Two-loop Integrands from the Riemann Sphere

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Amplitudes, Stockholm

arXiv:1507.00321, 1511.06315, 1607.?
YG, Lionel Mason, Ricardo Monteiro, Piotr Tourkine

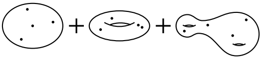
Motivation

Feynman diagrams

$$\mathscr{M} = \sum_{\text{graphs } \Gamma} \frac{\omega_\Gamma(k_i, \epsilon_i)}{\text{ord}(\Gamma)}$$

- graph combinatorics
- combinatorical problem

Worldsheet models

$\mathscr{M} =$  $+ \ \ldots$

- integration over moduli space
- geometric problem

CHY formulae and ambitwistor strings

$\mathscr{M} =$ $+ \ \ldots \Big|_{E_i(\sigma_j)=0}$

- localized on scattering equations $E_i(\sigma_j)$
- algebraic problem

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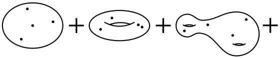
$\mathscr{M} =$  $+ \; \dots$

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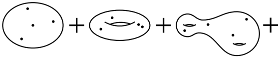
$\mathscr{M} =$  $+ \; \dots \;\Big|_{E_i(\sigma_j)=0}$

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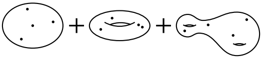

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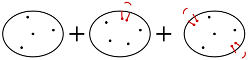

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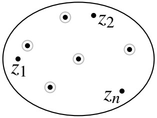
Scattering Equations and CHY formulae

The Scattering Equations

[Cachazo-He-Yuan, Mason-Skinner]

Construction: for n null momenta k_i, define
$P(\sigma, \sigma_i) \in \Omega^0(\Sigma, K_\Sigma)$

$$P(\sigma) = \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i} \, d\sigma \, .$$



Scattering Equations at tree-level

Enforce $P^2 = 0$ on Σ:

$$E_i \equiv \text{Res}_{\sigma_i} P^2(\sigma) = k_i \cdot P(\sigma_i) = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0 \, .$$

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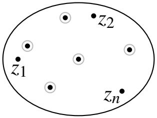

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- degeneration of sphere $\Leftrightarrow$ factorisation channel of amplitude
- Origin: Ambitwistor Strings
  $P(\sigma,\sigma_i) \in \Omega^0(\Sigma, K_\Sigma)$ solution to

$$\bar\partial P = 2\pi i \sum_i k_i \,\bar\delta(\sigma - \sigma_i) d\sigma\,.$$

Scattering Equations and CHY formulae

CHY formulae

[Cachazo-He-Yuan]

Tree-level S-matrix of massless theories:

$$\mathcal{M}_{n,0} = \int_{M_{0,n}} \frac{d\sigma^n}{\text{vol}\, G} \prod_i \bar{\delta}\left(\sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} \right) \mathcal{I}_n$$

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- Gravity: $\mathcal{I}_n = \text{Pf}\,'(M)\, \text{Pf}\,'(\tilde{M})$
- Yang-Mills theory: $\mathcal{I}_n = C_n\, \text{Pf}\,'(M)$
- Bi-adjoint scalar: $\mathcal{I}_n = C_n\, \tilde{C}_n$

with building blocks

- Parke-Taylor factor: $C_n(1, \ldots, n) = \frac{1}{\sigma_{1\,2} \ldots \sigma_{n-1\,n} \sigma_{n\,1}}$
- Reduced Pfaffian: $\text{Pf}'(M) = \frac{(-1)^{i+j}}{\sigma_{ij}} \text{Pf}(M^{ij}_{ij}), \qquad M = M(k_i, \epsilon_i, \sigma_i)$

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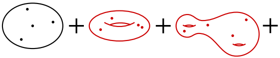
$\mathcal{M} =$  $+ \ \ldots$

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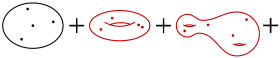
$\mathcal{M} =$  $+ \ \ldots \Big|_{E_i(\sigma_j)=0}$

