Renormalization Group and the S-matrix

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on work with Matthias Wilhelm, 1607.xxxxx

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Questions

-Can niceness of amplitudes help compute β-functions?

-How to think of RGE on shell, without a Lagrangian?

Outline

1. One-loop:
   -Yang-Mills beta function from tree S-matrix
   -review of related works

2. All-loops:
   -dilatation operator is phase of S-matrix

3. 1½-loops:
   -length changing effects in Yukawa theory
   -subtleties with masses, etc.

1-loop QCD β-function

- Start with form factor for Lagrangian $\mathrm{Tr}[F^2]/g^2$
- Remove IR divergences using IR-safe ratio:

$$\frac{\langle p_1, p_2|F^2|0\rangle}{\langle p_1, p_2|T^{\mu\nu}|0\rangle} \sim \left(\frac{-s}{\mu^2}\right)^{\frac{1}{2}\gamma_{F^2}}$$

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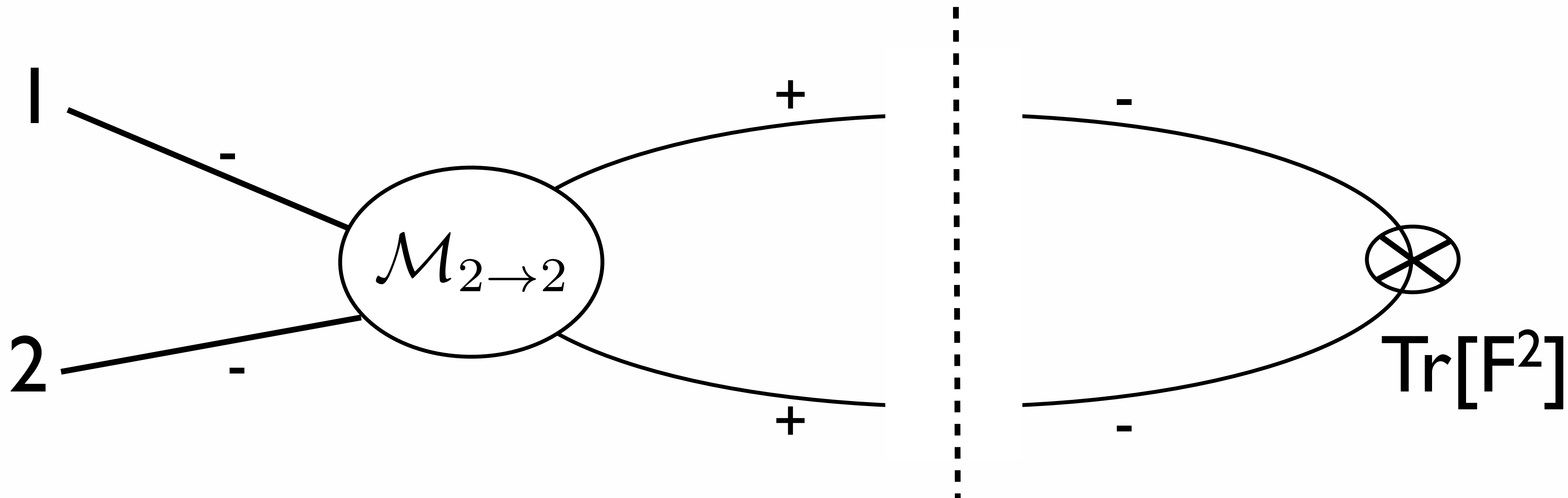
- Phase equal to anomalous dimension:

$$\gamma_{F^2}^{(1)} = -\frac{1}{\pi}\, 2\,\mathrm{Im}\left(\log \frac{\langle p_1, p_2 | F^2 | 0 \rangle^{(1)}}{\langle p_1, p_2 | T^{\mu\nu} | 0 \rangle^{(1)}} \right)$$

$$\gamma^{(1)}_{F^2} = -\frac{1}{\pi}\, 2\,\mathrm{Im}\left(\log \frac{\langle p_1, p_2|F^2|0\rangle^{(1)}}{\langle p_1, p_2|T^{\mu\nu}|0\rangle^{(1)}}\right)$$

- Optical theorem:



$$\mathrm{Im}\,\langle p_1, p_2|\mathrm{Tr}[F^2]|0\rangle^{(1)} \quad = \quad \langle p_1, p_2|\mathcal{M}^{\mathrm{tree}}|p_1', p_2'\rangle \;\otimes \;\otimes\; \langle p_1', p_2'|\mathrm{Tr}[F^2]|0\rangle^{\mathrm{tree}}$$

Tree amplitude x Polynomial

- Parke-Taylor tree amplitude for color-singlet pair:

$$\mathcal{M}^{abcd}_{1^-2^-3^+4^+}\delta^{cd} = -2g^2 C_A \delta^{ab}\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}$$

- Phase space: use nice spinor parametrization

[Zwiebel '11]

$$\begin{pmatrix} \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{i\phi} \\ \sin\frac{\theta}{2}e^{-i\phi} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

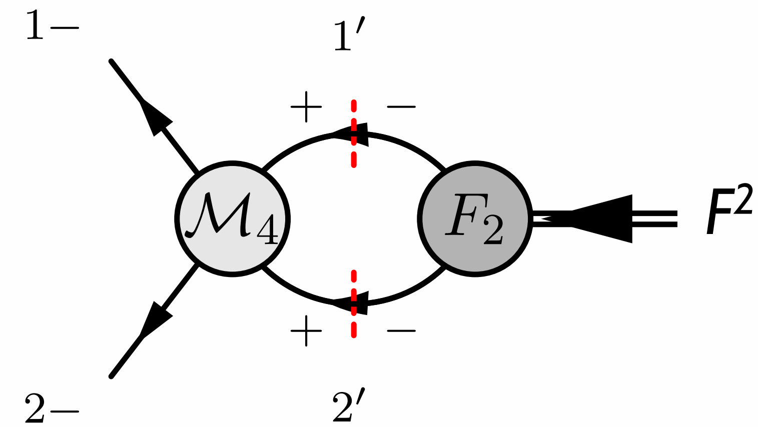
- θ = scattering angle. Plug into amplitude:

$$\langle 14\rangle = \langle 23\rangle = \cos\frac{\theta}{2},\ \text{etc.}$$

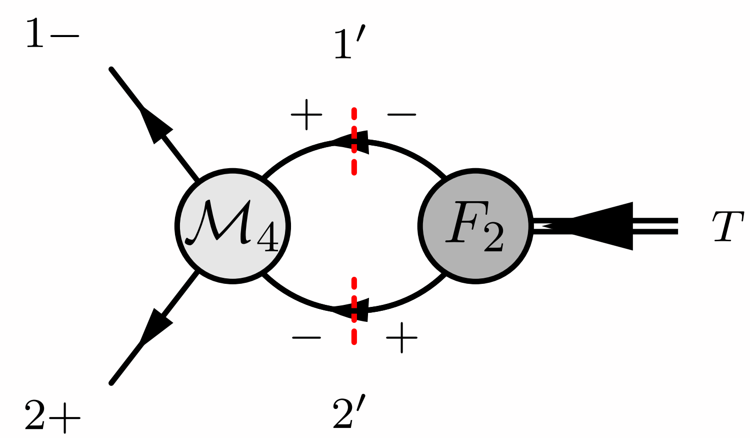
$$\mathcal{M}^{abcd}_{1^-2^-3^+4^+}\delta^{cd} = +\frac{2g^2 C_A \delta^{ab}}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}$$

$$\gamma^{(1)}_{F^2} = -\frac{1}{\pi} 2\,\mathrm{Im}\left(\log \frac{\langle 1_- 2_- | F^2 | 0\rangle^{(1)}}{\langle 1_- 2_+ | T^{\mu\nu} | 0\rangle^{(1)}}\right)$$

$$= -\frac{1}{16\pi^2}\int_0^\pi \frac{\sin\theta d\theta}{2}\left(+ \frac{2g^2 C_A}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}} - \frac{2g^2 C_A(\cos^8\frac{\theta}{2} + \sin^8\frac{\theta}{2})}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}\right)$$

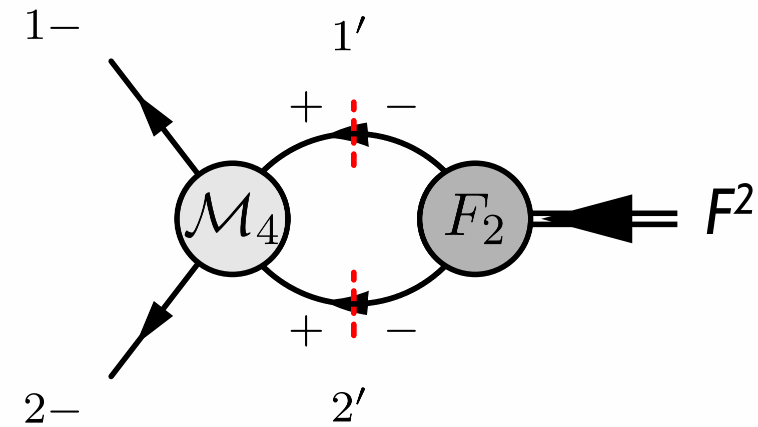


from T^{uv} helicities



$$\gamma_{F^2}^{(1)} = -\frac{1}{\pi} 2\,\mathrm{Im}\left(\log \frac{\langle 1_- 2_- | F^2 | 0\rangle^{(1)}}{\langle 1_- 2_+ | T^{\mu\nu} | 0\rangle^{(1)}}\right)$$

$$= -\frac{1}{16\pi^2}\int_0^\pi \frac{\sin\theta d\theta}{2}\left(+\frac{2g^2 C_A}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}} - \frac{2g^2 C_A(\cos^8\frac{\theta}{2} + \sin^8\frac{\theta}{2})}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}\right)$$



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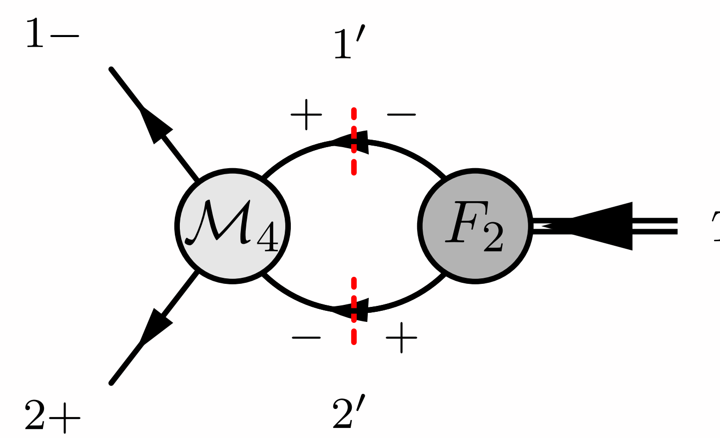


- IR/collinear divergences cancel, integral finite!

$$\gamma_{F^2}^{(1)} = -\frac{g^2}{16\pi^2} \times \frac{22C_A}{3}$$

- Running of F^2 equivalent to β-function

$$\gamma_{F^2} = g^2 \frac{\partial}{\partial g^2}\left(\frac{\beta(g^2)}{g^2}\right)$$

[Kluberg-Stern&Zuber '74]

$$\Rightarrow \beta(g^2) = -\frac{g^2}{16\pi^2} \times \frac{22 C_A}{3} \quad ✓$$

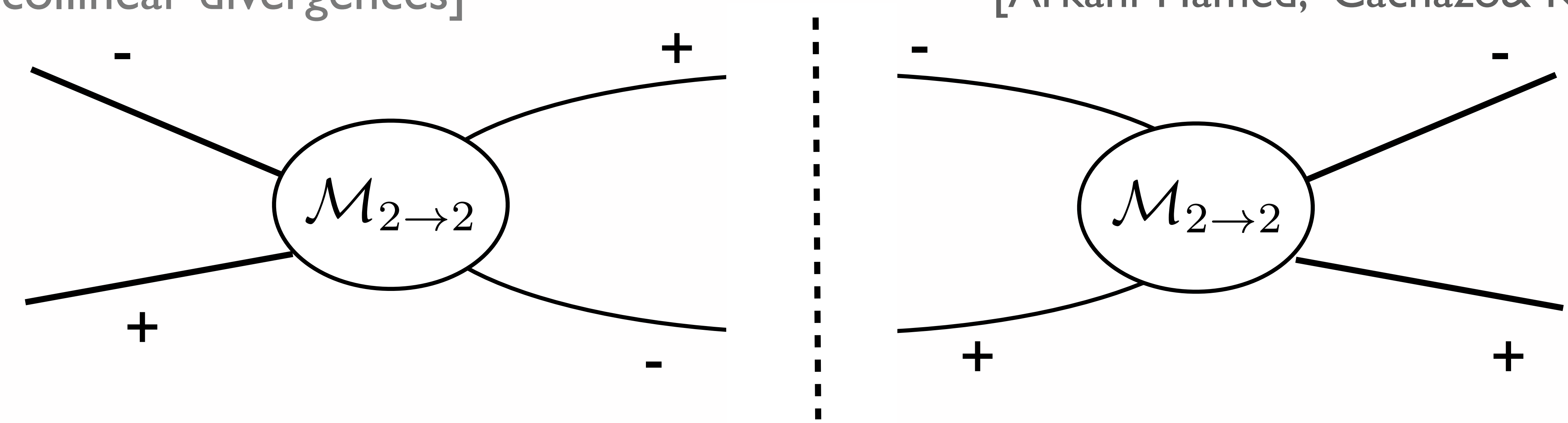
- One-loop unitarity ⇒ β-function as eigenvalue of 2→2 gluon tree amplitude. No ghosts, etc. ✓
- Sign: $\gamma \sim -\mathcal{M}/\pi$ negative because $\mathcal{M}$ is positive: γ probed by two gluons, which attract
- Goal: systematize and extend to higher loops

