New Relations for Einstein-Yang-Mills Amplitudes

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based on:

St.St., T.R. Taylor:

- **New Relations for Einstein-Yang-Mills Amplitudes,**
  arXiv:1606.09616 [hep-th]

- **Disk Scattering of Open and Closed Strings (I),**
  Nucl. Phys. B903 (2016) 104-117 [arXiv:1510.01774]

- **Subleading Terms in the Collinear Limit of Yang-Mills Amplitudes,**
  Phys. Lett. B750 (2015) 587-590 [arXiv:1508.01116]

- **Graviton as a Pair of Collinear Gauge Bosons,**
  Phys. Lett. B739 (2014) 457-461, [arXiv:1409.4771]

- **Graviton Amplitudes from Collinear Limits of Gauge Amplitudes,**
  Phys. Lett. B744 (2015) 160-162 [arXiv:1502.00655]

Einstein-Yang-Mills Amplitudes

basic building blocks for BCFW recursion relations:

$$A(1^-,2^-,3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

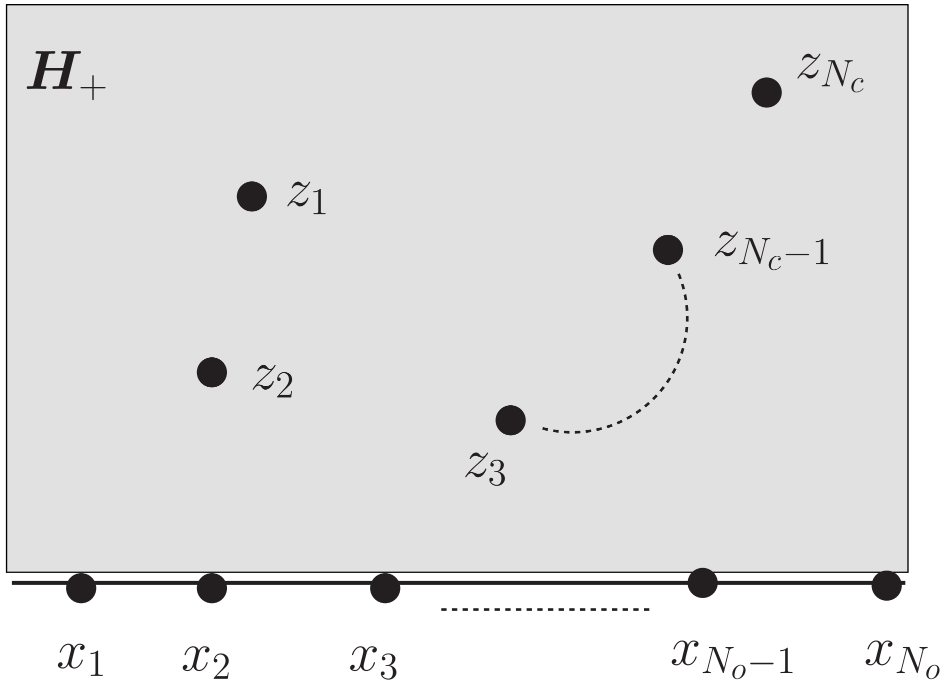
$$A(1^{--},2^{--},3^{++}) = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2} \qquad A(1^+,2^-,3^{--}) = \frac{\langle 23\rangle^4}{\langle 12\rangle^2}$$

standard Feynman diagram computation or BCFW recursion relations yields, e.g.:

$$A(1^+,2^+,3^-;q^{--}) = \frac{\langle 3q\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

EYM Amplitudes from String Theory

Mixed amplitudes involving open and closed strings:



$$\langle e^{ik_1^\mu X_\mu(x_1)} e^{ik_2^\nu X_\nu(x_2)} \rangle = (x_1 - x_2)^{2\alpha' k_1 k_2}$$

$$\langle e^{iq_1^\mu X_\mu(z_1)} e^{iq_2^\nu X_\nu(z_2)} \rangle = (z_1 - z_2)^{2\alpha' q_1 q_2}$$

$$\langle e^{iq_1^\mu X_\mu(z_1)} e^{iq_2^\nu X_\nu(\overline{z}_2)} \rangle = (z_1 - \overline{z}_2)^{2\alpha' q_1 q_2}$$

$$\langle e^{iq_1^\mu X_\mu(z_1)} e^{ik^\nu X_\nu(x_1)} \rangle = (z_1 - x_1)^{2\alpha' q_1^\mu k_\mu}$$