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The 1-loop Integrand and the SE on the torus [Adamo-Casali-Skinner, Casali-Tourkine]

One-loop integrand of type II supergravity from ambitwistor strings

$$\mathscr{M}_{\text{SG}}^{(1)} = \int d^d\ell \, d\tau \, \underbrace{\bar\delta(P^2(z_0)) \prod_{i=2}^{n} \bar\delta(k_i \cdot P(z_i))}_{\text{Scattering Equations}} \underbrace{\left(\sum_{\text{spin struct.}} Z^{(1)}(z_i) Z^{(2)}(z_i) \right)}_{\equiv \mathcal{I}_q, \text{ fermion correlator}}$$

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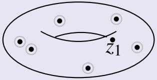
Solve $\bar\partial P = 2\pi i \sum_i k_i\, \bar\delta(z-z_i)\, dz$ on the torus $\Sigma_\tau = \mathbb{C}/\{\mathbb{Z}\oplus\mathbb{Z}\tau\}$:

$$P = 2\pi i\, \ell dz + \sum_i k_i\, \tilde S_1(z-z_i|\tau)\, dz\,.$$

Scattering Equations on the torus:

Enforce $P^2 = 0$ on Σ_τ:

$$\text{Res}_{z_i} P^2(z) := 2k_i\cdot P(z_i) = 0\,,$$
$$P^2(z_0) = 0\,.$$



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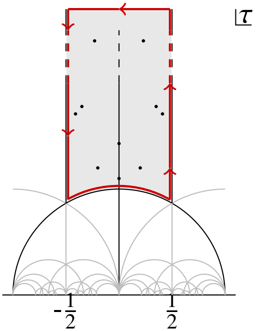
- modular invariant: $\tau \to \tau + 1$, $\tau \to -1/\tau$
- integrand localises on discrete set of solutions

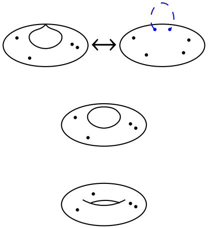
From the Torus to the Riemann Sphere
Contour argument

- localisation on scattering equations
  ⇒ contour integral argument in fundamental domain
- modular invariance: sides and unit circle cancel

⇒ localisation on $q = 0$



Contour argument in the fundamental domain



From the Torus to the Riemann Sphere

Mapping the fundamental domain to the Riemann sphere, we get

$$P = \left(\frac{\ell}{\sigma - \sigma_{\ell^+}} - \frac{\ell}{\sigma - \sigma_{\ell^-}} + \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i} \right) d\sigma \,,$$

Define $S = P^2 - \left(\frac{\ell}{\sigma-\sigma_{\ell^+}} - \frac{\ell}{\sigma-\sigma_{\ell^-}}\right)^2 d\sigma^2$.

One-loop off-shell scattering equations

$$\text{Res}_{\sigma_i} S = k_i \cdot P(\sigma_i) = \frac{k_i \cdot \ell}{\sigma_i - \sigma_{\ell^+}} - \frac{k_i \cdot \ell}{\sigma_i - \sigma_{\ell^-}} + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0 \,,$$

$$\text{Res}_{\sigma_{\ell^-}} S = - \sum_j \frac{\ell \cdot k_j}{\sigma_{\ell^-} - \sigma_j} = 0 \,,$$

$$\text{Res}_{\sigma_{\ell^+}} S = \sum_j \frac{\ell \cdot k_j}{\sigma_{\ell^+} - \sigma_j} = 0 \,.$$

From the Torus to the Riemann Sphere

One-loop integrand on the nodal Riemann sphere

$$\mathscr{M}^{(1)} = -\int \frac{d^d\ell}{\ell^2}\frac{d^{n+2}\sigma}{\text{vol}\,(G)} \quad \underbrace{\prod_{i,\ell^\pm} \bar{\delta}\left(\text{Res}_{\sigma_i} S\right)}_{\text{off-shell scattering equations}} \quad \mathcal{I}$$

- manifestly off-shell through loop momentum ℓ
- closely related to tree-level scattering equations at $n+2$ points
- manifestly rational

