Constrainting effective actions via scattering amplitudes

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Outline

- Introduction
- Supersymmetry constraints on the effective action of $\mathcal{N} = 4$ SYM in the Coulumb branch.
- Constraints from breaking conformal symmetry in the form of soft theorems.
- Scale invariance v.s. conformal symmetry.
- Conclusion and remarks

Introduction

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  For instance $\mathcal{L}_{\rm eff} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{c}{\Lambda^4}(\partial^\mu\phi\partial_\mu\phi)^2 + \dots$,
  Argument:

$$\frac{c}{\Lambda^4} = \frac{2}{\pi}\int_{s_0}^{\infty} ds \frac{\mathrm{Im}\mathcal{A}_4(s,0)}{s^3} = \frac{2}{\pi}\int_{s_0}^{\infty} ds \frac{s\sigma(s)}{s^3} > 0\,.$$

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$$S_{\text{eff}} = -\frac{1}{4}F^2 + \frac{c_4^{(0)}}{m_w^4}F^4 + \frac{c_6^{(0)}}{m_w^8}F^6 + \frac{c_4^{(2)}}{m_w^8}\partial^4 F^4 + \frac{c_4^{(3)}}{m_w^{10}}\partial^6 F^4 + \dots .$$

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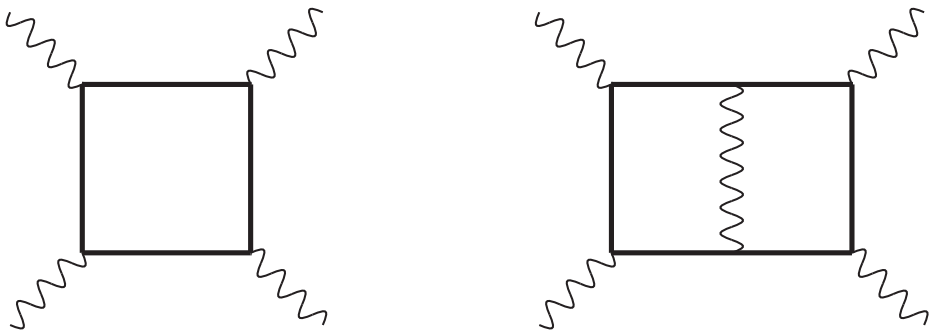
- Perturbation contributions:



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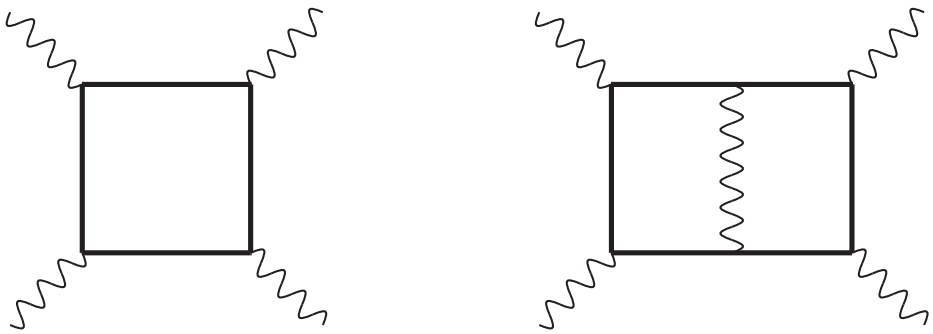

$$A_4^{1-\text{loop}} = F^4 \int \frac{d^4\ell}{(2\pi)^4} \frac{g^4 N}{(\ell^2+m_w^2)((\ell+p_1)^2+m_w^2)((\ell+p_{12})^2+m_w^2)((\ell-p_4)^2+m_w^2)} .$$

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$$\partial^k F^4 : \qquad \frac{g^4 N}{(4\pi)^2 m^4}\left[\frac{1}{2} + \frac{1}{240 m^4}(s^2+t^2+u^2) + \cdots + \right.$$
$$\left. + \frac{1}{192192 m^{16}}\left((s^6+t^6+u^6) - \frac{7}{75}(s^2\,t^2\,u^2)\right)\right] + \cdots$$

