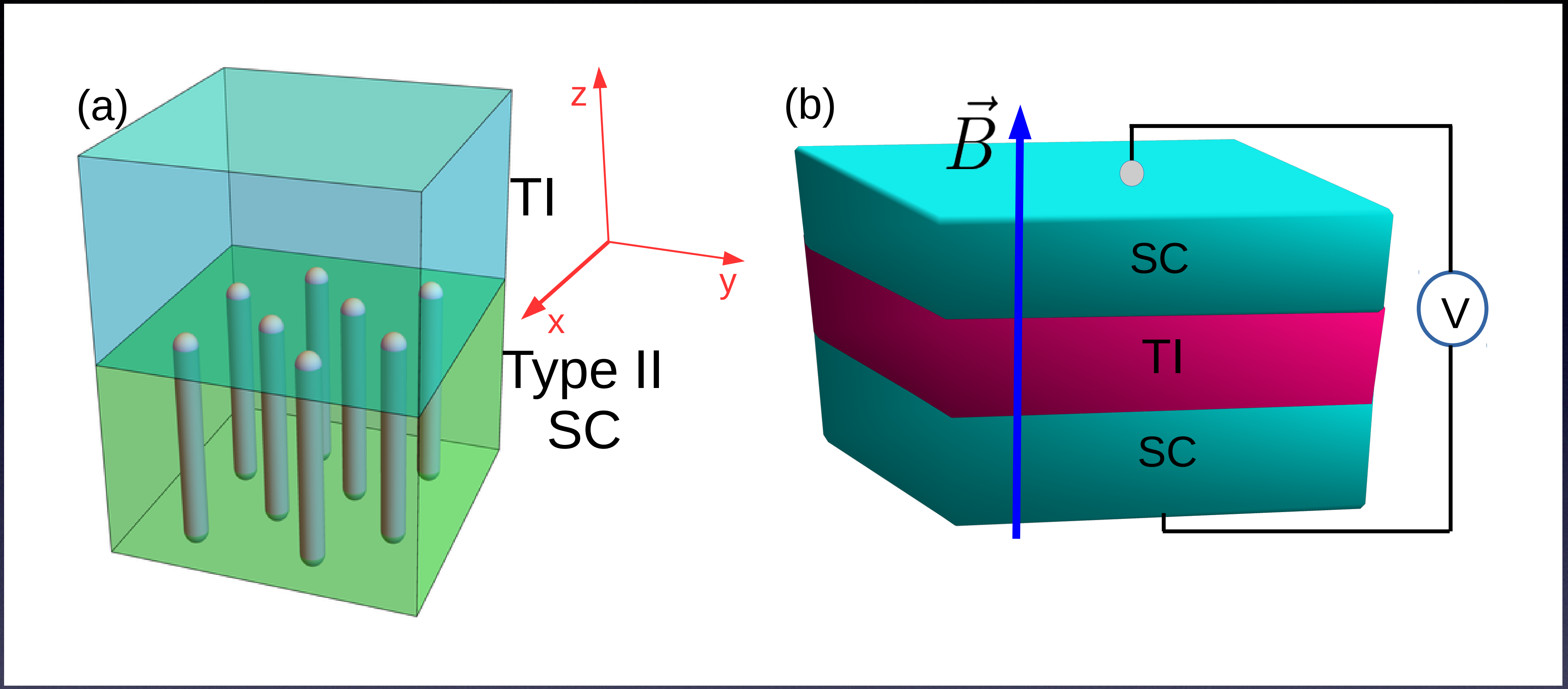
$$Q = e\left(n - \frac{\theta}{2\pi}\right)$$

$$\mathcal{L}_{\text{Axion}} = \frac{e^2\theta}{4\pi^2}\mathbf{E}\cdot\mathbf{B}$$





