Singularity structure of gravity amplitudes

with Enrico Herrmann (Caltech) 1604.03479 + in progress
related work by Paul Heslop and Arthur Lipstein 1604.03046

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Activities 2017

Scientific Programs

Amplitudes: Practical and Theoretical Developments
Fabrizio Cadol CERN, Herbert Gangl Univ. Durham, Jaroslav Trnka UC Davis, Johannes Henn, Stefan Müller-Stach, Stefan Weitzel JGU
February 6-17, 2017

Quantum Vacuum and Gravitation: Testing General Relativity in Cosmology
March 13-24, 2017

Topical Workshops

Quantum Methods for Lattice Gauge Theories Calculations
Ignacio Cirac MPI for Quantum Optics, Simone Montangero Univ. Ulm, Peter Zoller Univ. Innsbruck
February 6-10, 2017, Schloss Waldhausen

Women at the Intersection of Mathematics and High Energy Physics
Karin Wendlandt Un. Freiburg, Sylvie Paycha Un. Potsdam, Kasia Rejzner Un. York, Gabriele Honecker JGU
March 6-10, 2017

Geometry, Gravity and Supersymmetry
Vicente Cortés Un. Hamburg, José Figueroa-O’Farrill Un. Edinburgh, George Papadopoulos King’s College London
April 24-28, 2017

Low-Energy Probes of New Physics
Peter Fierlinger, Martin Jung TU Munich, Susan Gardner Un. of Kentucky
May 2-24, 2017

The TeV Scale: a Threshold to New Physics?
Csaba Csaki Comill, Christophe Grojean DESY, Andreas Weiler TU Munich, Pedro Schmaller JGU
June 12 - July 7, 2017

Diagrammatic Monte Carlo Methods for QFTs in Particle-, Nuclear-, and Condensed Matter Physics
September 18-29, 2017

MITP Summer School

Foundational and Structural Aspects of Gauge Theories
Claudio Dappiaggi Univ. Pavia, Klaus Fredenhagen DESY, Marco Benini Un. Potsdam
May 29 - June 6, 2017

Supernova Neutrino Observations: What can we learn and do?
Hans-Thomas Janka MPI for Astrophysics, Georg Raffelt MPI for Physics, Lutz Köpke, Michael Wurm JGU
October 9-13, 2017

For more details: http://www.mitp.uni-mainz.de
QMAP
FIELDS
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The first instance of this phenomenon is extremely simple and trivial. Consider an analog of the "factorization channel" diagram (2.22), but connecting two black vertices. Because these vertices require that all the $e$'s be parallel, it makes no physical difference how they are connected. And so, on-shell diagrams related by, (2.28) represent the same on-shell form. Thus, we can collapse and re-expand any chain of connected black vertices in any way we like; the same is obviously true for white vertices. Because of this, for some purposes it may be useful to define composite black and white vertices with any number of legs. By grouping black and white vertices together in this way, on-shell diagrams can always be made bipartite—with (internal) edges only connecting white with black vertices. We will, however, preferentially draw trivalent diagrams because of the fundamental role played by the three-particle amplitudes.

There is also a more interesting equivalence between on-shell diagrams that will play an important role in our story. We can see this already in the BCFW representation of the four-particle amplitude given above, (2.20). The picture is obviously not cyclically invariant—as a rotation would exchange its black and white vertices. But the four-particle amplitude of course is cyclically invariant; and so there is another generator of equivalences among on-shell diagrams, the "square move", [80]: (2.29)

The merger and square moves can be used to show the physical equivalence of many seemingly different on-shell diagrams. For instance, the following two diagrams generate physically equivalent on-shell forms:

- 1 6
- 3
- 2
- 7
- 4
- 5

Non-planar N=4 SYM

N=8 SUGRA
Questions

- Do non-planar integrands have special properties?
- N=8 supergravity: singularity structure, from IR to UV
Singularities of amplitudes