$$\langle e^{iq_1^\mu X_\mu(\overline{z}_1)} e^{ik^\nu X_\nu(x_1)} \rangle = (\overline{z}_1 - x_1)^{2\alpha' q_1^\mu k_\mu}$$

$$s_{ij} \equiv s_{i,j} = \alpha'(k_i + k_j)^2 = 2\alpha' k_i k_j$$

$$F(1, 2, \ldots, N-2; q_1, q_2)$$
$$= V_{\text{CKG}}^{-1}\, \delta\Big(\sum_{i=1}^{N-2} p_i + q_1 + q_2\Big) \int_{x_1 < \cdots < x_{N-2}} \Big(\prod_{i=1}^{N-2} dx_i\Big) \prod_{1 \leq r < s \leq N-2} (x_s - x_r)^{s_{rs}}\, (x_r - x_s)^{n_{rs}}$$
$$\times \int_{\mathcal{H}_+} d^2z\, (z - \overline{z})^{s_{N-1,N} + n} \prod_{i=1}^{N-2} (x_i - z)^{s_{i,N-1} + n_i}\, (x_i - \overline{z})^{s_{i,N} + \overline{n}_i}$$

"Doubling trick":

- *convert disk correlators to the **standard holomorphic** ones by extending the fields to the **entire complex plane**.*

"KLT trick":

- *integration over complex positions of closed string states can be **disentangled** into real ones by introducing **monodromy phases***

→ *monodromy problem on the complex plane*

N_o open & N_c closed strings:

$N_o + 2N_c$ - point **pure** open string amplitude

→ relations between amplitudes involving **open & closed strings** and **pure open** string amplitudes

$$N_c = 1$$

$$A(1,2,\ldots,N-2;q_1,q_2) = (-1)^N \; e^{-\pi i(s_{1,N}+s_{2,N-1})} \sum_{l=2}^{N-2} (-1)^l \; \sin(\pi s_{l,N-1}) \; e^{\pi i(-1)^l \; s_{l,N-1}}$$

$$\times \sum_{\rho\in\{OP(\alpha,\beta^t),l\}} e^{\pi i \sum_{k=1}^{\lfloor\frac{N-3}{2}\rfloor} \tau_{2k+1}(\rho)} \; \mathcal{S}(\rho) \; A(1,\rho,N-1,N)$$

$$s_{ij} \equiv s_{i,j} = \alpha'(k_i+k_j)^2 = 2\alpha' k_i k_j$$

E.g.: N=5

$$A(1,2,3;q_1,q_2) = e^{-\pi i s_{24}} \; \Big[\; e^{-\pi i s_{51}} \; \sin(\pi s_{34}) \; A(1,2,3,4,5) - \sin(\pi s_{24}) \; A(1,3,2,4,5) \; \Big]$$

$$\mathcal{S}(\rho) \equiv \mathcal{S}[\rho(2,\ldots,N-2)\,] = \prod_{i=2}^{N-2} \prod_{j=i+1}^{N-2} \exp\left\{\pi i \; \theta(\rho^{-1}(i)-\rho^{-1}(j)) \; s_{i,j}\right\}$$

$$\tau_i(\rho) = \begin{cases} \mathrm{sign}(\rho^{-1}(i)-\rho^{-1}(i+1)) \; (s_{i,N-1}+s_{i+1,N-1}) \; , & 3 \le i \le N-3 \; , \\ s_{N-2,N-1} \; , & i = N-2 \end{cases}$$

take collinear limit: $q_1 = k_{N-1} = \frac{1}{2}\, q,$ $q_2 = k_N = \frac{1}{2}\, q$

graviton is replaced by two gluons in collinear configurations

E.g.:

$$A(1,2,3;q) = \sin(\pi s_{24})\ A(1,5,2,4,3)\ ,$$
$$A(1,2,3,4;q) = \sin(\pi s_{25})\ A(1,6,2,5,3,4) + \sin(\pi s_{45})\ A(1,2,3,5,4,6)\ ,$$
$$A(1,2,3,4,5;q) = \sin(\pi s_{26})\ A(1,7,2,6,3,4,5) + \sin(\pi s_{36})\ A(1,2,7,3,6,4,5) + \sin[\pi(s_{36}+s_{26})]\ A(1,7,2,3,6,4,5) + \sin(\pi s_{56})\ A(1,2,3,4,6,5,7)$$

take field-theory limit: yields **Einstein-Yang-Mills** for any kinematical configuration