Josephson-Witten effect



$$H_J = \frac{2e^2}{C}\left(\Delta n + \frac{e\theta}{8\pi^2}\Phi_B\right)^2 - E_J \cos\Delta\phi$$

$$i[H_J, \Delta\phi] = \partial_t \Delta\phi = 2e\Delta V_{\text{ind}} \Longrightarrow \Delta V_{\text{ind}} = \frac{1}{C}\left(2e\Delta n + \frac{e^2\theta}{4\pi^2}\Phi_B\right)$$

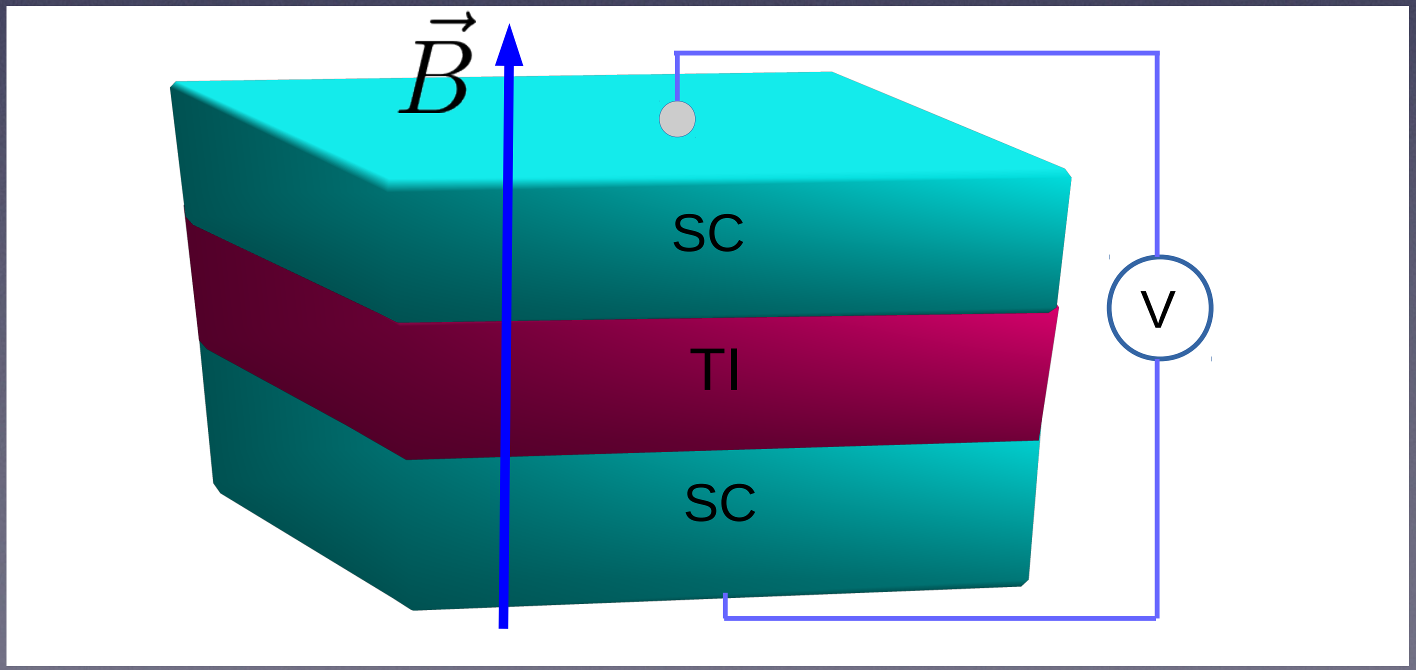
Invariance: $\theta \to \theta + 8\pi, \quad \Delta n \to \Delta n - N_v$

AC Josephson effect induced by the Witten effect

Witten effect induces an AC Josephson effect by applying a magnetic field perpendicular to the junction

$$I_J(\Delta\phi, t) = 2eE_J \sin\left(\Delta\phi + \omega_\theta t\right)$$

$$\omega_\theta = \frac{e^3\theta}{2\pi^2 C}\Phi_B,$$



AC Josephson effect induced by the Witten effect

Junction Lagrangian:

$$L_J = \Delta n \partial_t \Delta\phi - H_J$$

Partition function:

$$Z = \int \mathcal{D}\Delta n \mathcal{D}\Delta\phi e^{-\int_0^\beta d\tau (i\Delta n \partial_\tau \Delta\phi + H_J)}$$