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  - Two loops:

$$\partial^k F^4 : \qquad \frac{g^6 N(N+1)}{(4\pi)^4 m^4}\left[0 + \frac{1}{24 m^4}(s^2+t^2+u^2) + \cdots + \right.$$
$$\left. + \frac{7541}{59875200 m^{16}}\left((s^6+t^6+u^6) - \frac{249}{7541}(s^2\,t^2\,u^2)\right)\right] + \cdots$$

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  - One-instanton for Sp(2N) guage group:

$$S_{\rm eff}^{1-inst} = c' \, \frac{g^4}{\pi^6} \, e^{2\pi i \tau} \int \frac{d^4x \, d^8\theta \, \sqrt{\det_{4N} 2\bar{\Phi}_{Au,Bv}}}{\sqrt{\det_{2N}\left(\Phi^{AB}\bar{\Phi}_{AB} + \frac{1}{g}\bar{\mathcal{F}} + \frac{1}{\sqrt{2}g}\bar{\Lambda}_A(\Phi^{-1})^{AB}\bar{\Lambda}_B\right)_{\dot{\alpha}u,\dot{\beta}v}}} \, ,$$

where $\tau = (\vartheta/2\pi) + (4\pi i/g^2)$, and

$$\begin{aligned} \Phi &= v + \varphi + \theta\,\psi + \theta^2 F^- \, , \\ \bar{\Lambda} &= \bar{\psi} + \Theta\partial\varphi - \Theta^2\partial\psi + \Theta^3\partial F^- \, , \\ \mathcal{F} &= F^+ + \theta\partial\bar{\psi} + 4\theta^2\partial^2\varphi + \theta^3\partial^2\psi + \theta^4\partial^2 F^- \, . \end{aligned}$$

  - Integrating out θ and expanding in $1/v$ leads to:

$$S_{\rm eff}^{1-inst} = e^{2\pi i \tau} \sum_{\ell} \left(0 \times (F_-^2 F_+^{2\ell} + F_+^2 F_-^{2\ell}) + c_1 \, F_-^4 F_+^{2\ell} + c_2 \, \partial^4 F_-^2 F_+^{2\ell} + \ldots \right)$$

Suggesting nonrenormalization theorem for $F_-^2 F_+^{2\ell}$!

Non-renormalization theorem from $\mathcal{N} = 4$ SUSY

- As for F^4 ($F_-^2 F_+^2$), this is a known non-renormalization theorem by [Dine, Seiberg (1997)] that F^4 is one-loop exact in $\mathcal{N} = 4$ SYM.

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- F^6 ($F_-^2 F_+^4$ and its conjugate) was found is not generated at one loop, but appears at two loops, and it was conjectured to be two-loop exact! [Buchbinder, Petrov, Tseytlin (2001)]

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- Indeed all the "MHV" operator $F_-^2 F_+^{2\ell}$ is in fact ℓ-loop exact due to $\mathcal{N} = 4$ SUSY. [Chen, Huang, C.W. (2015)]

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- Scattering amplitudes generated from F^n:

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\begin{aligned}
F_-^2 F_+^2 : &\quad \langle 12\rangle^2 [34]^2 \,, \\
F_-^2 F_+^4 : &\quad \langle 12\rangle^2 ([34]^2[56]^2 + [35]^2[46]^2 + [36]^2[45]^2) \,.
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- Now, we supersymmetrize the results via SUSY Ward identity:

$$\begin{aligned} \mathcal{F}^4 : &\quad \frac{\delta^8(Q)}{\langle 12\rangle^4} \langle 12\rangle^2 [34]^2 \,, \\ \mathcal{F}^6 : &\quad \frac{\delta^8(Q)}{\langle 12\rangle^4} \langle 12\rangle^2 ([34]^2[56]^2 + [35]^2[46]^2 + [36]^2[45]^2) \,, \end{aligned}$$

with supercharge $Q^{\alpha,A} = \sum \lambda_i^\alpha \eta_i^A$.