"graviton appears as a pair of collinear gauge bosons"

$$A_{EYM}(1^+,2^+,3^-;q^{--}) = \pi\ s_{24}\ A_{\rm YM}(1^+,5^-,2^+,4^-,3^-)$$

with SYM amplitude:

$$A_{\rm YM}(1^+,5^-,2^+,4^-,3^-) = 4\ \frac{[12]^4}{[1q][q3][13][2q]^2}$$

generalization to arbitrary collinear configuration:

graviton is replaced by two gluons
*in **arbitrary** collinear configurations*

$$q_1 = k_{N-1} = x\ q,$$
$$q_2 = k_N = (1-x)\ q$$

x is a free real parameter !

E.g.:

$$A(1,2,3;q_1,q_2) = \frac{\kappa\,(1-x)}{g^2}\ s_{24}\ A(1,5,2,4,3)\ ,$$

$$A(1,2,3,4;q_1,q_2) = \frac{\kappa\,(1-x)}{g^2}\ \Big\{\ s_{25}\ A(1,6,2,5,3,4) + s_{45}\ A(1,2,3,5,4,6^{)}\ \Big\}\ ,$$

$$A(1,2,3,4,5;q_1,q_2) = \frac{\kappa\,(1-x)}{g^2}\ \Big\{\ s_{26}\ A(1,7,2,6,3,4,5) + s_{36}\ A(1,2,7,3,6,4,5)$$
$$+ (s_{36}+s_{26})\ A(1,7,2,3,6,4,5) + s_{56}\ A(1,2,3,4,6,5,7)\ \Big\}$$

highly non-trivial !

can explicitly be checked !

New Relations between Einstein-Yang-Mills and Yang-Mills Amplitudes

based on these results we may take:

$$x = 0 \begin{cases} q_1 = k_{N-1} \to 0, \\ q_2 = k_N \to q \end{cases}$$

soft-gluon limit of (N-1)-th gluon

with:

$$\lim_{x\to 0} x\,(1-x)\,A(\ldots,m,xq^+,n,\ldots) = g\frac{\langle mn\rangle}{\langle mq\rangle\langle qn\rangle} A(\ldots,m,n,\ldots)\,,$$

$$\lim_{x\to 0} x\,(1-x)\,A(\ldots,m,xq^-,n,\ldots) = g\frac{[mn]}{[mq][qn]} A(\ldots,m,n,\ldots)$$

$$A_{EYM}(1,2,\ldots N;q^{++}) = \frac{\kappa}{g}\sum_{l=2}^{N-1}\frac{\langle 1|x_l|q]}{\langle 1q\rangle} A(1,2,\ldots,l,q^+,l{+}1,\ldots,N)$$

$$A_{EYM}(1,2,\ldots N;q^{--}) = \frac{\kappa}{g}\sum_{l=2}^{N-1}\frac{[1|x_l|q\rangle}{[1q]} A(1,2,\ldots,l,q^-,l{+}1,\ldots,N)$$

$$x_l = \sum_{k=1}^{l} p_k$$

(N+1)-point on both sides !

in any dimension D:

$$A_{EYM}(1,2,\ldots,N;q^{\pm\pm}) \;=\; \frac{\kappa}{g}\sum_{l=1}^{N-1}(\epsilon_P^{\pm}\cdot x_l)\; A(1,2,\ldots,l,q^{\pm},l{+}1,\ldots,N)$$

the graviton is inserted into partial gauge amplitudes in the same way as a vector boson of a **spectator group** commuting with the group associated to N gauge bosons.

Note:

$$A_{EYM}(1,2,\ldots N;q^{++}) \;=\; \frac{\kappa}{g}\sum_{l=2}^{N-1}\frac{\langle 1|x_l|q]}{\langle 1q\rangle}A(1,2,\ldots,l,q^{+},l{+}1,\ldots,N)$$

factors appear in Mason-Skinner representation of multi-graviton MHV amplitudes

$$A_E = \frac{1}{\langle 12\rangle\langle 23\rangle\langle 13\rangle}\;\sum_{\mathcal{P}} A(1,\ldots,N)\prod_{k=4}^{N}\frac{\langle 2|3+\ldots+(k-1)|k]}{\langle 2k\rangle}$$