Shift:

Phase factor:

$$\Delta n \to \Delta n - e\theta\Phi_B/(8\pi^2) \Longrightarrow e^{ie\theta\Phi_B/(8\pi^2)\int_0^\beta d\tau \partial_\tau \Delta\phi}$$

$$\Longrightarrow \theta = \frac{8\pi m}{N_v} \qquad m \in \mathbb{Z} \Longrightarrow \omega_\theta = \frac{(2e)^2 m}{C}$$

Electromagnetic duality with superconductors

Axionic superconductors with magnetic monopoles:

$$\mathcal{L} = \frac{1}{4}\mathcal{F}_{\mu\nu}^2 + \frac{ie^2\theta}{16\pi^2}\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu} + \frac{\rho^2}{2}(\partial_\mu\varphi + 2eA_\mu)^2$$

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + \frac{\pi}{e}\tilde{M}_{\mu\nu} \qquad \tilde{\mathcal{F}}_{\mu\nu} = \tilde{F}_{\mu\nu} + \frac{\pi}{e}M_{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \tilde{F}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}$$

$$M_{\mu\nu} = \partial_\mu M_\nu - \partial_\nu M_\mu \qquad \tilde{M}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\lambda\rho}M_{\mu\nu}$$

Monopole gauge field: $M_\mu(x) = \int d^4x' G(x-x')m_\mu(x')$

$$G(x) = 1/(4\pi^2x^2) \qquad \partial_\mu m_\mu = 0$$

$$\partial_\mu\mathcal{F}_{\mu\nu} = j_\nu \qquad \partial_\mu\tilde{\mathcal{F}}_{\mu\nu} = (\pi/e)m_\nu$$

Electromagnetic duality with superconductors

$$\mathcal{L} = \frac{1}{4}\mathcal{F}_{\mu\nu}^2 + \frac{ie^2\theta}{16\pi^2}\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu} + \frac{\rho^2}{2}(\partial_\mu\varphi + 2eA_\mu)^2$$

$$\Updownarrow$$

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}(F_{\mu\nu}^2 + f_{\mu\nu}^2) + i\frac{e^2\theta}{16\pi^2}F_{\mu\nu}\tilde{F}_{\mu\nu} \\ &+ \frac{\rho^2}{2}(\partial_\mu\varphi + 2eA_\mu)^2 + \frac{\rho_V^2}{2}\left(\partial_\mu\varphi_V + \frac{\pi}{e}h_\mu + \frac{e\theta}{4\pi}A_\mu\right)^2\end{aligned}$$