- Scattering amplitudes of massless particles in D=4
  Fixed by singularities
  \[ A = \sum_j \int dI_j = \int dI \]

  No loop momenta, complicated functions

  Powerful unitarity

  Loop momenta, rational function
Generalized unitarity

(Bern, Dixon, Kosower)  (Britto, Cachazo, Feng)

- Iterative use of cut equation
- Cuts of loops are products of tree-level amplitudes
- Cut everything you can: maximal cuts
- Factorization into three point amplitudes

On-shell diagrams
On-shell diagrams

(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT 2012)

- Draw planar graph with three point vertices

Cuts of loop integrands
Product of 3pt amplitudes

- Exist in all theories, special in planar N=4 SYM
Recursion relations
(Britto, Cachazo, Feng, Witten, 2005) (Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, JT, 2010)

Recursion relations for $\ell$-loop integrand

Example: 4pt 1-loop

5-loop on-shell diagram = 1-loop off-shell box
Momentum conservation

- Deep connection: on-shell diagrams vs Grassmannian
- Simple motivation: linearize momentum conservation
  \[ \delta (P) = \delta \left( \sum_a \lambda_a \tilde{\lambda}_a \right) \]
- The Grassmannian as matrix of coefficients
  \[ \delta \left( C_{ab} \tilde{\lambda}_b \right) \delta \left( C_{ab}^{\perp} \lambda_b \right) \]

Positive Grassmannian

(Postnikov, 2006)

• Building positive matrix: face or edge variables

\[ C = \begin{pmatrix} 1 & \alpha_1 & 0 & -\alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix} \]

• Connection to mathematics for planar diagrams

Choose \( \alpha_i > 0 \): positive minors \(\rightarrow\) Positive Grassmannian

Area of research in algebraic geometry, combinatorics
Connection to amplitudes

(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT, 2012)

• Building positive matrix: face or edge variables

\[
C = \begin{pmatrix}
1 & \alpha_1 & 0 & -\alpha_4 \\
0 & \alpha_2 & 1 & \alpha_3
\end{pmatrix}
\]

• Same function as a product of 3pt amplitudes equal to

\[
\Omega = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{d\alpha_3}{\alpha_3} \frac{d\alpha_4}{\alpha_4} \delta(C \cdot Z)
\]

\[
\delta(C \cdot Z) = \delta(C \cdot \tilde{\lambda}) \delta(C^\perp \cdot \lambda) \delta(C \cdot \tilde{\eta})
\]

Solves for \(\alpha_i\) in terms of \(\lambda_i, \tilde{\lambda}_i\) and gives \(\delta(P)\delta(Q)\)
Hidden properties

- Dual conformal symmetry: absence of poles at infinity
  - Cuts never localized at $\ell \to \infty$
  - Relation to UV behavior

- Logarithmic singularities $\frac{dx}{x}$
  - Statement about types of poles in the cut structure
  - Link to the uniform transcendentality

- Recursion relations, complete geometry picture?
Amplituhedron
(Arkani-Hamed, JT, 2013)

- Motivation: Grassmannian + polytope picture

Definition of the space $Y = C \cdot Z$

Loop integrand = logarithmic volume on this space

Amplitudes are volumes!

external data: momentum twistors
Grassmannian and generalizations

(Hodges 2009)
Scattering inequalities

- Amplitudes are fixed by cuts: unique object
- Amplituhedron: provides the list of all legal cuts in the form of inequalities (for both loops and helicity)
  \[ P_j(z_k) \geq 0 \quad \text{for } z_k \text{ parametrize loop momenta} \]
  Furthermore the inequalities define a nice region in Grassmannian
- Implication: homogeneous equations $\text{Cut } I = 0$
  