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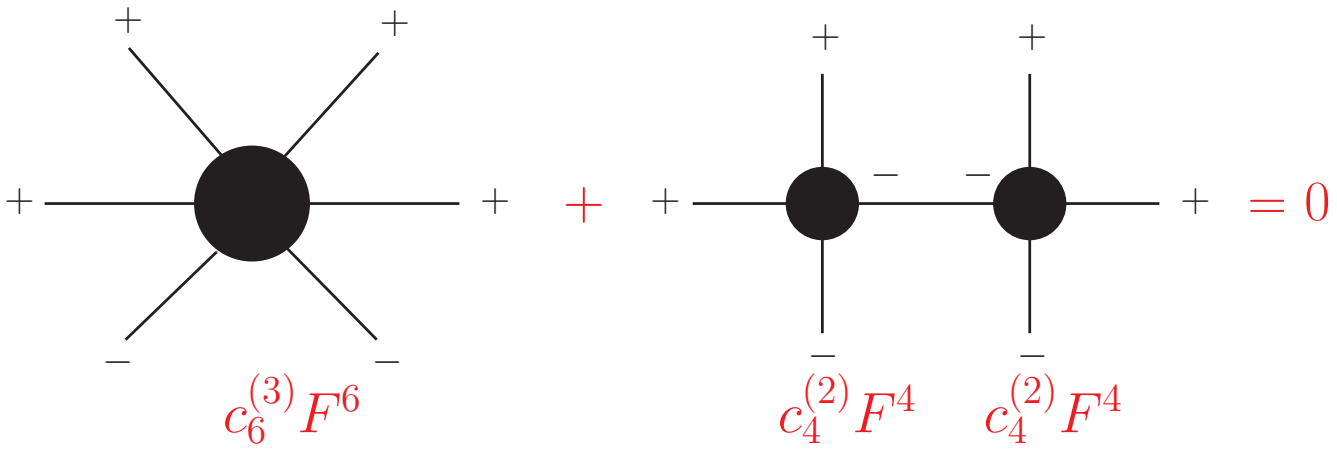
- For $\mathcal{F}^4$, $\frac{1}{\langle 12\rangle^2}$ is NOT a pole; while for $\mathcal{F}^6$ it is a pole, thus the amplitude must vanish!
- The vanishing of the amplitude can be achieved by following cancellation:



The F-F contractions [Boels, Larsen, Obers, Vonk(2008)]:

$$\langle F_-^{\alpha\beta} F_-^{\gamma\delta}\rangle = i(\epsilon_{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\alpha\delta}\epsilon_{\beta\gamma}), \quad \langle F_+^{\dot\alpha\dot\beta} F_-^{\gamma\delta}\rangle = -\frac{i(p_{\alpha\dot\gamma}p_{\beta\dot\delta} + p_{\beta\dot\gamma}p_{\alpha\dot\delta})}{p^2}.$$

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- It leads to an exact relation:

$$c_6^{(3)} = -4\left(c_4^{(2)}\right)^2 ,$$

in $\mathcal{S}_{\rm eff} = -\frac{1}{4}F^2 + \frac{c_4^{(2)}}{m_w^4}F^4 + \frac{c_6^{(3)}}{m_w^8}F^6 + \cdots$ Because $c_4^{(2)}$ is one-loop exact, $c_6^{(3)}$ is two-loop exact.

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- The argument can be extended to all MHV operator $F_-^2 F_+^{2\ell}$ by showing that there is no consistent local superamplitude corresponding to $F_-^2 F_+^{2\ell}$: $F_-^2 F_+^{2\ell}$ is ℓ-loop exact,

$$\sum_{\ell=1}^{\infty}(4)^{\ell-1}\left(-\frac{\lambda}{2(4\pi)^2}\right)^{\ell}\frac{(F_-)^2(F_+)^{2\ell}}{m_w^{2\ell}} .$$

Non-renormalization theorem from (2,0), (1,0) SUSY

- Similar generalization to theories in other space-time dimensions: for instance in 6D CFT with (2,0), (1,0) SUSY, leads to a-theorem [Cordova, Dumitrescu, Yin (2015); Chen, Huang, C.W. (2015); Cordova, Dumitrescu, Intriligator (2015)]: SUSY forces four and five-point amplitudes at order s^3 to vanish, namely

$$\left(\frac{3\Delta a}{2} - \frac{b^2}{f^4}\right) \frac{2}{f^{10}} \sum_{i<j} s_{ij}^3 = 0\,,$$

thus $\Delta a = \frac{2b^2}{3f^4} > 0$.