Related works

- Generalized unitarity:

  1-loop β-function ⇔ 1/ε poles ⇔ sum of bubble coefficients

  [+ collinear divergences] [Arkani-Hamed, Cachazo& Kaplan '08]



- 1-loop dilatation operator in N=4SYM as 2→2 S-matrix

  originally motivated by symmetries! [Zwiebel '11; Wilhelm '14] [Brandhuber et al '15]

- Vanishing of M_{++++} and M_{+++-} leads to helicity selection rules: explains many 'zeros' in dimension-6 SM EFT operator mixing

  [Alonso,Jenkins,Manohar&Trott, '14] [Cheung&Shen, '15]

- Here: only standard unitarity; one tree and one form factor; use T^{uv} to control IR divergences: works for QCD too!

More on stress tensor

- IR-safe ratio: $$\frac{1}{g^2}\beta(g^2)^{(1)} = -\frac{1}{\pi}\, 2\,\mathrm{Im}\left(\log \frac{\langle p_1, p_2|F^2|0\rangle^{(1)}}{\langle p_1, p_2|T^{\mu\nu}|0\rangle^{(1)}}\right)$$
- Denominator is only matter contribution in QCD (& QED!)
- Tree-level T^{uv} form factors: polynomials fixed by symmetry:
  - -normalization is physical: $\langle p|T^{\mu\nu}|p\rangle = 2p^\mu p^\nu$
  - -transverse (momentum conservation)
  - -little group weights

gluons: $$\langle 1^a_- 2^b_+|T^{\alpha\beta,\dot\alpha\dot\beta}|0\rangle = 2\lambda_1^\alpha\lambda_1^\beta\tilde\lambda_2^{\dot\alpha}\tilde\lambda_2^{\dot\beta}$$

quarks: $$\langle 1_\psi 2_{\bar\psi}|T^{\alpha\beta,\dot\alpha\dot\beta}|0\rangle = \tfrac{1}{2}\left(\lambda_1^\alpha\lambda_1^\beta\tilde\lambda_1^{\dot\alpha}\tilde\lambda_2^{\dot\beta} + \lambda_1^\alpha\lambda_1^\beta\tilde\lambda_1^{\dot\beta}\tilde\lambda_2^{\dot\alpha} - \lambda_1^\alpha\lambda_2^\beta\tilde\lambda_2^{\dot\alpha}\tilde\lambda_2^{\dot\beta} - \lambda_1^\beta\lambda_2^\alpha\tilde\lambda_2^{\dot\alpha}\tilde\lambda_2^{\dot\beta}\right)$$

scalars: $$\langle 1_\phi 2_{\bar\phi}|T^{\alpha\beta,\dot\alpha\dot\beta}|0\rangle = \tfrac{1}{3}\left(p_1^{\alpha\dot\alpha}p_1^{\beta\dot\beta}+p_2^{\alpha\dot\alpha}p_2^{\beta\dot\beta}-p_1^{\alpha\dot\alpha}p_2^{\beta\dot\beta}-p_1^{\beta\dot\alpha}p_2^{\alpha\dot\beta}-p_1^{\alpha\dot\beta}p_2^{\beta\dot\alpha}-p_1^{\beta\dot\beta}p_2^{\alpha\dot\alpha}\right)$$

Finite coupling, I: Analyticity

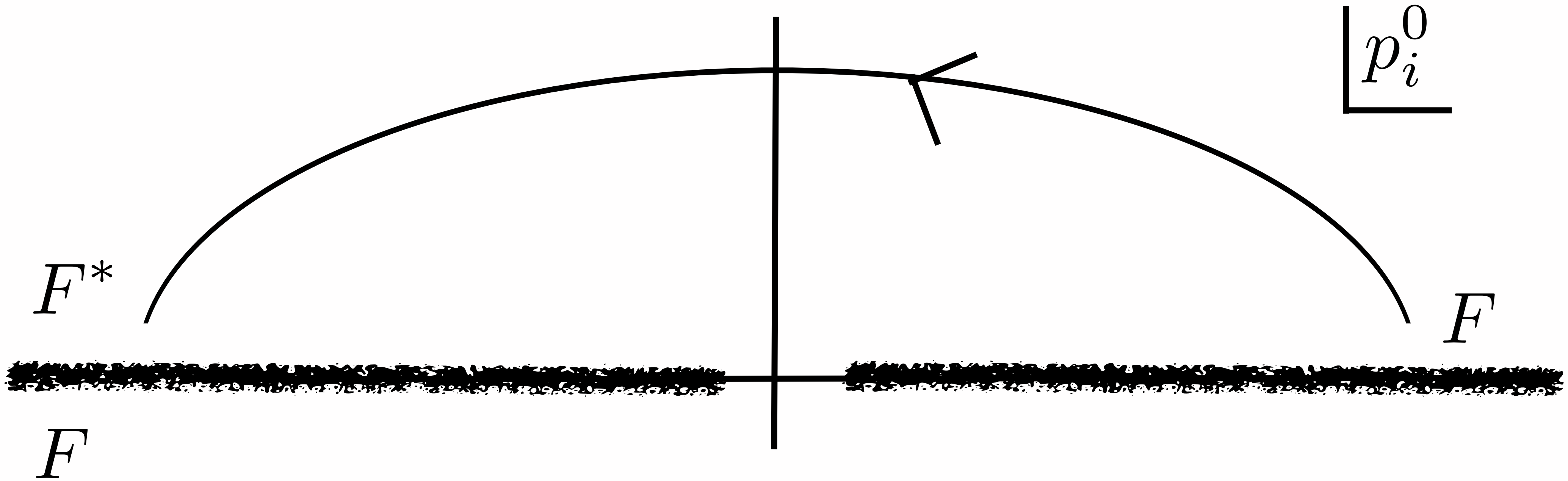
- Energy dependence ⇔ phase of amplitude