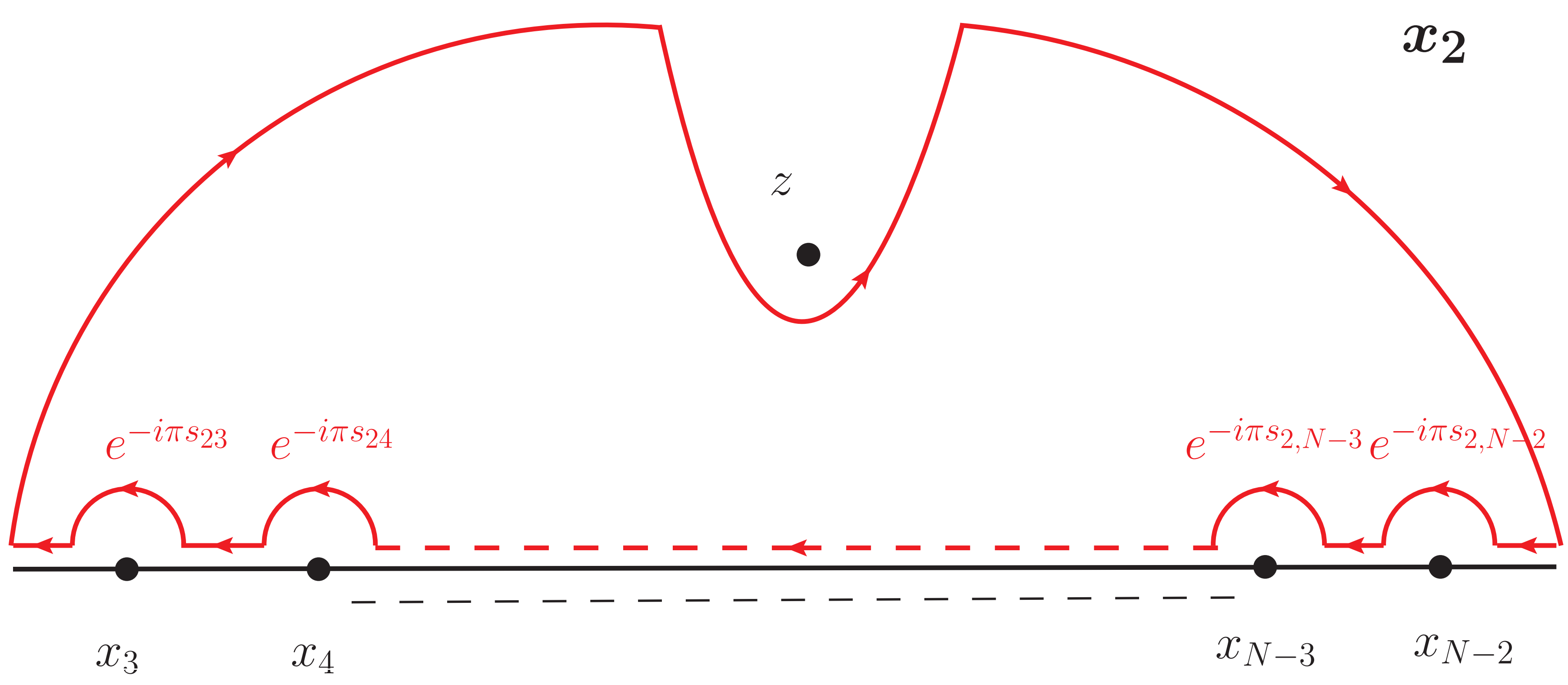
hints towards multi-graviton generalization

back to string theory:

generic form factor

$$F(1,2,\ldots,N-2;q_1,q_2)$$
$$= V_{\rm CKG}^{-1}\,\delta\Big(\sum_{i=1}^{N-2} p_i + q_1 + q_2\Big) \int_{x_1<\cdots<x_{N-2}} \Big(\prod_{i=1}^{N-2} dx_i\Big) \prod_{1\le r<s\le N-2} (x_s - x_r)^{s_{rs}}\ (x_r - x_s)^{n_{rs}}$$
$$\times \int_{\mathcal{H}_+} d^2z\ (z-\overline{z})^{s_{N-1,N}+n} \prod_{i=1}^{N-2} (x_i - z)^{s_{i,N-1}+n_i}\ (x_i - \overline{z})^{s_{i,N}+\overline{n}_i}$$