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- Can we go beyond SUSY? Further constraints from the spontaneously broken symmetries in the form of Soft theorems.

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- Due to the (broken) symmetries, amplitudes with Goldstone boson π satisfy soft theorems:

$$A_n(\phi_1, \cdots, \phi_{n-1}, \pi_n)|_{p_n \to 0} = \sum_{i=1}^{n-1} \langle \phi_1, \cdots, \delta\phi_i, \cdots, \phi_{n-1} \rangle|_{\mathrm{LSZ}}$$

$\delta\phi_i$ is the infinitesimal transformation of the hard particle ϕ_i under the generators of the symmetry.

Soft theorems from spontaneously breaking conformal symmetry and R-symmetry

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- In the case of dilaton of conformal symmetry breaking,

$$\begin{aligned}\delta\phi &= [\mathcal{D},\phi] = i(d + x^\mu\partial_\mu)\phi\,,\\ \delta_\mu\phi &= [\mathcal{K}_\mu,\phi] = i\left((2x_\mu x_\nu - \eta_{\mu\nu}x^2)\partial^\nu + 2d\,x_\mu\right)\phi\end{aligned}$$

- It leads to the soft theorems of dilatons [Boels, Wormsbecher (2015), Huang, C.W. (2015), Di Vecchia, Marotta, Mojaza, Nohle (2015)] (Talk by Mojaza)

$$v\,A_n\big|_{p_n\to\tau p_n} \to \left(\cdots + \mathcal{S}_n^{(0)} + \tau\mathcal{S}_n^{(1)}\right)A_{n-1} + \mathcal{O}(\tau^2)\,,$$

with soft factors,

$$\begin{aligned}\mathcal{S}_n^{(0)} &= \sum_{i=1}^{n-1}\left(p_i\cdot\frac{\partial}{\partial p_i} + \frac{d-2}{2}\right) - d \quad \text{From } \mathcal{D}\\ \mathcal{S}_n^{(1)} &= p_n^\mu\sum_{i=1}^{n-1}\left[\frac{1}{2}\left(2\,p_i^\nu\frac{\partial^2}{\partial p_i^\nu\partial p_i^\mu} - p_{i\mu}\frac{\partial^2}{\partial p_{i\nu}\partial p_i^\nu}\right) + \frac{d-2}{2}\frac{\partial}{\partial p_i^\mu}\right] \quad \text{From } \mathcal{K}_\mu\,.\end{aligned}$$

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where ϕ_i represents either a dilaton φ or R-symmetry pion ϕ^I, with $\delta^I \varphi = \phi^I$ and $\delta_I \phi^J = -\delta_I{}^J \varphi$ under $\mathrm{SO}(6)_R$.

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- We verified the soft theorems via explicitly computing one-loop amplitudes in the Coulomb branch up to six points, and amplitudes generated from one-instanton effective action.

Soft theorems from spontaneously breaking conformal symmetry and R-symmetry: explicit verification

- One-loop: use integrands of 10D SYM [Mafra, Schlotterer (2014)] or 6D SYM [Brandhuber, Korres, Koschade, Travaglini (2010), Bianchi, Guerrieri, Huang, Lee, C.W. (2016)]

$$\mathcal{L}^{\mathrm{SU}(4)}_{1-\mathrm{loop}} = \frac{g^4 N}{32 m^4 \pi^2}\left(\mathcal{O}_{F^4} + \frac{\mathcal{O}_{D^4F^4}}{2^3\times 15 m^4} - \frac{2\mathcal{O}_{D^2F^6}}{15 m^6} + \frac{\mathcal{O}_{D^4F^6}}{2^3\times 21 m^8} - \frac{\mathcal{O}_{D^6F^6}}{2\times 15^2 m^{10}} + \dots\right)$$