- Start from form factor with all outgoing momenta

- Use complex rescaling:

$$F(p_1, \ldots, p_n) \to F(p_1 e^{i\alpha}, \ldots, p_n e^{i\alpha})$$

Finite coupling, I: Analyticity



$$F(p_1, \ldots, p_n) \to F(p_1 e^{i\alpha}, \ldots, p_n e^{i\alpha}) \\ = e^{i\alpha D} F(p_1, \ldots, p_n)\,,$$

- Rotation by π gives complex conjugate:

$$\boxed{F^* = e^{i\pi D} F}$$

$$D \equiv \sum_i p_i^\mu \frac{\partial}{\partial p_i^\mu}$$ dilatation operator

Finite coupling, 2: Optical theorem

- To compute 'imaginary parts'
- 'Standard' optical theorem: $SS^\dagger = 1, S = 1 + i\mathcal{M}$

$$\Rightarrow -i(\mathcal{M} - \mathcal{M}^\dagger) = \mathcal{M}\mathcal{M}^\dagger$$

- Formally, form factor = deformation of S-matrix $\delta S = i\mathcal{F}$

$$0 = \delta(SS^\dagger) = i(\mathcal{F}S^\dagger - S\mathcal{F}^\dagger)$$

$$\Rightarrow \mathcal{F} = S\mathcal{F}^\dagger S$$

- Restrict to final-state particles: $F = SF^*$

Finite coupling

Analyticity + Unitarity

$F = e^{-i\pi D} F^*$

$F = S F^*$

$$\boxed{e^{-i\pi D} F^* = S F^*}$$

Dilatation operator
= minus the phase of the S-matrix,
divided by π.

Partial waves ⇔ twist-2

- Twist-2 operators $\mathrm{Tr}[\bar{\phi}(\overleftrightarrow{D}_+)^m \phi]$
- Mod out total derivatives: go forward p_2=-p_1
- Minimal form factor:

$$\langle p, -p|\mathcal{O}_m|0\rangle = (p_+)^m$$

- Zwiebel's phase space parametrization: [works for spacelike channels!]

$$p'^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} (\cos\tfrac{\theta}{2} - \sin\tfrac{\theta}{2} e^{i\phi})(\cos\tfrac{\theta}{2} + \sin\tfrac{\theta}{2} e^{-i\phi})$$

- Azimuthal integral gives Legendre polynomial!

$$P_m(\cos\theta) = \int_0^{2\pi} \frac{d\phi}{2\pi} (\cos\tfrac{\theta}{2} - \sin\tfrac{\theta}{2} e^{i\phi})^m (\cos\tfrac{\theta}{2} + \sin\tfrac{\theta}{2} e^{-i\phi})^m$$

twist-two operators ⇔ partial wave amplitude

Partial waves ⇔ twist-2

- For QCD: gluons have spin, partial waves are some generalization of Legendre polynomials
- Made simple with Zwiebel's parametrization:

$$\langle 1_- 2_+ | F_{11}(D_{1\dot{1}})^m \bar{F}_{\dot{1}\dot{1}} | 0 \rangle = (\lambda_1)^2 (\tilde{\lambda}_2)^2 (\lambda_1 \tilde{\lambda}_1)^{m-2}$$

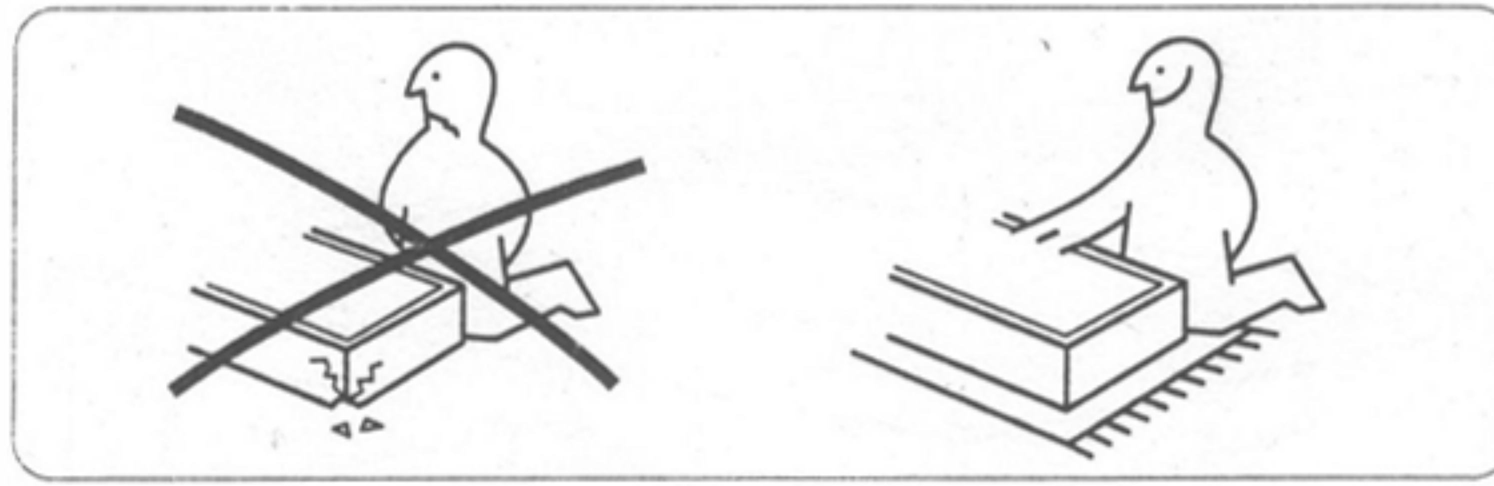
- Rotated form factor gives partial wave projector:

$$\gamma_m \sim \frac{-1}{16\pi^2} \int \frac{d\Omega}{4\pi} \mathcal{M}(\cos\theta) \int_0^{2\pi} \frac{d\phi}{2\pi} (\cos\tfrac{\theta}{2} - i \sin\tfrac{\theta}{2} e^{i\phi})^{m-2} (\cos\tfrac{\theta}{2} + i \sin\tfrac{\theta}{2} e^{-i\phi})^{m+2}$$

