On scattering amplitudes in higher spin theories

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"Scalar scattering via conformal higher spin exchange"
with E. Joung and S. Nakach   arXiv:1512.08896

"On quantum corrections in higher-spin theory in flat space"
with D. Ponomarev   arXiv:1603.06273

with M. Beccaria and S. Nakach, to appear

with R. Roiban, in progress

Why Higher Spins?

• massive HS fields in string theory
may become massless in “tensionless” limit
interacting massless HS theory in flat space ?

• massless HS theory in AdS:
important role in (vectorial) AdS/CFT
• closely related “shadow”: conformal higher spin theory

• infinite dimensional HS symmetry:
implications for S-matrix? trivial?
(cf. Coleman-Mandula, Weinberg’s soft theorem)

Consistent HS theories:

• massless HS theory in AdS$_{d+1}$:
2-derivative (unitary) but non-flat vac
dual to free CFT$_d$: e.g. scalar in vector rep of $U(N)$
S-matrix is "simple":
reproduces correlators of currents in free CFT

• conformal higher spin theory (CHS):
higher derivatives (non-unitary) but flat vacuum
closely related to AdS/CFT
S-matrix is "trivial": constrained by HS symmetry

- massless higher spin (MHS) theory in flat space:

existence of interacting theory presently unclear

S-matrix should be trivial to comply with

large gauge/global symmetry?

locality of 4-point and higher terms in action?

limit of AdS HS theory?

Aim: study simple amplitudes in CHS and MHS

and implications of HS symmetry

Free massless HS theory in flat space

• collection of free massless spins $s = 0, 1, 2, 3, ..., \infty$

gauge-invariance $\quad \delta\phi_{m_1...m_s} = \partial_{(m_1}\epsilon_{m_2...m_s)}$

Fronsdal action $\quad S = \int d^4x \, \phi^{m_1...m_s}\partial^2\phi_{m_1...m_s} + ...$

• massless vector, graviton, etc.: for $s > 0$ have 2 d.o.f. in 4d

• "trivial" theory: total no. of d.o.f. =0

$$1 + \sum_{s=1}^{\infty} 2 = 1 + 2\zeta(0) = 0$$

• free massless spin s partition function:

$$Z_{\mathrm{MHS},s} = \Big[\frac{\det \Delta_{s-1\,\perp}}{\det \Delta_{s\,\perp}}\Big]^{1/2} = \Big[\frac{(\det \Delta_{s-1})^2}{\det \Delta_s \det \Delta_{s-2}}\Big]^{1/2} = \Big(\frac{1}{\sqrt{\det(-\partial^2)}}\Big)^2$$

$\Delta_s = -\partial^2$ on symmetric rank s traceless tensor

Total partition function: [Beccaria, AT 15]

$$Z_{\rm MHS} = \prod_{s=0}^{\infty} Z_{{\rm MHS},s}$$

$$= \Big[\frac{1}{\det \Delta_0}\Big]^{1/2} \Big[\frac{\det \Delta_0}{\det \Delta_{1\perp}}\Big]^{1/2} \Big[\frac{\det \Delta_{1\perp}}{\det \Delta_{2\perp}}\Big]^{1/2} \Big[\frac{\det \Delta_{2\perp}}{\det \Delta_{3\perp}}\Big]^{1/2} ... = 1$$

• cancellation of physical spin s det and ghost det for spin $s+1$ reflects hidden simplicity due to large gauge symmetry

• ∞ product is a priori ambiguous: requires regularization that should be consistent with underlying symmetry

• ζ-func. reg. is equiv to cancellation of factors in Z

[cf. ζ-func. reg. of vac energy in string theory consistent with symmetries – massless vector in $d=26$]

Lesson: theories of ∞ number of fields:
require specific definition at quantum level
maintain symmetries – regularization or defn of $\sum_s^\infty$
[cf. ambiguities in defn of string field theory may be fixed by
corresp with underlying 1-st quantised w-sheet formulation]

Conformal higher spin theory

• generalization of Maxwell ($s=1$) and Weyl ($s=2$) theories:

$F^2_{mn} \sim h_1 \partial^2 h_1$

$C^2_{mnkl} \sim h_2 \partial^4 h_2 + \partial^4 h_2 h_2 h_2 + \partial^4 h_2 h_2 h_2 h_2 + ...$

dimensionless coupling

differential + algebraic (Weyl) gauge symmetry

$\delta h_s = \partial \epsilon_{s-1} + \eta\, \alpha_{s-2}$

can gauge-fix h_s to be transverse and traceless

• totally symmetric $h_{m_1 ... m_s}$ describes pure spin s states off shell

has maximal gauge symm consistent with locality

at expense of higher-deriv kin terms (non-unitary)

[Fradkin, AT 85; AT 02; Segal 02]

$$S_s = \int d^4x\, h_s P_s \partial^{2s} h_s$$

$P_s \sim (\delta^m_n - \frac{\partial^m \partial_n}{\partial^2})^s$ – transv. traceless projector

• CHS field h_s has dim $\Delta = 2 - s$

Interacting theory: conformally invariant in flat space

number of derivatives in vertices fixed by dimension

$$S_s = \kappa \int d^4x \Big(h_s \partial^{2s} h_s + \partial^{s_1+s_2+s_3-2} h_{s_1} h_{s_2} h_{s_3} \\ + \partial^{s_1+s_2+s_3+s_4-4} h_{s_1} h_{s_2} h_{s_3} h_{s_4} + \dots \Big)$$

interacting action consistent with symmetries: induced theory

Some properties of free theory indicating hidden simplicity:

- free partition function ($\Delta_s = -\partial^2$)

$$Z_{\mathrm{CHS},s} = \Big[\frac{(\det \Delta_{s-1})^{s+1}}{(\det \Delta_s)^s} \Big]^{1/2} = \prod_{k=0}^{s-1} \Big[\frac{\det \Delta_{k\,\perp}}{\det \Delta_{s\,\perp}} \Big]^{1/2}$$

$$Z_{\mathrm{CHS}} = \prod_{s=1}^{\infty} Z_{\mathrm{CHS},s} = \Big[\frac{\det \Delta_0}{\det \Delta_1} \Big]^{1/2} \Big[\frac{(\det \Delta_1)^3}{(\det \Delta_2)^2} \Big]^{1/2} \Big[\frac{(\det \Delta_2)^4}{(\det \Delta_3)^3} \Big]^{1/2} \dots$$

no naive cancellation but if define as counting effective d.o.f.

$$Z_{\mathrm{CHS},s} = (Z_0)^{\nu_s} = (\det \Delta_0)^{-\nu_s/2} \ , \qquad \nu_s = s(s+1) = 2, 6, ...$$

$$Z_{\mathrm{CHS}} = \prod_{s=0}^{\infty} (Z_0)^{\nu_s} = (Z_0)^{\nu_{\mathrm{tot}}} \ , \qquad \nu_{\mathrm{tot}} = \sum_{s=0}^{\infty} \nu_s$$

$$\text{regularization}: \qquad \sum_{s=0}^{\infty} f(s) \ \to \ \sum_{s=0}^{\infty} f(s)\, e^{-\epsilon(s+\frac{1}{2})}\Big|_{\mathrm{fin.}}$$

then $\nu_{\mathrm{tot}} = \sum_{s=0}^{\infty} \nu_s = 0$ or $(Z_{\mathrm{CHS}})_{\mathrm{tot}} = 1$ as in MHS case

- same regularization is implied by relation

of massless HS in AdS_{d+1} and CHS at the boundary

- 1-loop $Z_{\mathrm{CHS}}(S^4)$ in same regularization is again =1

consistent with relation to MHS part. funct. in AdS_5

[Giombi et al 13; AT 13; Beccaria, Bekaert, AT 14]

• regularization consistent with symmetries of CHS theory: vanishing of conformal anomaly [Giombi,Klebanov; AT 13]

$$T^m_m = -\mathrm{a} R^* R^* + \mathrm{c}\, C^2$$

$$\mathrm{a}_s = \tfrac{1}{720}\nu_s^2(14\nu_s + 3)$$

$$\mathrm{c}_s - \mathrm{a}_s = \tfrac{1}{720}\nu_s(15\nu_s^2 - 45\nu_s + 4)\ , \qquad \nu_s = s(s+1)$$