$$\mathcal{L}^{\mathrm{Sp}(4)}_{1-\mathrm{loop}} = \frac{g^4 N}{4\pi^2 m^4}\left[\frac{\partial^4\varphi^4}{4} + \frac{\partial^8\varphi^4}{240 m^4} + \frac{\partial^4\varphi^5}{m^2} + \frac{\partial^8\varphi^5}{120 m^6} - \frac{5\partial^4\varphi^6}{m^2} - \frac{\partial^8\varphi^6}{120 m^6} + \frac{\partial^{10}\varphi^6}{2^8 3^5 m^8} + \frac{\partial^{12}\varphi^6}{2^9 3^2 m^{10}} + \frac{\partial^4\varphi^2\phi^2}{2} - \frac{5\partial^4\varphi^2\phi^4}{m^2} + \frac{\partial^4\varphi^4\phi^2}{m^2}\right] + \dots$$

All soft theorems are satisfied!

Soft theorems from spontaneously breaking conformal symmetry and R-symmetry: explicit verification

- One-instanton: use the one-instanton effective action [Bianchi, Maroles, C.W. (2015)]

$$
\begin{aligned}
v^8 \mathcal{A}_4^{inst} &= \frac{1}{32}\left(s_{12}^2 + \mathcal{P}_4\right)^2 , \quad v^9 \mathcal{A}_5^{inst} = -\frac{1}{36}\left(s_{12}^2 + \mathcal{P}_5\right)^2 , \\
v^{10} \mathcal{A}_6^{inst} &= -\frac{2}{3} s_{12}^4 - 6 s_{12}^2 s_{23}^2 + \frac{17}{18} s_{123}^4 + \frac{15}{2} s_{123}^2 s_{124}^2 + \mathcal{P}_6 , \\
v^{11} \mathcal{A}_7^{inst} &= 4 s_{12}^4 + 40 s_{12}^2 s_{23}^2 - \frac{5}{3} s_{123}^4 - 25 s_{123}^2 s_{124}^2 + \mathcal{P}_7 , \\
v^{12} \mathcal{A}_8^{inst} &= -\frac{809}{144} s_{12}^4 - \frac{395}{8} s_{12}^2 s_{13}^2 + \frac{1339}{576} s_{123}^4 + \frac{595}{32} s_{123}^2 s_{124}^2 + \frac{535}{32} s_{123}^2 s_{145}^2 + \mathcal{P}_8 , \\
v^{13} \mathcal{A}_9^{inst} &= \frac{3935}{294} s_{12}^4 + \frac{846}{7} s_{12}^2 s_{13}^2 - \frac{475}{126} s_{123}^4 - \frac{491}{14} s_{123}^2 s_{124}^2 - \frac{535}{14} s_{123}^2 s_{145}^2 + \mathcal{P}_9 .
\end{aligned}
$$

All soft theorems are satisfied!

Systematics of soft theorems: recursion relations

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$$p_{\hat{i}} = (1 - a_i z)\, p_i \,,$$

with $\sum_{i=1}^{n} a_i\, p_i = 0$. When $z \to 1/a_i$, we realize the soft limit on particle i.

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- The amplitude can be expressed as a contour integral,

$$A_n = A_n(0) = \frac{1}{2\pi i} \oint_{z=0} \frac{dz}{z} \frac{A_n(z)}{F_n^{(\sigma)}(z)} \,,$$

where an additional function $F_n^{(\sigma)}(z)$ is introduced,

$$F_n^{(\sigma)}(z) = \prod_{i=1}^{n} (1 - a_i z)^{\sigma} \,,$$

with certain positive value σ such that $\frac{A_n(z)}{F_n^{(\sigma)}(z)} \sim 0$ at large z.

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- If $A_n(z) \sim z^m$ at $z \to \infty$,

$$\frac{A_n(z)}{F_n^{(\sigma)}(z)} \sim z^{m-n\sigma}$$

so we require $m - n\sigma < 0$: true for theories such as DBI, NLSM, Akulov-Volkov theory, DBI in AdS-space.