$$f_{\mu\nu} = \partial_\mu h_\nu - \partial_\nu h_\mu$$

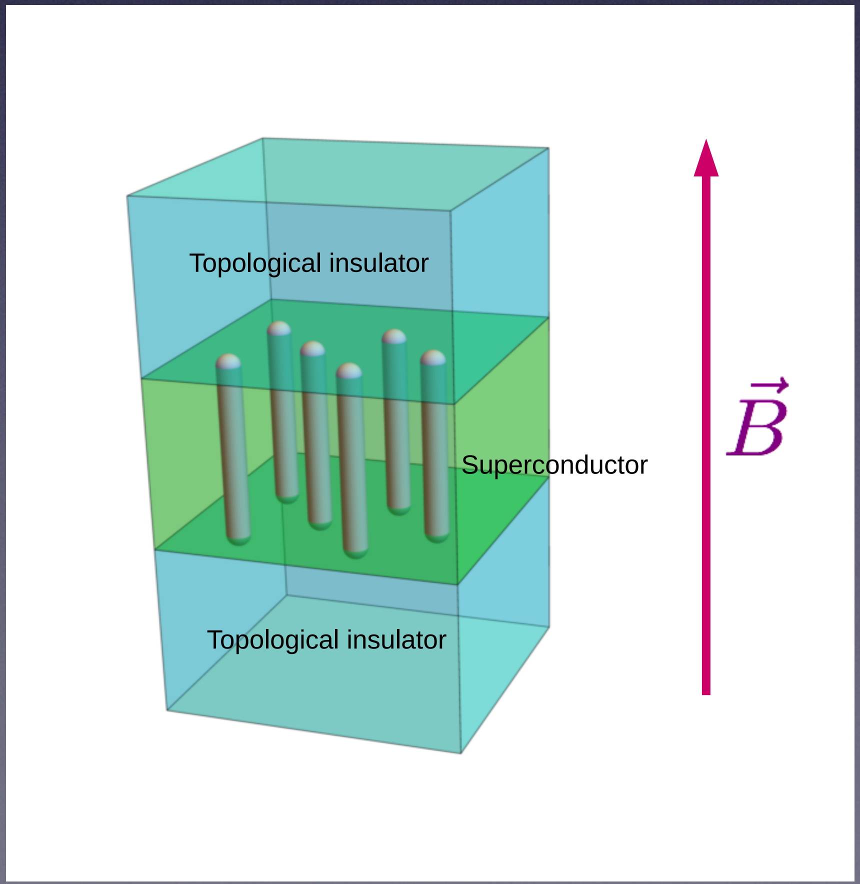
$\rho = \theta = 0 \Longrightarrow$ Higgs model $\Longleftrightarrow$ Compact Maxwell

Electromagnetic duality with superconductors

$$\mathcal{L} = \frac{1}{4}(F_{\mu\nu}^2 + f_{\mu\nu}^2) + i\frac{e^2\theta}{16\pi^2}F_{\mu\nu}\tilde{F}_{\mu\nu}$$
$$+ \frac{\rho^2}{2}(\partial_\mu\varphi + 2eA_\mu)^2 + \frac{\rho_V^2}{2}\left(\partial_\mu\varphi_V + \frac{\pi}{e}h_\mu + \frac{e\theta}{4\pi}A_\mu\right)^2$$

$\Longrightarrow$ Topological field theory for vortices and bosons

For $\theta = 0$ this is a model for superconducting cosmic strings



Vortex charge:

$$Q_V = S\int_L ds\left[2e\rho^2(\partial_t\varphi + 2eA_0) + \frac{e\theta}{4\pi}\rho_V^2\left(\partial_t\varphi_V + \frac{\pi}{e}h_0 + \frac{e\theta}{4\pi}A_0\right)\right]$$

Electromagnetic duality with superconductors

Shift: $h_\mu \to h_\mu - \frac{e^2\theta}{4\pi^2} A_\mu$ Rescalings: $h_\mu \to 2e h_\mu,\ A_\mu \to (\pi/e) A_\mu$

$$\mathcal{L} = \frac{1}{4} \left[\begin{array}{cc} F_{\mu\nu} & f_{\mu\nu} \end{array}\right] \left[\begin{array}{cc} \frac{\pi^2}{e^2} + \frac{e^2\theta^2}{16\pi^2} & -\frac{e^2\theta}{2\pi} \\ -\frac{e^2\theta}{2\pi} & 4e^2 \end{array}\right] \left[\begin{array}{c} F_{\mu\nu} \\ f_{\mu\nu} \end{array}\right] + i\frac{\theta}{16} F_{\mu\nu}\tilde{F}_{\mu\nu}$$

$$+ \quad \frac{\rho^2}{2}(\partial_\mu \varphi + 2\pi A_\mu)^2 + \frac{\rho_V^2}{2}(\partial_\mu \varphi_V + 2\pi h_\mu)^2$$

Duality:

$$e' = \frac{1}{2}\left(\frac{\pi^2}{e^2} + \frac{e^2\theta^2}{16\pi^2}\right)^{1/2} \qquad \theta' = -\frac{4\theta e^2}{\frac{\pi^2}{e^2} + \frac{e^2\theta^2}{16\pi^2}}$$