  (Arkani-Hamed, Hodges, JT, 2014)
Example of inequalities

- Consider 4pt two-loop amplitude
- Inequalities: \( z_1, z_2, z_3, z_4 \geq 0 \)
  \( z_5, z_6, z_7, z_8 \geq 0 \)
  \((z_1 - z_5)(z_6 - z_2) + (z_3 - z_7)(z_8 - z_4) \geq 0\)
- Check: one-loop cut
  \( z_1 = 0 \)
  \( z_2 = 0 \)
  \( z_3 = 0 \)
  \( z_4 = 0 \)
  \(-z_5 z_6 - z_7 z_8 \geq 0\)
  \( \Omega \) vanishes on this cut
Non-planar amplitudes in $\mathcal{N}=4$ SYM

(Arkani-Hamed, Bourjaily, Cachazo, JT, 2014)
(Bern, Herrmann, Litsey, Stankowicz, JT, 2014, 2015)
Non-planar problems

- No unique integrand, labeling problem

- No momentum twistors, no known hidden symmetries

- On-shell diagrams for singularities

(Arkani-Hamed, Bourjaily, Cachazo, Postnikov, JT, 2014) (Franco, Galloni, Penante, Wen 2015)

Connection to Grassmannian, logarithmic form

\[ \Omega = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{d\alpha_3}{\alpha_3} \frac{d\alpha_4}{\alpha_4} \frac{d\alpha_5}{\alpha_5} \frac{d\alpha_6}{\alpha_6} \delta(C \cdot Z) \]

see Jake’s talk
Non-planar amplitudes

- No unique integrand, no recursion relations
- On-shell diagrams: cuts of amplitudes
- Conjecture: amplitudes have the same properties
  - Logarithmic singularities
  - No poles at infinity
  - Only homogeneous cuts
  - Volumes of something?
  - Analogue od DCI?
  - Scattering inequalities?
  - Amplituhedron?
Non-planar amplitudes

• Conservative approach: sum of integrals

\[ A = \sum \limits_i a_i \cdot C_i \cdot I_i \]

Fix by homogeneous conditions

Basis of integrals:
- Logarithmic singularities
- No poles at infinity

• Some diagrams forbidden
Explicit checks

- Construct basis for 4pt at 2-loop and 3-loop, 5pt 2-loop

Expand the amplitude:
Non-planar conclusion

- Same properties of amplitudes as in planar sector
- Open questions: identify new symmetries, role of color factor, complete geometric formulation.....
- What about gravity?
Gravity on-shell diagrams

(Herrmann, JT 2016)
(Heslop, Lipstein 2016)
First look: gravity on-shell diagrams

- Let us just get some data
- Simplest class: MHV leading singularities

\[
\begin{align*}
\text{Yang-Mills} & \quad \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle \\
\text{Gravity} & \quad [13][24] \langle 12 \rangle \langle 13 \rangle \langle 14 \rangle \langle 23 \rangle \langle 24 \rangle \langle 34 \rangle
\end{align*}
\]
First look: gravity on-shell diagrams

- Let us just get some data
- Simplest class: MHV leading singularities

\[
\begin{align*}
\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle \\
\langle 12 \rangle \langle 13 \rangle \langle 15 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \\
\end{align*}
\]

\[
\begin{align*}
[12][23][45]^2 \\
\end{align*}
\]
First look: gravity on-shell diagrams

- Let us just get some data
- Simplest class: MHV leading singularities

\[
\begin{align*}
|12\rangle |23\rangle |14\rangle |15\rangle |34\rangle |35\rangle \\
|13\rangle
\end{align*}
\]

**Yang-Mills**

\[
|12\rangle |23\rangle |14\rangle |15\rangle |34\rangle |35\rangle
\]

**Gravity**

\[
|12\rangle |23\rangle |45\rangle^2 \\
|12\rangle |14\rangle |15\rangle |23\rangle |34\rangle |35\rangle
\]
Based on these results: in Yang-Mills we could already conjecture the structure