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$$F_n^{(2)}(z) = \prod_{i=1}^{n} (1 - a_i z)^2 \sim z^{2n} \,.$$

- Thus for order-s^k amplitudes (go as z^{2k}) with n dilatons the recursion is valid when $n > k$.

Constraints on pure dilaton effective actions

- In other words, knowing the k-point amplitude at order s^k for $k \leq n$ allows us to construct all the amplitudes upto the order s^n!

$s^k \setminus$ # of points	4	5	6	7	$\cdots$
2	$\times$	✓	✓	✓	✓
3	$\times$	✓	✓	✓	✓
4	$\times$	✓	✓	✓	✓
5	✓	$\times$	✓	✓	✓
6	✓	✓	$\times$	✓	✓
7	✓	✓	✓	$\times$	✓
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table: The $\times$ is to indicate the amplitudes that have to be computed by other means, then all other amplitudes marked with ✓ are completely determined by the soft theorems as well as the soft-BCFW.

Constraints on pure dilaton effective actions

- If each scalar mostly carries one derivative (analogous to the constant-field approximation): at $2n$ or $(2n+1)$ points, amplitudes mostly go as s^n:

s^n \ # of points	4	5	6	7	8	$\cdots$
2	$\times$	✓	✓	✓	✓	✓
3	0	0	✓	✓	✓	✓
4	0	0	0	0	✓	✓
5	0	0	0	0	0	✓
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Thus uniquely fixes it to be the conformal DBI.

Constraining effective action with soft theorems and SUSY

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- $\mathcal{N}=4$ SUSY forces that 4 and 5-point superamplitudes to take the form,

$$\begin{aligned}\mathcal{A}_4 &= \delta^8(Q)\frac{[12]^2}{\langle 34\rangle^2}\sum_{k=0}^{\infty}\mathcal{P}_4^{(k)}(s_{ij})\,,\\ \mathcal{A}_5 &= \delta^8(Q)\frac{m_{1,2,3}^{(1)}m_{1,2,3}^{(2)}+m_{1,2,3}^{(3)}m_{1,2,3}^{(4)}}{\langle 45\rangle^2}\sum_{k=0}^{\infty}\mathcal{P}_5^{(k)}(s_{ij})\end{aligned}$$

$m_{1,2,3}^{(1)}=[12]\eta_3^1+[23]\eta_1^1+[31]\eta_2^1$, and $\delta^8(Q)\sim\lambda^8\sim s^2$.
$\mathcal{P}_4^{(k)}(s_{ij}),\mathcal{P}_5^{(k)}(s_{ij})$ are degree-k symmetric polynomials of s_{ij}.

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- $\mathcal{A}_4$ at order s^2,

$$\mathcal{P}_4^{(0)}(s_{ij})=c_4^{(2)}=\frac{Ng^4}{32\pi^2 m_w^4}\,,$$

  which is one-loop exact: soft theorems determine all higher-point amplitudes at order s^2, and they are one-loop exact, and in fact identical to conformal DBI.

Constraining effective action with soft theorems and SUSY

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- The six-point s^3 amplitude receives contributions from local term $c_6^{(3)}\partial^6\phi^6$ as well as the factorization diagrams with two $c_4^{(2)}\partial^4\phi^4$. These two should combine such that satisfy soft theorems:

$$c_6^{(3)} = -(c_4^{(2)})^2 = -\frac{N^2 g^8}{32^2\pi^4 m_w^8}\,,$$

which is two-loop exact since $c_4^{(2)}$ is one-loop exact.

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- Using recursion relations, we find $c_n^{(3)}\partial^6\phi^n$ are all two-loop exact. In fact, identical to those of conformal DBI.

Constraining effective action with soft theorems and SUSY

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- Move on to the order s^4, four-point amplitude has an unique kinematics structure but with a highly non-trivial coefficient

$$\mathcal{P}_4^{(2)}(s_{ij}) = c_4^{(4)}(g, N)(s^2 + t^2 + u^2)\,.$$

$c_4^{(4)}(g, N)$ is expected to receive all-loop and instanton contributions, although should be constrained by SL(2, Z) symmetry.