$\Longrightarrow$ generalizes Dirac duality

Electromagnetic duality with superconductors

Dual action in terms of charge and monopole currents:

$$\begin{aligned}\tilde{S} &= \frac{1}{2}\left(\frac{\pi^2}{e^2}+\frac{e^2\theta^2}{16\pi^2}\right)\int d^4x\int d^4x' G(x-x')m_\mu(x)m_\mu(x') \\ &+ \frac{(2e)^2}{2}\int d^4x\int d^4x' G(x-x')j_\mu(x)j_\mu(x') \\ &+ \frac{e^2\theta}{2\pi}\int d^4x\int d^4x' G(x-x')j_\mu(x)m_\mu(x')\end{aligned}$$

Continuum equivalent of the dual lattice model (Cardy1982)

Self-duality: $e' = \frac{1}{2}\left(\frac{\pi^2}{e^2}+\frac{e^2\theta^2}{16\pi^2}\right)^{1/2}$ $\qquad \theta' = -\frac{4\theta e^2}{\frac{\pi^2}{e^2}+\frac{e^2\theta^2}{16\pi^2}}$

$m_\mu \to j_\mu \qquad j_\mu \to -m_\mu$

Renormalization aspects of S-duality

$$e'^2\theta' = -e^2\theta \Longrightarrow e'e = \frac{1}{2}\left(\pi^2 + \frac{e^4\theta^2}{16\pi^2}\right)^{1/2} = \text{invariant}$$

$$\Longrightarrow e'_r e_r = e'e$$

Renormalized fields: $A_{r,\mu} = Z_A^{-1/2} A_\mu$ $\qquad h_{r,\mu} = Z_h^{-1/2} h_\mu$

Duality + Ward identities: $Z_A Z_h = 1 \Longrightarrow \theta_r = \sqrt{Z_A Z_h}\,\theta = \theta$

$$\Longrightarrow e_r^2 \theta_r F_{r,\mu\nu}\tilde{F}_{r,\mu\nu} = e^2\theta F_{\mu\nu}\tilde{F}_{\mu\nu}$$

Axion term is an RG invariant!

Consisten with topological character = insensitive to scale transformations

Boundary dual theory at strong coupling

For $e^2 \to \infty$ the bulk and the boundary decouple

Witten charge Q is constrained to vanish,
implying $\theta = -8\pi n/m$

$$
\begin{aligned}
\mathcal{L}_\infty &= i\frac{mn}{\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + |(\partial_\mu - i2mA_\mu)\phi|^2 \\
&+ |(\partial_\mu + 2inA_\mu)\,\phi_V|^2 + U(\phi,\phi_V)
\end{aligned}
$$

Dual Lagrangian:

$$
\begin{aligned}
\tilde{\mathcal{L}}_\infty &= \frac{1}{2\rho^2}(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2 + \frac{1}{2\rho_V^2}(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda)^2 \\
&+ i\frac{2\pi^2}{\theta}\epsilon_{\mu\nu\lambda}\left(2a_\mu + \frac{\theta}{4\pi}b_\mu\right)\partial_\nu\left(2a_\lambda + \frac{\theta}{4\pi}b_\lambda\right) + \ldots
\end{aligned}
$$

Physical consequences

$\theta = -8\pi n/m$ corresponds to a critical point

Reminiscent of the quantization of the AC Josephson-Witten effect when quantum fluctuations of the phase are accounted for

- Quantized photon mass:

$$\widetilde{M}_A(m,n) = 2\pi\left(\frac{n}{m}\rho^2 + \frac{m}{n}\rho_V^2\right)$$

- Quantized critical exponents (easier to calculate in the dual theory)

Conclusions

- Electromagnetic duality combined with particle-vortex duality allows to establish a Witten effect using vortices instead of magnetic monopoles
- Consequences for capacitance in Josephson effect using 3D TI as junction: AC Josephson effect is induced by just applying a magnetic field perpendicular to the junction
- Boundary theory at strong coupling quantizes θ