Integrands – Colour/kinematics again

One-loop integrand on the nodal Riemann sphere

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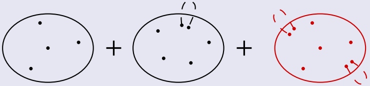
- supergravity: $\mathcal{I} = \frac{1}{(\sigma_{\ell^+\,\ell^-})^4}\ \mathcal{I}_0\ \widetilde{\mathcal{I}}_0$
- super Yang-Mills theory: $\mathcal{I} = \frac{1}{(\sigma_{\ell^+\,\ell^-})^4}\ \mathcal{I}_0\ \mathcal{I}^{PT}$

with building blocks

- Parke-Taylor: $\mathcal{I}^{PT} = \sum_{i=1}^n \frac{\sigma_{\ell^+\,\ell^-}}{\sigma_{\ell^+\,i}\sigma_{i+1\,i}\sigma_{i+2\,i+1}\ldots\sigma_{i+n\,\ell^-}}$
- Pfaffian: $\mathcal{I}_0 = \sum_r \text{Pf}\,'(M^r_{\text{NS}}) - \frac{c_d}{\sigma^2_{\ell^+\,\ell^-}}\text{Pf}\,(M_2)$

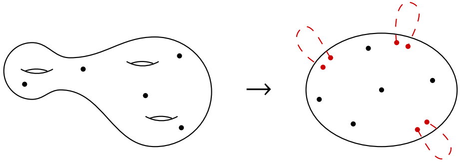
Checks: numerical, factorisation
Also: non-supersymmetric

Question: Beyond one loop?

$\mathscr{M} =$  $+ \dots \Big|_{E_i(\sigma_j)=0}$

Heuristics: Higher genus

Riemann surface Σ_g $\xrightarrow[\text{contract } g \text{ } a\text{-cycles}]{\text{residue theorems}}$ nodal RS



- fixes g moduli
- remaining $2g-3 \Leftrightarrow 2g$ new marked points mod $SL(2,\mathbb{C})$
- 1-form P_μ

$$P = \sum_{r=1}^{g} \ell_r \omega_r + \sum_i k_i \frac{d\sigma}{\sigma - \sigma_i}\,,$$

$\omega_r = \frac{(\sigma_{r^+} - \sigma_{r^-})\, d\sigma}{(\sigma - \sigma_{r^+})(\sigma - \sigma_{r^+})}$: basis of g global holomorphic 1-forms

Two loops

[Adamo-Casali]

n=4 g=2 supergravity amplitude

$$\mathcal{M}_4^{(2)} = N \int_{\mathcal{M}_{2,4}} \underbrace{\prod_{r\leq s} d\Omega_{rs}}_{\text{period matrix}} \prod_{i=1}^{n} \bar{\delta}(E_i) \prod_{a=1}^{3} \bar{\delta}\left(\int \mu P^2\right) Y^2$$

where

- $P_\mu(z) = \sum_{r=1}^{2} \ell_\mu^r \omega_r + \sum_{i=1}^{n} k_i \tilde{S}_2(z - z_i|\Omega)$
- abelian differentials ω_r dual to a-cycles
  $\oint_{a_r} \omega_s = \delta_{rs}, \oint_{b_r} \omega_s = \Omega_{rs}$
- modular invariant

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where

- $P_\mu(z) = \sum_{r=1}^{2} \ell_\mu^r \omega_r + \sum_{i=1}^{n} k_i \, \tilde{S}_2(z - z_i|\Omega)$
- Scattering equations: $E_i \equiv \text{Res}_{z_i} P^2 = k_i \cdot P(z_i) = 0$
- Choice of Beltrami differential: $P^2(x_a) = 0$

  Or: $u_{rs} = 0$

  where $P^2 = \sum_{r\leq s} u_{rs} \omega_r \omega_s$

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where

- $P_\mu(z) = \sum_{r=1}^{2} \ell_\mu^r \omega_r + \sum_{i=1}^{n} k_i \tilde{S}_2(z - z_i|\Omega)$
- integrand:

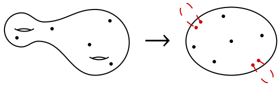
$$3Y = (k_1 - k_2) \cdot (k_3 - k_4)\Delta_{12}\Delta_{34} + \text{perm}$$

where $\Delta_{ab} = \omega_{[1}(z_a)\omega_{2]}(z_b)$

From $g = 2$ to the Riemann sphere

Two-loop amplitude:

$$\mathcal{M}^{(2)} = \frac{1}{\ell_1^2 \ell_2^2} \int \frac{d\sigma_A}{\text{vol}G} \prod_A \bar{\delta}\left(E_A^{(2)}\right) \mathcal{I}$$



with the two-loop off-shell scattering equations:

$$E_A^{(2)} \equiv \text{Res}_{\sigma_A} S = 0 \,, \quad A = 1, \ldots, n + 2g$$

$$S(\sigma) \equiv P^2 - \sum_{r=1}^{2} \ell_r^2 \omega_r^2 \,.$$

and $P = \sum_{r=1}^{g} \ell_r \omega_r + \sum_i k_i \frac{d\sigma}{\sigma - \sigma_i}$.

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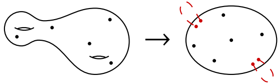

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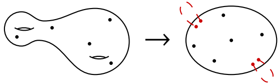
and $P = \sum_{r=1}^{g} \ell_r \omega_r + \sum_i k_i \frac{d\sigma}{\sigma - \sigma_i}$.
a_{12}: different choices of μ.

From $g=2$ to the Riemann sphere



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with the two-loop off-shell scattering equations:

$$E_{1^\pm}^{(2)} = \pm\frac{1}{2}\left(a_{12}(\ell_1^2+\ell_2^2) + 2\ell_1\cdot\ell_2\right)\left(\frac{1}{\sigma_{1^\pm 2^+}} - \frac{1}{\sigma_{1^\pm 2^-}}\right) \pm \sum_i \frac{\ell_1\cdot k_i}{\sigma_{1^\pm i}}$$

$$E_i^{(2)} = \ell_1\cdot k_i\left(\frac{1}{\sigma_{i\,1^+}} - \frac{1}{\sigma_{i\,1^-}}\right) + \ell_2\cdot k_i\left(\frac{1}{\sigma_{i\,2^+}} - \frac{1}{\sigma_{i\,2^-}}\right) + \sum_j \frac{k_i\cdot k_j}{\sigma_{ij}}$$

a_{12}: different choices of μ.

Question: What is a_{12}?

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- Still unclear from the ambitwistor string.
- Physicists' approach: What properties could fix a_{12}?

Factorisation

[YG-Mason-Monteiro-Tourkine]

All poles (and residues) of $\mathscr{M}^{(1)}$ are determined from

- localising on the boundary of the moduli space

$$\Leftrightarrow$$

- factorising the nodal Riemann sphere

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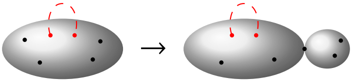
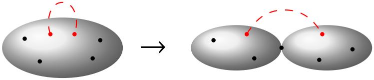
There are two cases of interest:

Separating:



Q-cut:



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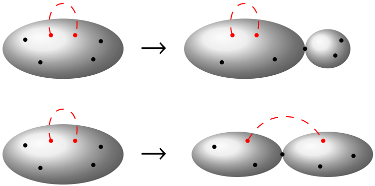


Pole structure:

$$\tilde{s}_I = \begin{cases} k_I^2 & \text{separating} \\ 2\ell \cdot k_I + k_I^2 & \text{non-separating} \end{cases}$$

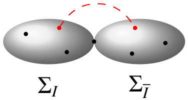
Why?