- branching (monodromies) from non-integer part
- make integrand single-valued and consider contour integral



mixed monodromy relations:

$$A(1,2,\ldots,N-2;q_1,q_2)+e^{-i\pi s_{23}}\ A(1,3,2,\ldots,N-2;q_1,q_2)$$
$$+\,e^{-i\pi(s_{23}+s_{24})}\ A(1,3,4,2,\ldots,N-2;q_1,q_2)$$
$$+\ldots+e^{-i\pi(s_{23}+s_{23}+\ldots+s_{2,N-2})}\ A(1,3,\ldots,N-2,2;q_1,q_2)=T(3,\ldots,N-2)$$

system can be solved
for mixed amplitudes
in terms of pure open string amplitudes

tube contribution T(3,...,N-2)
string effect

T(3,...,N-2) can be expressed
in terms of pure open string subamplitudes

$$T(3)=2i\ e^{-i\pi s_{51}}\ \sin(\pi s_{24})\ \{\ \sin(\pi s_{34})\ A(1,5,4,3,2)+\sin[\pi(s_{23}+s_{34})]\ A(1,5,4,2,3)\ \}$$

E.g.:

$$A(1,2,3;q_1,q_2)+e^{-i\pi s_{23}}\ A(1,3,2;q_1,q_2)=-2i\ e^{-i\pi s_{51}}\ \sin(\pi s_{24})\ \sin(\pi s_{35})\ A(1,2,4,5,3)$$

Extract various amplitude relations:

$$A(1,\ldots,N-2,q)=\alpha'\ A_{EYM}(1,\ldots,N-2,q)+i\alpha'^2\ A^{(1)}+\alpha'^3\ A^{(2)}+\mathcal{O}(\alpha'^4)$$

$$A_{EYM}(1,2,\ldots,N-2;q)+A_{EYM}(1,3,2,\ldots,N-2;q)+A_{EYM}(1,3,4,2,\ldots,N-2;q)+\ldots+A_{EYM}(1,3,\ldots,N-2,2;q)=0$$

KK type relations

E.g. N=6:

$$A^{(1)}(1,2,3,4;q)+A^{(1)}(1,3,2,4;q)+A^{(1)}(1,3,4,2;q)-\pi s_{23}\ A_{EYM}(1,3,2,4;q)-\pi\ (s_{23}+s_{24})\ A_{EYM}(1,3,4,2;q)=\Im\ T(3,4)|_{\alpha'^2}$$

BCJ type relations

Actually: $A^{(1)}(1,2,3;q)-A^{(1)}(1,3,2;q)=0$

Collinear gauge limit from EYM

Recall:

$$A(1,2,\ldots,N-2;q_1,q_2) = (-1)^N\ e^{-\pi i(s_{1,N}+s_{2,N-1})}\ \sum_{l=2}^{N-2}(-1)^l\ \sin(\pi s_{l,N-1})\ e^{\pi i(-1)^l\ s_{l,N-1}}$$

$$\times \sum_{\rho\in\{OP(\alpha,\beta^t),l\}} e^{\pi i \sum_{k=1}^{\lfloor\frac{N-3}{2}\rfloor}\tau_{2k+1}(\rho)}\ \mathcal{S}(\rho)\ A(1,\rho,N-1,N)\ .$$

take field-theory limit:

$$A_{EYM}(1,2,\ldots,N-2;q) = (-1)^N\ \pi \sum_{l=2}^{N-2}(-1)^l\ s_{l,N-1}\ A(1,\rho,N-1,N)$$

collinear limit
of gluon N-1 and N

$$q_1 = k_{N-1} = x\ q$$
$$q_2 = k_N = (1-x)\ q$$

$$s_{ij} = 2\alpha' k_i k_j$$

N=5

$$s_{3q}\ A(1,2,3,4^{\pm},5^{\pm}) - s_{2q}\ A(1,3,2,4^{\pm},5^{\pm}) = \frac{g^2}{\kappa\, x}\ A(1,2,3;q^{\pm 2})$$

with: $$A(1,\ldots,N{-}1^{+},N^{+}) = \frac{1}{\langle N{-}1\, N\rangle\, \mathbf{sc}}\ A(1,\ldots,q^{+}) + \epsilon^{0}\ \mathrm{Sub}^{++} + \ldots$$