- We checked that this matches moments of gluon DGLAP equation! $\quad \gamma_m = \int_0^1 dx x^m P_{gg}(x)$

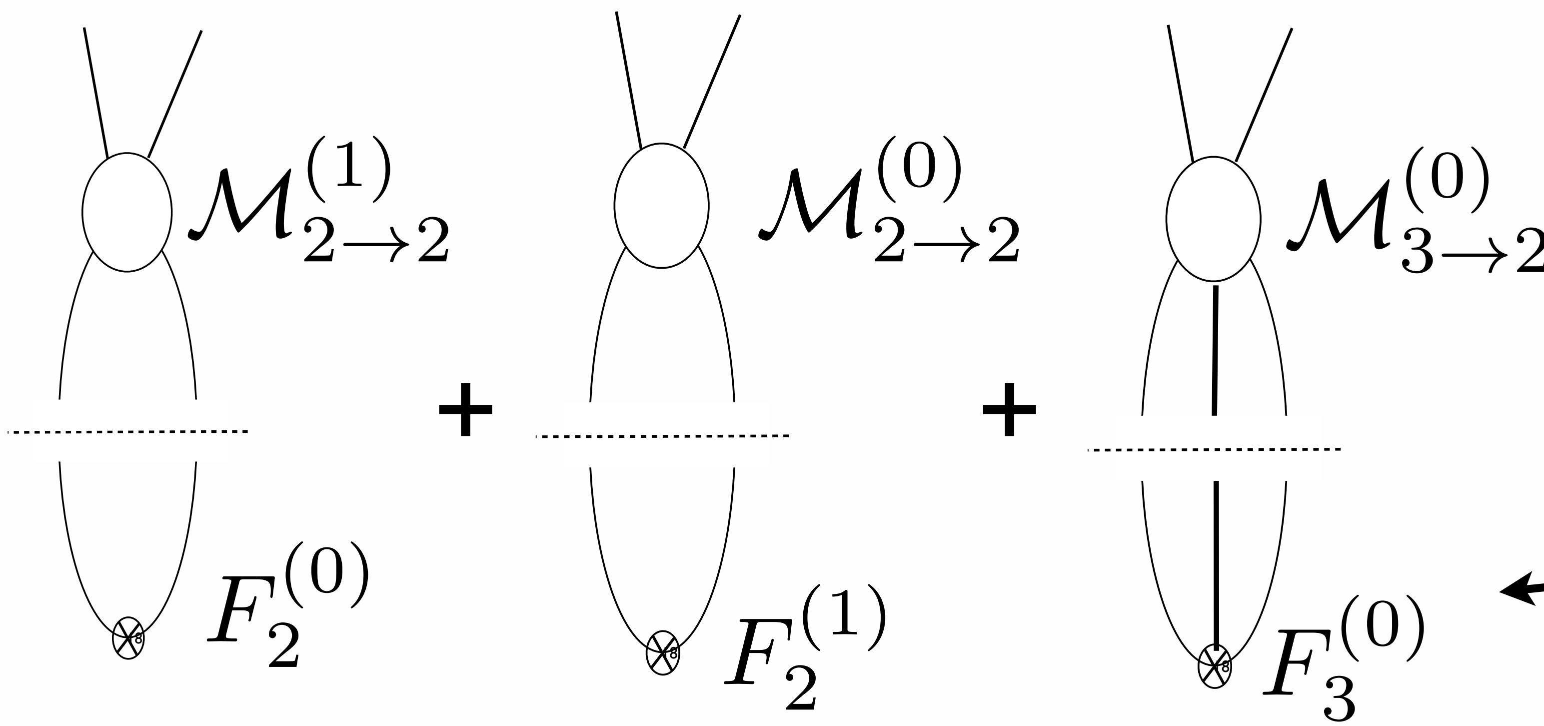
QCD phase shifts (times -1/π) ⇔ DGLAP equation!

Varning!



- At higher loops: Phase of S-<u>matrix</u> $\not\Leftrightarrow$ phase of $S_{2\to 2}$!!

2 loops:



non-minimal form factor

- If coupling runs, phase gives anomalous dimension averaged over complex circle!

$$e^{-i\pi D} F^* = S F^* \qquad D \simeq \left(\gamma_{\mathcal{O}} + \gamma_{\rm IR} + \beta(g^2) \frac{\partial}{\partial g^2} \right)$$

Yukawa theory

- Motivation:
  -mixing between different lengths
  -investigate possible subtleties with formalism
  -see nontrivial interplay between cuts

- 1 real scalar + 1 Weyl fermion

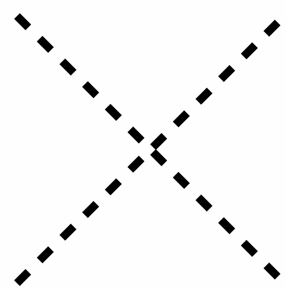
$$\mathcal{L}_{\rm int} = \lambda \mathcal{O}_\lambda + y \mathcal{O}_y \quad \text{with} \quad \mathcal{O}_\lambda = -\frac{1}{4!}\phi^4 \quad \text{and} \quad \mathcal{O}_y = \frac{1}{2}(\psi\psi\phi + \text{h.c.})\,.$$

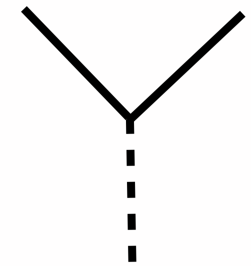
- Anomalous dimension matrix:

$$-\mu \frac{\partial}{\partial \mu} \begin{pmatrix} \mathcal{O}_\lambda \\ \mathcal{O}_y \end{pmatrix} = \begin{pmatrix} \gamma_{yy} & \gamma_{y\lambda} \\ \gamma_{\lambda y} & \gamma_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} \mathcal{O}_\lambda \\ \mathcal{O}_y \end{pmatrix}$$

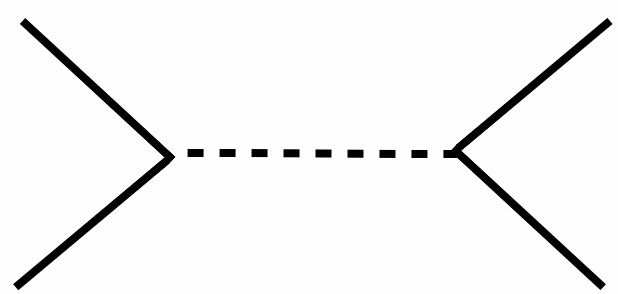
- Basic matrix elements:

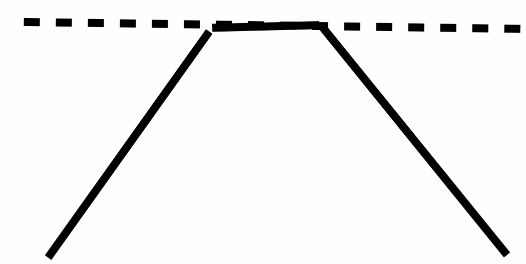
$$\mathcal{M}_{1_\phi 2_\phi 3_\phi 4_\phi} = -\lambda\,, \qquad \mathcal{M}_{1_\psi 2_\psi 3_\phi} = y\langle 12\rangle$$





- From these, simple 2→2 tree amplitudes



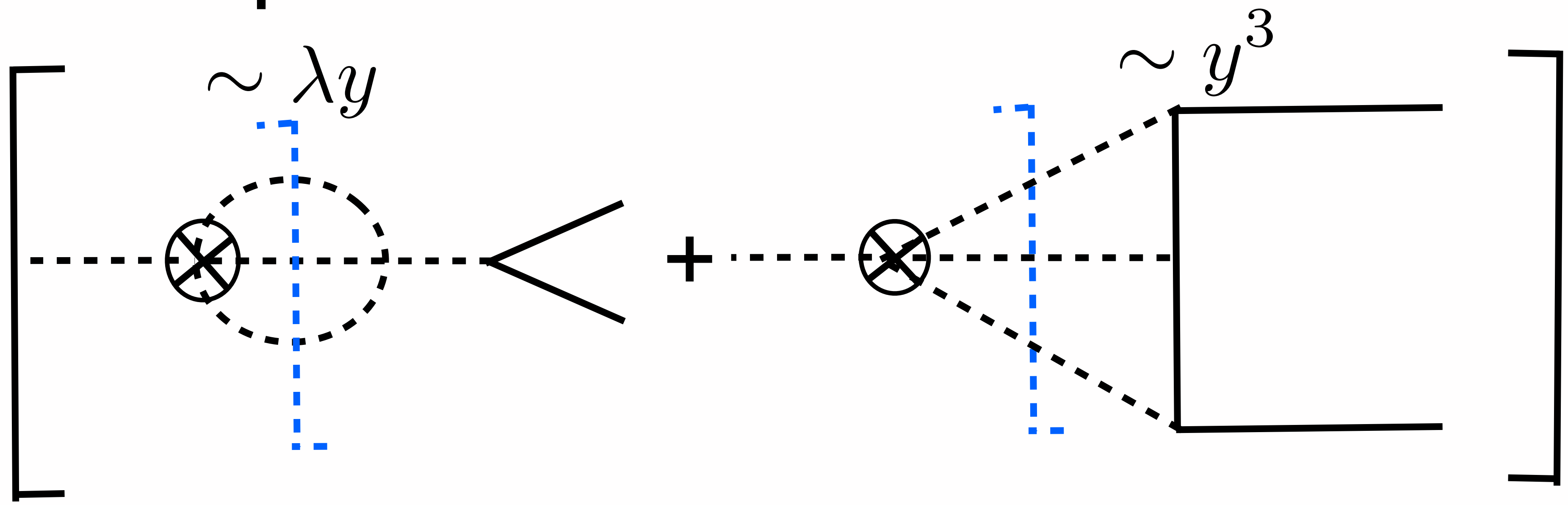


& permutations

- Diagonal elements, same recipe:
  -act with 2→2 amplitude on all pairs
  -subtract IR (collinear) divs. using stress tensor

- Lower-triangular elements: two-loop integrals with 3→2 tree amplitude:

$\gamma_{y\lambda} \sim -\frac{1}{\pi}$



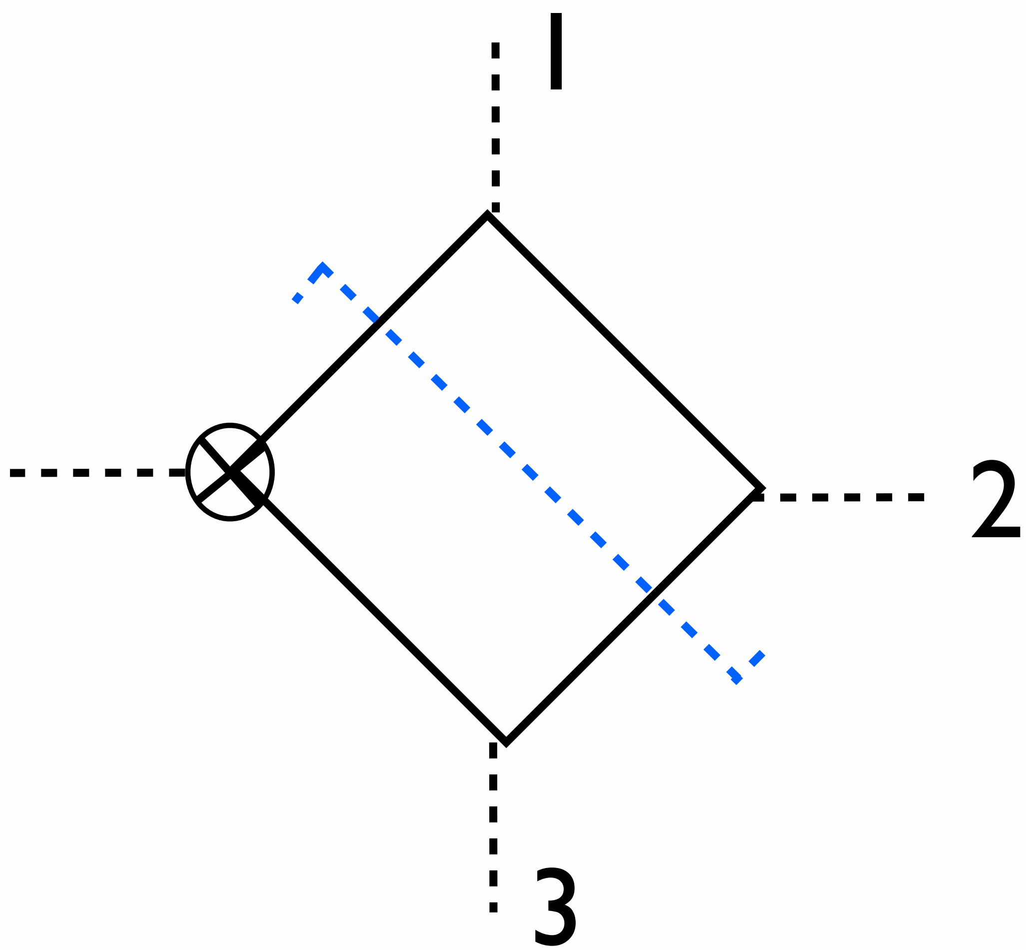
- Lower-triangular vanish at 1-loop: can't have 2→1 cut!