Scattering equations on degenerate Riemann sphere, $\sigma_i = \sigma_I + \varepsilon x_i$ for $i \in I$:

$$P(\sigma) = \frac{1}{\varepsilon} P_I(x) + \mathcal{O}(1) \qquad \sigma \in I$$

$$P(\sigma) = P_{\bar{I}}(\sigma) + \mathcal{O}(\varepsilon) \qquad \sigma \notin I$$

where $P_I(x) = \sum_{i \in I} \frac{k_i}{x - x_i} dx$



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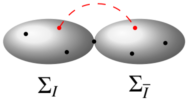


Thus the pole is given by (with $k_{n+1,n+2} = \pm \ell$):

$$\tilde{s}_I = \frac{1}{2} \sum_{i,j \in I} k_i \cdot k_j = \sum_{i,j \in I} x_i \frac{k_i \cdot k_j}{x_i - x_j} = \sum_{i \in I} x_i \, k_i \cdot P_I(x_i) = \mathcal{O}(\varepsilon)$$

Q-cuts

[Baadsgaard, Bjerrum-Bohr, Bourjaily, Caron-Huot, Damgaard, Feng]

Starting from the Feynman diagram expansion,

$$\mathscr{M}_{FD}^{(1)} = \frac{N(\ell)}{D_1(\ell)...D_m(\ell)},$$

- shift $\ell \to \ell + \eta$, such that $\eta \cdot \ell = \eta \cdot k_i = 0$, and $\eta^2 = z$.
  $\Rightarrow D_i(\ell) \to D_i(\ell) + z$
- Cauchy Residue theorem
- shift ℓ in each term by appropriate sum of external momenta k_I
- remove forward limit singularities

Q-cuts

[Baadsgaard, Bjerrum-Bohr, Bourjaily, Caron-Huot, Damgaard, Feng]

Starting from the Feynman diagram expansion,

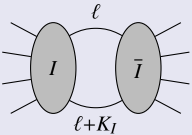
$$\mathcal{M}^{(1)}_{FD} = \frac{N(\ell)}{D_1(\ell)...D_m(\ell)}\,,$$

- shift $\ell \to \ell + \eta$, such that $\eta \cdot \ell = \eta \cdot k_i = 0$, and $\eta^2 = z$.
  $\Rightarrow D_i(\ell) \to D_i(\ell) + z$
- Cauchy Residue theorem
- shift ℓ in each term by appropriate sum of external momenta k_I
- remove forward limit singularities

$\Downarrow$

'Q-cut' expansion

$$\mathcal{M}^{(1)}_{FD} = \sum_I$$



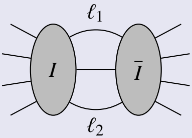
$$= \sum_I \frac{\mathcal{M}^{(0)}_I \mathcal{M}^{(0)}_{\bar{I}}}{\ell^2(2\ell \cdot K_I + K_I^2)}$$

Q-cuts at two loops

[Baadsgaard, Bjerrum-Bohr, Bourjaily, Caron-Huot, Damgaard, Feng]

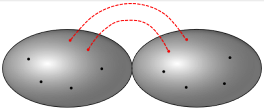
Repeat at two loops:

'Q-cuts' at two loops



$$= \frac{\mathcal{M}_I^{(0)}\,\mathcal{M}_{\bar{I}}^{(0)}}{\ell_1^2\,\ell_2^2\,(\ell_1+\ell_2+k_I)^2}$$

Fixing the ambiguity



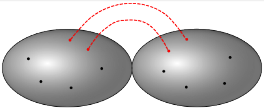
Fix the ambiguity in a_{12}:
correct factorisation on Q-cut

From Scattering Equations:

$$\tilde{s}_I = \begin{cases} a_{12}(\ell_1^2 + \ell_2^2) + 2\ell_1 \cdot \ell_2 + 2(\ell_1 + \ell_2) \cdot k_I + k_I^2 \\ 2\ell_{1,2} \cdot k_I + k_I^2 \end{cases} \overset{!}{=} \begin{cases} (\ell_1 + \ell_2 + k_I)^2 \\ 2\ell_{1,2} \cdot k_I + k_I^2 \end{cases}$$

Fixing the ambiguity

Fix the ambiguity in a_{12}:
correct factorisation on Q-cut



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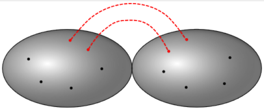
$$a_{12} = 1$$