leading contributions cancel due to BCJ relations

$$\begin{aligned} k_{N-1} &= \mathbf{c}^2 q - \epsilon\ \mathbf{sc}(\lambda_q\tilde{\lambda}_r + \lambda_r\tilde{\lambda}_q) + \epsilon^2\mathbf{s}^2 r\ , \\ k_N &= \mathbf{s}^2 q + \epsilon\ \mathbf{sc}(\lambda_q\tilde{\lambda}_r + \lambda_r\tilde{\lambda}_q) + \epsilon^2\mathbf{c}^2 r \end{aligned}$$

$$\mathbf{c} = \sqrt{x}\ , \qquad \mathbf{s} = \sqrt{1-x}$$

→ Subleading terms are related to EYM amplitude

N=6

$$s_{4q}A(1,2,3,4,\ 5^{\pm},6^{\pm}) - s_{3q}[A(1,2,4,3,5^{\pm},6^{\pm}) + A(1,4,2,3,5^{\pm},6^{\pm})]$$
$$+ s_{2q}A(1,4,3,2,5^{\pm},6^{\pm}) = \frac{g^2}{\kappa\, x}A(1,2,3,4;q^{\pm 2})$$

$\frac{1}{2}\,(N-3)!$ independent constraints for subleading terms

Graviton amplitudes from gauge amplitudes

express N-graviton amplitude in **Einstein's gravity**
as collinear limits of
certain linear combinations of **pure SYM amplitudes**
in which each graviton is represented by two gauge bosons

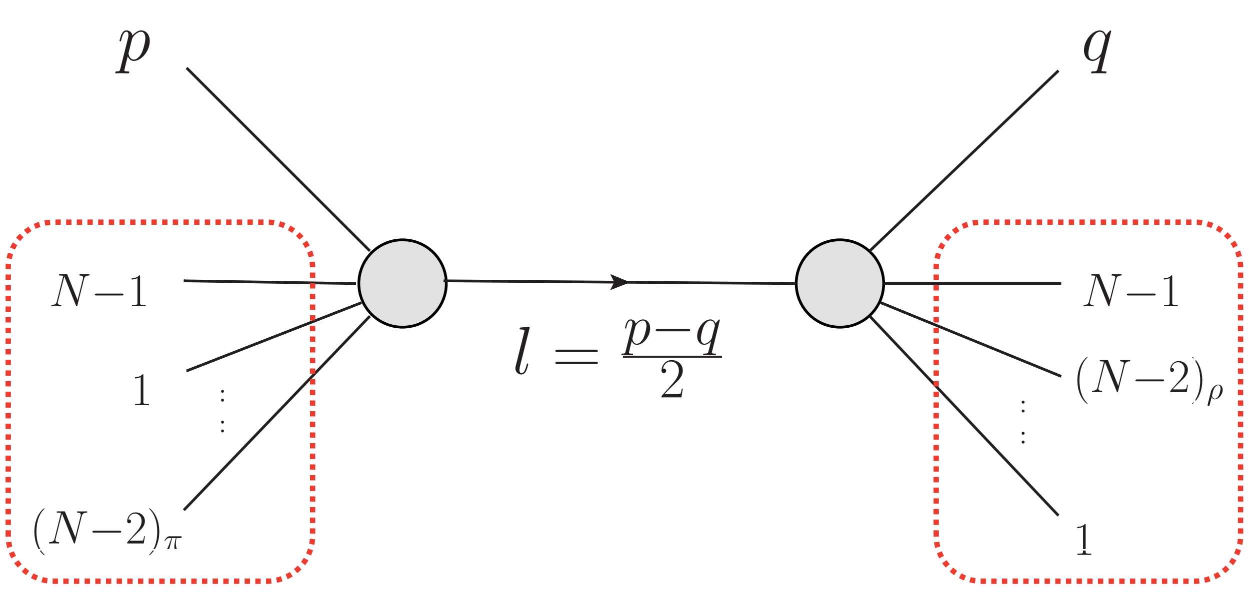
no string theory !
but motivated from string theory