- Keep all graphs in 3→2 amplitude: 1PI or not

- Only single-scale integrals. Propagators& measure factorize, integrals ok with explicit parametrization:

$$
\begin{aligned}
{\lambda'_1}^{\alpha} &= \lambda_1^\alpha \cos\theta_2 - \mathrm{e}^{i\phi}\,\lambda_2^\alpha \cos\theta_1 \sin\theta_2\,, \\
{\lambda'_2}^{\alpha} &= \lambda_1^\alpha \sin\theta_2 \cos\theta_3 + \mathrm{e}^{i\phi}\,\lambda_2^\alpha \left(\cos\theta_1 \cos\theta_2 \cos\theta_3 - \mathrm{e}^{i\rho} \sin\theta_1 \sin\theta_3\right)\,, \\
{\lambda'_3}^{\alpha} &= \lambda_1^\alpha \sin\theta_2 \sin\theta_3 + \mathrm{e}^{i\phi}\,\lambda_2^\alpha \left(\cos\theta_1 \cos\theta_2 \sin\theta_3 + \mathrm{e}^{i\rho} \sin\theta_1 \cos\theta_3\right)\,.
\end{aligned}
$$

- Upper-triangular elements (length-increasing): formally one-loop, but individual cuts harder



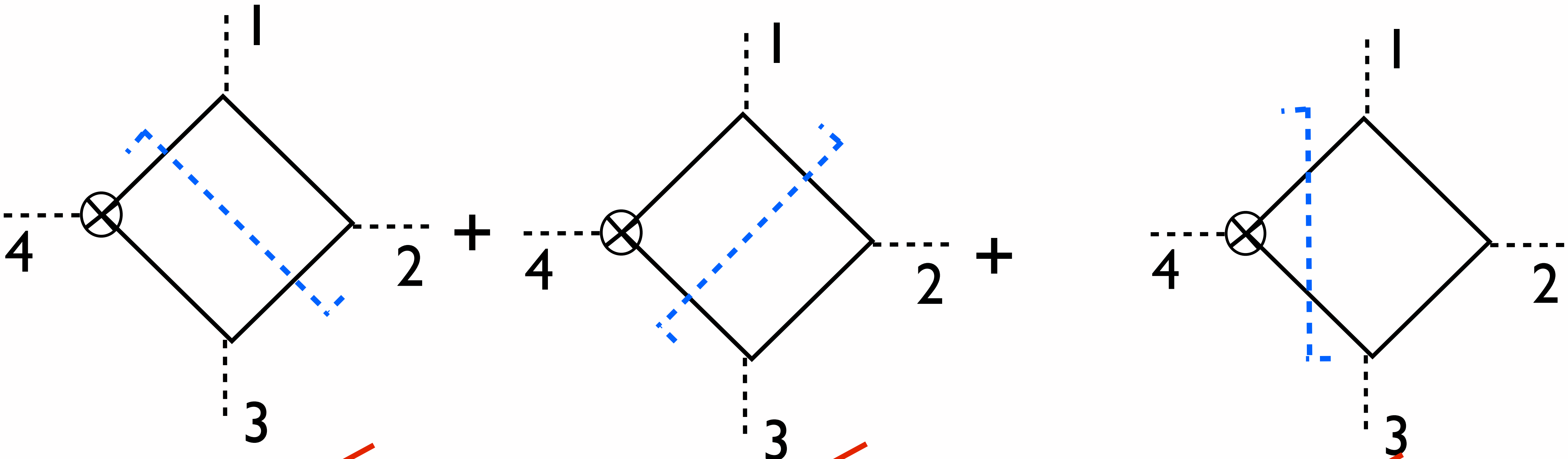
$$-\frac{1}{\pi}\int d\mathrm{Lips}\,\frac{\langle l_2 p_1\rangle\langle l_1 p_3\rangle}{\langle l_1 p_1\rangle\langle l_2 p_3\rangle}$$

New feature:
integral produces a log

$$\propto 1+\log\frac{p_2\cdot p_3}{(p_1+p_2)\cdot p_3}$$

Generic feature beyond LO for final states with >= 3 particles! But RG should give a polynomial!

Cancellation between cuts:



$$1 + \cancel{\log \frac{p_2\cdot p_3}{(p_1+p_2)\cdot p_3}} \;\; + 1 + \cancel{\log \frac{p_1\cdot p_2}{p_1\cdot(p_2+p_3)}} \;\; + \cancel{\log \frac{(p_1+p_2)\cdot p_3 \, p_1\cdot(p_2+p_3)}{p_1\cdot p_2 \, p_2\cdot p_3}} \;\; = 2$$

Generically expected!

- In summary [up to irrelevant typos]

$$\begin{array}{c} \begin{array}{cc} n=3 & n=4 \end{array} \\ \begin{pmatrix}\gamma_{yy} & \gamma_{y\lambda}\\ \gamma_{\lambda y} & \gamma_{\lambda\lambda}\end{pmatrix}\end{array} = \frac{1}{16\pi^2}\begin{pmatrix} \overset{\text{1-loop, 'easy'}}{12y^2} & \overset{\text{1-loop } 3\to 4 \text{ 'hard'}}{8\lambda y - 96y^3} \\ \underset{\text{2-loop, 'easy'}}{0 + \frac{-2y^3+\lambda y/6}{16\pi^2}} & \underset{\text{1-loop, 'easy'}}{6\lambda+4y^2}\end{pmatrix}$$

- Length-changing effect only affect eigenvalues at two-loops: '1½loops'

(because of the one-loop zero for length-decreasing effects)

- To get β-function:
  use generalization of QCD formula for F^2!