Natural conjecture for gravity

\[
\frac{[\ldots][\ldots][\ldots][\ldots]}{\langle\ldots\rangle\langle\ldots\rangle\langle\ldots\rangle\langle\ldots\rangle\langle\ldots\rangle}
\]

anti-holomorphic numerator

single poles in the denominator
First look: gravity on-shell diagrams

- Higher poles, more complicated numerators

\[
\frac{\langle 5|6 + 1|2\rangle\langle 2|3 + 4|5\rangle[16]^2[34]^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle\langle 25\rangle^2}
\]
First look: gravity on-shell diagrams

- Higher poles, more complicated numerators

Each numerator factor:
collinear condition in the vertex

\[ \frac{\langle 5|6 + 1|2 \rangle \langle 2|3 + 4|5 \rangle [16]^2 [34]^2}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 45\rangle \langle 56\rangle \langle 61\rangle \langle 25\rangle^2} \]

Higher poles:
infinite momentum
\[ \langle 25 \rangle = 0 \]
blows up the loop in hexagon
Grassmannian formula

- Edge variables for each edge

![Diagram with variables α1, α2, α3, α4]

\[ C = \begin{pmatrix} 1 & \alpha_1 & 0 & -\alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix} \]

- The value of the diagram in Yang-Mills

\[ \Omega = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \ldots \frac{d\alpha_m}{\alpha_m} \cdot \delta(C \cdot Z) \]

\[ \delta(C \cdot Z) = \delta(C \cdot \tilde{\lambda}) \delta(C^\perp \cdot \lambda) \delta(C \cdot \tilde{\eta}) \]

Solves for \( \alpha_i \) in terms of \( \lambda_i, \tilde{\lambda}_i \) and gives \( \delta(P)\delta(Q) \)
Grassmannian formula

(Herrmann, JT 2016)

★ Edge variables for each edge

★ The value of the diagram in gravity

\[
\begin{align*}
\Omega &= \frac{d\alpha_1}{\alpha_1^3} \frac{d\alpha_2}{\alpha_2^3} \cdots \frac{d\alpha_m}{\alpha_m^3} \prod_v \Delta_v \cdot \delta(C \cdot Z) \\
\delta(C \cdot Z) &= \delta(C \cdot \tilde{\lambda}) \delta(C^\perp \cdot \lambda) \delta(C \cdot \tilde{\eta})
\end{align*}
\]
Grassmannian formula

(Herrmann, JT 2016)

Analyzing: \[ \Omega = \frac{d\alpha_1}{\alpha_1^3} \frac{d\alpha_2}{\alpha_2^3} \cdots \frac{d\alpha_m}{\alpha_m^3} \prod_v \Delta_v \cdot \delta(C \cdot Z) \]

- Higher poles at infinity
- Single poles for finite momentum
- Vanishes if for collinear momenta in vertex

Similar formula for N<8 SUGRA
Relation to other work
(Heslop, Lipstein 2016)

- Gravity on-shell diagrams in the context of BCFW recursion relations

- Gravity tree amplitudes not just sum of on-shell diagrams

\[ A_n = \sum_{L,R} \frac{1}{s_{1n}} \]

Extra kinematical factors
Singularities of gravity amplitudes

(Herrmann, JT 2016)
Conjectures for the amplitude

- No loop recursion using on-shell diagrams
  - Absence of variables, generic non-planar problem

- Conjectures for loop integrands
  - Logarithmic poles for cuts
  - Collinearity conditions
  - Poles at infinity

→ IR

→ UV
Logarithmic poles

- On-shell diagrams: all finite cuts logarithmic
  
  \[
  \int \to \text{Cut} \to \text{Cut} \to \frac{d\alpha}{\alpha^2} \quad \text{never happens}
  \]