Fixing the ambiguity



Fix the ambiguity in a_{12}:
correct factorisation on Q-cut

From Scattering Equations:

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$$a_{12} = 1$$

Comments:

- resulting scattering equations Möbius invariant
- unphysical potential pole $\tilde{s} = -(\ell_1 + \ell_2)^2 + 2(\ell_1 - \ell_2) \cdot k_I + k_I^2$ in scattering equations, but never realised in integrand

2-loop amplitudes

Two-loop amplitudes:

$$\mathcal{M}_4^{(2)} = \frac{1}{\ell_1^2 \ell_2^2} \int \frac{\prod d\sigma_A}{\text{vol}G} \underbrace{\prod_A \bar{\delta}\left(E_A^{(2)}\right)}_{\text{Scattering equations}} \quad \mathcal{I}$$

Two-loop scattering equations:

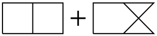
$$E_A^{(2)} \equiv \text{Res}_{\sigma_A} S = 0 \qquad S(\sigma) \equiv P^2 - \sum_{r=1,2} \ell_r^2 \omega_r^2 + (\ell_1^2 + \ell_2^2)\omega_1\omega_2 \,.$$

- Möbius invariant
- factorisation properties

2-loop amplitudes

Two-loop amplitudes:

$$\mathcal{M}_4^{(2)} = \frac{1}{\ell_1^2 \ell_2^2} \int \frac{\prod d\sigma_A}{\text{vol}G} \underbrace{\prod_A \bar{\delta}\left(E_A^{(2)}\right)}_{\text{Scattering equations}} \; \mathcal{I}$$

- Supergravity: $\mathcal{I}^{(2)}_{\text{sugra}} = \left(\frac{Y}{J}\right)^2 \frac{\sigma_{1^+2^-}\sigma_{1^-2^+}}{\sigma_{1^+1^-}\sigma_{2^+2^-}}$ 

  $\Downarrow$ C/K
- super Yang-Mills: $\mathcal{I}^{(2)}_{\text{sYM}} = \left(\frac{Y}{J}\right) \mathcal{I}^{(2)}_{PT}$ 

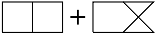
where
$$\mathcal{I}^{(2)}_{PT} = \sum_{\text{cycl},\pm,1/2} \frac{1}{\sigma_{1^+2^+}\sigma_{2^+2^-}\sigma_{2^-1^-}\sigma_{1^-1}\ldots\sigma_{41^+}}$$

$$J = \sigma_{1^+2^+}\sigma_{1^+2^-}\sigma_{1^-2^+}\sigma_{1^-2^-}$$

2-loop amplitudes

Two-loop amplitudes:

$$\mathscr{M}_4^{(2)} = \frac{1}{\ell_1^2 \ell_2^2} \int \frac{\prod d\sigma_A}{\mathrm{vol}\,G} \underbrace{\prod_A \bar{\delta}\left(E_A^{(2)}\right)}_{\text{Scattering equations}} \; \mathcal{I}$$

- Supergravity: $\mathcal{I}^{(2)}_{\text{sugra}} = \left(\frac{Y}{J}\right)^2 \frac{\sigma_{1^+ 2^-}\sigma_{1^- 2^+}}{\sigma_{1^+ 1^-}\sigma_{2^+ 2^-}}$ 

  $\Downarrow$ C/K

- super Yang-Mills: $\mathcal{I}^{(2)}_{\text{sYM}} = \left(\frac{Y}{J}\right) \mathcal{I}^{(2)}_{PT}$ 

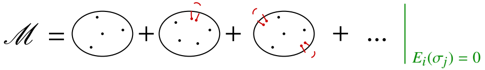
Checks:

- numerics
- factorisation

Conclusion and Outlook

Summary:

- two loop integrands for supergravity and super Yang-Mills
- supported on new, manifestly off-shell scattering equations
- derivation relying on:
  - ambitwistor string origin
  - localisation on the scattering equations
  - colour/kinematics duality
  - factorisation properties of the scattering equations



Outlook:

- sharpen ambitwistor string derivation
- non-supersymmetric theories
- higher loops?

Thank you!