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- At order s^4, all higher-point amplitudes are completely determined by a single coefficient $c_4^{(4)}(g, N)$ via recursions. It leads to non-renormalization theorems that relate all the coefficients $c_n^{(4)}(g, N)$ to $c_4^{(4)}(g, N)$.

Constraining effective action with soft theorems and SUSY

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- At order s^4, we have

$$\sum_{m\leq 8} \mathcal{L}_{\partial^m \phi^n} = c_4^{(4)}(g, N)\mathcal{L}^{(1)}_{\partial^8 \phi^n} + \sum_{m\leq 8} \mathcal{L}^{\mathrm{DBI}}_{\partial^m \phi^n} \,.$$

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  $\mathcal{L}^{(1)}_{\partial^8\phi^n}$ is the one-loop Lagrangian (fixed by soft theorems!).

- Amplitudes at order s^5 are all completely determined by two four-point coefficients:

$$\mathcal{L}_{\partial^{10}\phi^n} = c_4^{(5)}(g,N)\mathcal{L}^{(1)}_{\partial^{10}\phi^n} + c_4^{(4)}(g,N)\mathcal{L}^{(2)}_{\partial^{10}\phi^n} + \mathcal{L}^{\mathrm{DBI}}_{\partial^{10}\phi^n}\,,$$

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  $\mathcal{L}^{(2)}_{\partial^{10}\phi^n}$ is the two-loop Lagrangian (fixed by soft theorems!).

- Similar results can be obtained for 6D (2,0) theory in the Coulomb branch.

Scale invariance v.s. Conformal symmetry

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Scale invariance v.s. Conformal symmetry

- Scale invariance v.s. Conformal in relativistic QFT.
- In our language, the question can be framed as, to what extent does the subleading soft theorem due to conformal boost, follows from the leading behaviour stemming from dilation symmetry?
- We find that any five-point amplitude constrained by the leading soft theorem automatically satisfy the subleading soft theorem.

Scale invariance v.s. Conformal symmetry

A seven-point example of order s^3:

Scale invariance v.s. Conformal symmetry

A seven-point example of order s^3:

- Using both leading and sub-leading soft theorems:

$$
\begin{aligned}
A_6^{(3)} &= -c_5^{(3)}(s_{12}^3 + \mathcal{P}_6) - \left(\frac{c_5^{(3)}}{2} + (c_4^{(2)})^2\right)(s_{123}^3 + \mathcal{P}_6) \\
&+ (c_4^{(2)})^2 \left((s_{12}^2 + s_{13}^2 + s_{23}^2)\frac{1}{s_{123}}(s_{45}^2 + s_{46}^2 + s_{56}^2) + \mathcal{P}_6\right) .
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- Now, the leading soft theorem alone allows us to determine order-s^3 7-point amplitude

$$A_7^{(3)} = c_5^{(3)}(s_{12}^3+\mathcal{P}_7) + \left(c_5^{(3)}+3(c_4^{(2)})^2\right)(s_{123}^3+\mathcal{P}_7) - (c_4^{(2)})^2 A_7^{\mathrm{fac}} .$$

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- $A_7^{(3)}$ with particular parameters fixed by the leading soft theorem does satisfy the sub-leading soft theorem automatically.

Conclusions and remarks

- The requirement of having consistent S-matrix can impose highly non-trivial constraints on the theory.
- Both SUSY and soft theorems can strongly constrain the effective action of $\mathcal{N} = 4$ SYM (as well as (2,0)) in the Coloumb branch, and lead to new non-renormalization theorems.
- We observe amplitudes determined by leading soft automatically satisfy the subleading soft theorem.
- It would be interesting to explore other possible constraints such as SUSY at higher points as well as SL(2,Z) symmetry.
- We may eventually require explicit higher-loop as well as multi-instanton data, at least for some lower-point cases.

Thank you!