[Kluberg-Stern&Zuber '74]

$$\partial_a \beta_b = \gamma_{ab}$$

[interpretation:

- Form factor = variation of S-matrix

$$\partial_a \mathcal{M}(g_a(\mu), \mu) = \mathcal{F}_a$$

- Commute this variation with RGE:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_a \frac{\partial}{\partial g_a}\right) \mathcal{M}(g_a(\mu), \mu)$$

- 'anomalous dimension of couplings controls perturbations of RG flow']

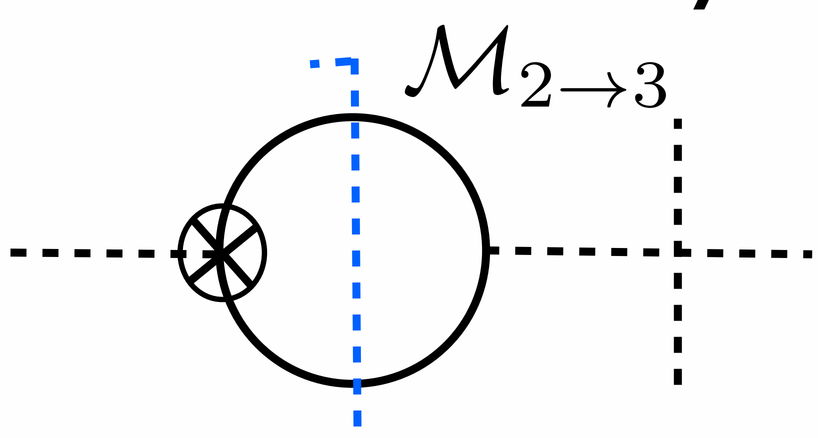
Symmetry of derivatives not manifest!

$$\begin{pmatrix} \gamma_{yy} & \gamma_{y\lambda} \\ \gamma_{\lambda y} & \gamma_{\lambda\lambda} \end{pmatrix} = \frac{1}{16\pi^2} \begin{pmatrix} 12y^2 & 8\lambda y - 96y^3 \\ 0 + \frac{-2y^3+\lambda y/6}{16\pi^2} & 6\lambda + 4y^2 \end{pmatrix}$$

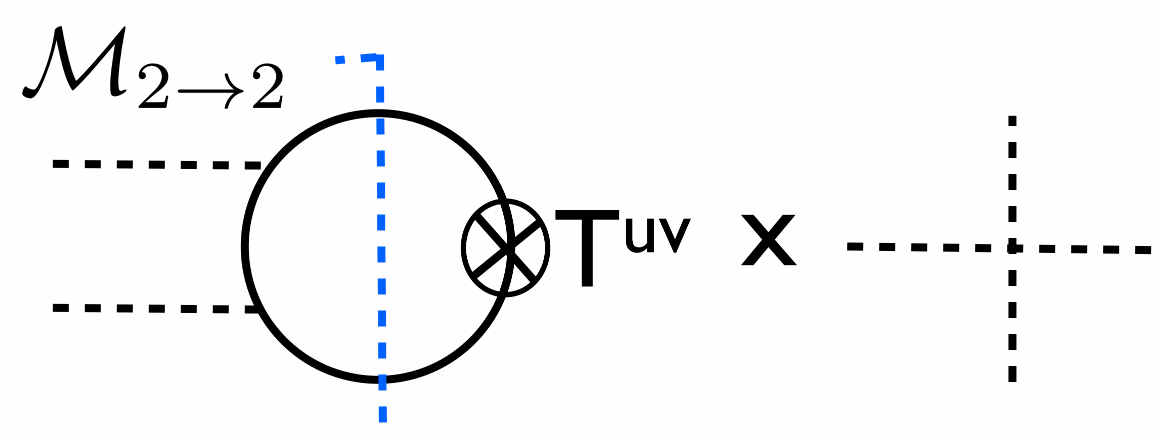
$$\partial_a \beta_b = \gamma_{ab} \Rightarrow \quad \beta(\lambda) = \frac{1}{16\pi^2}\left(3\lambda^2 + 4\lambda y^2 - 24y^4\right)$$

$$\beta(y) = \ldots$$

Nice consistency check!



versus

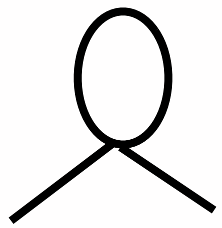


Which property of the S-matrix ensures this??

On masses

- In conventional RG applications, a particle is either light or heavy and integrated out:
  it is appropriate to use massless S-matrix
- Example: running of heavy quark mass
  operator $\bar{\psi}_Q \psi_Q$ is only relevant at energies >> m_Q

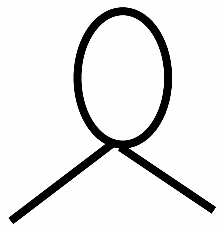
- Does unitarity 'misses' mass logarithms?



$$\propto \lambda m^2 \log\frac{m^2}{\mu^2} \Rightarrow \mu\frac{\partial}{\mu}(m^2) \overset{?}{\propto} \lambda m^2$$

No cut!

- Does unitarity 'misses' mass logarithms?



$$\propto \lambda m^2 \log \frac{m^2}{\mu^2} \quad \Rightarrow \mu \frac{\partial}{\mu}(m^2) \overset{?}{\propto} \lambda m^2$$

No cut!

- No! RG can answer two distinct questions:
  1. Optimal bare parameters to use at a given cutoff scale (e.g. lattice, or putative UV completion of SM)?
  2. Optimal running couplings to use to minimize large logs in a physical observable at given energy scale?

- Constant logarithms affect only first question [and quadratic divergences]

⇒Unitarity correctly answers second question

[see evanescent effects: Bern,Cheung,Chi,Davies,Dixon&Nohle '15]

Summary

- Dilatation operator is minus the phase of the S-matrix, divided by π. ($e^{-i\pi D}F^* = SF^*$)

- One-loop YM β-function "$-\frac{11}{3}$" is an eigenvalue of Parke-Taylor amplitude:

$$\mathcal{M}_4 = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

- IR divergences: cancel using stress tensor

- Analyzed Yukawa to 1½ loops

Outlook

- Dilatation operator in N=4/QCD at higher loops? (Twist-two in QCD: all spins from same matrix elements?)
  [Beisert,Ferretti,Heise&Zarembo '04]
  [(Loebbert), Nandan,Sieg,Wilhelm&Yang, '15]
  [Brandhuber, Kostacinska,Penante,Travaglini&Young,'16]

- Length-changing: mysterious cancellations of logs between cuts? N=4 Yangian? Role of *F** in $e^{-i\pi D}F^* = SF^*$?
  [Brandhuber, Heslop,Travaglini&Young,'15]

- Which questions can be answered with spectrum of S (now already known in planar N=4 SYM?)

- Exponentiation of logs: Derive RGE from on-shell ideas?