- Strong hint from BCJ relations

  \[
  A^{(YM)} = \sum_i \frac{n_i^{(BCJ)} c_i}{s_i} = \sum_i \frac{n_i^{(dlog)} c_i}{s_i}
  \]

  cancels double poles

  \[
  A^{(GR)} = \sum_i \frac{n_i^{(dlog)} n_i^{(BCJ)}}{s_i}
  \]

  Some diagrams prohibited
Collinearity conditions

- On-shell diagrams: any cut of the form

\[ \sim [\ell_1 \ell_2] \]

- In special case of external legs: collinear limit

\[ A \sim \frac{[12]}{\langle 12 \rangle} \cdot (\ldots) \]
Collinearity conditions

- Example: prediction for cut

\[ \ell = \lambda_1 \tilde{\lambda}_\ell \]

\[ \sim [\ell \ 1] \quad \Rightarrow \quad A^{1-\text{loop}}_{GR} \sim \frac{1}{\epsilon} \]

- Expansion of the 4pt 1-loop amplitude: three boxes

\[ I^1_4(s; t) = \]

\[ I^1_4(t; u) = \]

\[ I^1_4(u; s) = \]

Each scales \( \sim \frac{1}{[\ell \ 1]} \)

Sum \( \sim 1 \)
Collinearity conditions

- We made a choice in labeling diagrams
- Consider another labels, and sum over both options
- Two loop checks more complicated, symmetrization!

Total sum
\[ \sim [\ell 1] \]

Also important in the pre-integrand story
(Carrasco, 2015)
Poles at infinity and UV behavior of \( N=8 \) supergravity
Poles at infinity

- They are present starting at 3-loops

\[ \frac{dz}{z} \sim \text{Pole at } z \to \infty \]
\[ \ell(z) \to \infty \]

- Higher poles at higher loops

- Generically everything complex, the detailed description of space of poles at infinity needed
Poles at infinity

- Expected divergence at 7-loops

\[ \ell_1 \quad \ell_2 \]

matches the on-shell diagram with pole at infinity

- What is the relation between poles at infinity and UV?

- The problem: real vs complex kinematics
Poles at infinity

- Lower cuts: real kinematics, no poles at infinity

- Requires cancelations between different diagrams

- Higher cuts: kinematics complex, poles at infinity
Cancelations of UV divergencies

- Expansion of the amplitude in terms of integrals
  Enhanced cancelations
- Understand these cancelations at the integrand level
  (Bern, Herrmann, Martinez, Stankowicz, JT, in progress)
  - Relation to poles at infinity
  - Lower supersymmetry -> lower loops
- We saw these cancelation in IR in the collinear regions
Rigidity of N=8 SUGRA

- Related to the UV question:

  What is special about loop integrands in N=8 SUGRA?

- For N=4 SYM: simple singularity structure (logarithmic) + all singularities captured by inequalities = Amplituhedron

- If N=8 SUGRA is special, it must be the rigidity of poles at infinity which are absent in N=4 SYM
Gravitational polytopes

(Herrmann, JT, in progress)

- The key to gravity is hidden in tree-level amplitudes
- Amplituhedron: first sign in NMHV tree amplitudes

![Diagrams of gravitational polytopes]

- Encouraging preliminary results: gravity volume forms

\[
\frac{(\langle 12 \rangle [45] \langle 3 | 4 + 5 | 6 \rangle )^2}{s_{345} \cdot s_{34} s_{45} \cdot s_{61} s_{12}} \quad \rightarrow \quad \frac{(\langle 12 \rangle [45] \langle 3 | 4 + 5 | 6 \rangle )^4}{s_{345} \cdot s_{34} s_{45} s_{53} \cdot s_{61} s_{12} s_{26}}
\]

Translate to momentum space

Look at gravity: different than BCJ
Conclusion
Conclusion

- Grassmannian formula for gravity on-shell diagrams
- Conjecture for IR properties of the gravity integrands
- Poles at infinity: key to UV behavior
- First signs of new structures in tree-level amplitudes
Thank you for your attention