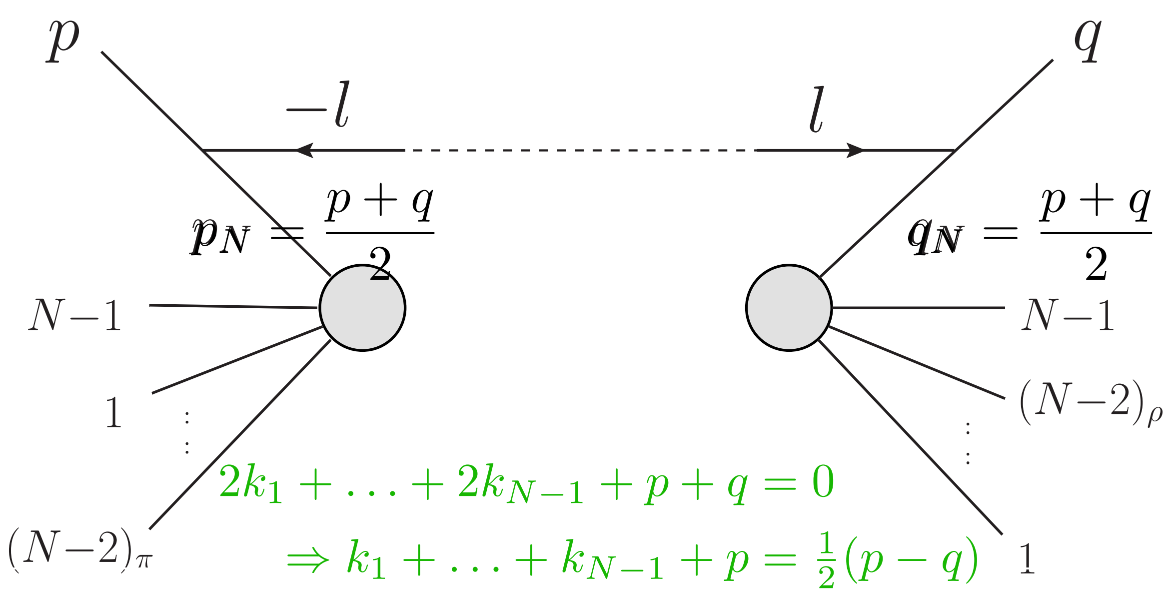
$$A_E[k_1,\lambda_1;\ldots;k_{N-1},\lambda_{N-1};k_N=p+q,\lambda_N=+2]=\lim_{[pq]\to 0}\left(\frac{1}{2x}\right)^4\frac{[pq]}{\langle pq\rangle}s_{pq}^2$$

$$\times\sum_{\pi,\rho\in S_{N-3}}S[\pi|\rho]\;A_{YM}[p,N-1,1,\pi(2,3,\ldots,N-2),1,\rho(2,\ldots,N-2),N-1,q]$$

(2N-2 gluons become collinear without producing poles)

Proof: contributions from factorization on **triple** pole $s_{pq}^3 \sim (p-q)^6$





$2k_1 + \ldots + 2k_{N-1} + p + q = 0$

$\Rightarrow k_1 + \ldots + k_{N-1} + p = \frac{1}{2}(p-q)$

$\Rightarrow k_1 + \ldots + k_{N-1} = -\frac{1}{2}(p+q)$

$$A_{YM}[p, N-1, 1, \pi(2,3,\ldots,N-2), 1, \rho(2,\ldots,N-2), N-1, q] \to \left(\frac{4}{s_{pq}}\right)^3 \times$$
$$\times A_{YM}[p^+, -l^-, -p_N^-] \times A_{YM}[p_N, \mu_N = +1; N-1, 1, \pi(2,3,\ldots,N-2)]$$
$$\times A_{YM}[1, \rho(2,\ldots,N-2), N-1; q_N, \nu_N = +1] \times A_{YM}[q^-, l^+, -q_N^-]$$

$$\left\{ \begin{aligned} A_{YM}[p^+, -l^-, -p_N^-] &= \frac{x^3}{2}\langle pq\rangle, \\ A_{YM}[q^-, l^+, -q_N^-] &= \frac{x}{2}\langle pq\rangle \end{aligned} \right.$$

yields:

$$A_E[k_1, \lambda_1; \ldots; k_{N-1}, \lambda_{N-1}; k_N, \lambda_N = +2]$$
$$= \sum_{\pi,\rho \in S_{N-3}} S[\pi|\rho]\; A_{YM}[p_N, \mu_N = +1; N-1, 1, \pi(2,3,\ldots,N-2)]$$
$$\times A_{YM}[1, \rho(2,\ldots,N-2), N-1; q_N, \nu_N = +1]$$

Concluding remarks

- **new** expression for EYM amplitudes
- monodromy relations for mixed string amplitudes giving rise to various amplitude relations
- **linear** expressions in contrast to KLT (linearization of KLT relations)
- graviton scattering **unified** into gauge amplitudes
- underlying gauge structure in quantum gravity