• sums to 0 in same regularization $\sum_{s=0}^{\infty} f(s)\, e^{-\epsilon(s+\frac{1}{2})}\Big|_{\text{fin.}}$

$$\sum_{s=1}^{\infty}(\mathrm{c}_s - \mathrm{a}_s) = 0\ , \qquad \sum_{s=1}^{\infty}\mathrm{a}_s = 0$$

• total c- and a- anomaly vanish: 1-loop quantum consistency

• novel mechanism of UV finiteness:

summation of ∞ number of bosonic fields (cf. string theory)

CHS as "induced" theory

consider free $U(N)$ complex scalar CFT

$\int d^dx\, \Phi_i^* \partial^2 \Phi_i$

• exists tower of on-shell conserved traceless HS currents

$J_s = \Phi_i^* \, \mathcal{J}_s \, \Phi_i \sim \Phi_i^* \partial_{(m_1} ... \partial_{m_s)} \Phi_i + ...$

• implies existence of infinite tower of conserved charges:

symmetries of free equation $\partial^2 \Phi = 0 \to$ HS symmetry

(conf Killing tensors) [Eastwood, Vasiliev]

• generating functional for correlators of currents:

add $h_s J_s$ and integrate out Φ_i

$$\Gamma[h] = N \, \log\det \Big(-\partial^2 + \sum_s h_s \, \mathcal{J}_s \Big) \,, \qquad\qquad \mathcal{J}_s \sim \partial^s$$

• source fields h_s are CHS fields

CHS: gauge th of symm of Laplace eq (cf. Poincare $\to$ diffs)

h_s are gauge fields for symm of free scalar theory:

$\delta h_{m_1\cdots m_s} = \partial_{(m_1}\varepsilon_{m_2\cdots m_s)} + \eta_{(m_1 m_2}\,\alpha_{m_3\cdots m_s)}$

generalizing diffs and Weyl symmetry of Weyl gravity

have dim $\Delta = 2 - s$, i.e. "shadow" counterparts

of dim $s+2$ currents J_s in scalar CFT

• vectorial AdS/CFT:

J_s dual to massless HS fields in AdS_{d+1}

$\Gamma[h]$ should follow from Vasiliev-type theory in AdS$_{d+1}$

upon integrating over AdS$_{d+1}$ fields ϕ_s with Dirichlet b.c.

$$e^{-\Gamma[h]} = \int_{\phi_s|_{\partial \mathrm{AdS}}=h_s} [d\phi_s]\,\exp\big(-N\bar{S}[\phi]\big)$$

• $\Gamma[h]$ is non-local and does not have CHS symmetries but its logarithmically UV divergent part is local and invariant

• natural defn of CHS action as "induced" [AT 02; Segal 02]

$$S_{\rm CHS} \sim \log\det \Delta(h)\Big|_{\log\epsilon}\ , \qquad \Delta(h) = -\partial^2 + \sum_s \mathcal{J}_s h_s$$

or $\quad S_{\rm CHS} \sim\ {\rm tr}\ e^{-\epsilon\,\Delta(h)}\Big|_{\epsilon\to 0,\ {\rm fin}}$

• familiar low-spin case in manifestly covariant form ($d=4$)

$$L = \sqrt{g}\, g^{mn} D_m\Phi^* D_n\Phi + (\tfrac{1}{6}R + h_0')\Phi^*\Phi\ , \qquad D_m = \partial_m + iA_m$$

related to $\partial\Phi^*\partial\Phi + h_s\Phi^*\mathcal{J}_s\Phi$ by redefs
$h_0 = h_0' + A_m A^m + \frac{1}{6}R$, etc.
coeff of log UV divergence – from standard Seeley coeff:

$$S_{0+1+2} = \int d^4x \sqrt{g}\Big(h_0'^2 + \tfrac{1}{6}F_{mn}^2 + \tfrac{1}{60}C_{mnkl}^2\Big)$$

set $g_{mn} = \eta_{mn} + h'_{mn}$ and extract cubic, quartic, etc. couplings
then can compute CHS scattering amplitudes: 1111, 2222, etc.

Strategy:

• compute quadratic, cubic, quartic CHS couplings directly from UV singular part of corresponding scalar 1-loop diagrams

• use them to compute tree-level CHS 4-point scattering amps

• they turn out to be zero after non-trivial summation over all spin s CHS intermediate states

• this appears to be a consequence of CHS global symmetry

• this may serve as a lesson for attempts to understand what may happen in 2-derivative MHS theory in flat space

First illustrate this on simplest example:
scattering of external scalars via exchange of
tower of CHS fields

Scalar scattering via conformal HS exchange

[Joung, Nakach, AT 15]

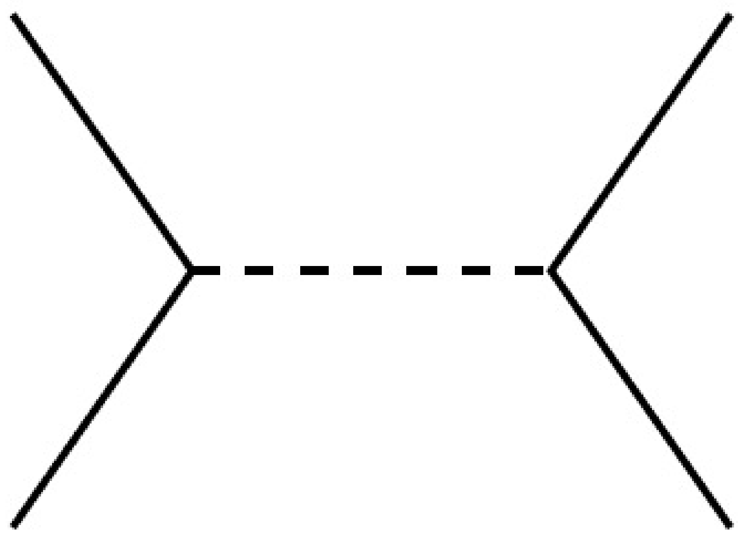
external scalar scattering via exchange of tower of CHS fields

$$S[\Phi, h] = \int d^4x \big[\Phi^* \partial^2 \Phi + \sum_{s=0}^{\infty} h_s\, J_s(\Phi)\big] + S[h]$$

$$S[h] = \kappa \sum_{s=0}^{\infty} \int h_s\, P_s \partial^{2s}\, h_s + \mathcal{O}(h^3)$$

- h_0 coupled to $\Phi^*\Phi$; h_μ to $i\Phi^*\partial_\mu\Phi + c.c.$; $h_{\mu\nu}$ to $T_{\mu\nu}$, etc.
- h_s exchange with propagator $\sim \frac{1}{p^{2s}}$ and p^s in the vertices:

scale invariance, no dimensional parameters



Four-scalar tree-level scattering amplitude

t-channel amplitude

$$A^{(\mathsf{t})}(\mathsf{s},\mathsf{t},\mathsf{u}) = \kappa^{-1} F\left(\tfrac{\mathsf{s}-\mathsf{u}}{\mathsf{s}+\mathsf{u}}\right) , \qquad F(z) \equiv \sum_{s=0}^{\infty} (s+\tfrac{1}{2}) P_s(z)$$

$\mathsf{s},\mathsf{t},\mathsf{u}$ are Mandelstam variables: $\mathsf{s}+\mathsf{t}+\mathsf{u}=0$

$P_s(z)$ – Legendre polynomial

- amplitude is scale-invariant: depends on ratios $\mathsf{s},\mathsf{t},\mathsf{u}$
- summing over spins: natural cutoff prescription

$$\sum_{s=0}^{\infty} f(s) \to \sum_{s=0}^{\infty} f(s)\, e^{-\varepsilon(s+\alpha_d)}\Big|_{\epsilon\to 0,\ \mathrm{fin}} , \qquad \alpha_d = \frac{d-3}{2} = \frac{1}{2}$$

$$F(z) = \delta(z-1)$$

same found using gen function for Legendre polynomials

- surprising result: amplitude is δ-function in phase space

Total amplitude:

- $\Phi\,\Phi \;\to\; \Phi\,\Phi$: t-channel plus u-channel

$$A_{\Phi\Phi\to\Phi\Phi} = \kappa^{-1}\,\left[\,\delta(\tfrac{\mathsf{s}}{\mathsf{t}}) + \delta(\tfrac{\mathsf{s}}{\mathsf{u}})\,\right]$$

in c.o.m. frame $\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_3 + \vec{p}_4$
scattering angle: $\frac{\mathsf{s}}{\mathsf{t}} = -(\sin^2\frac{\theta}{2})^{-1}, \qquad \frac{\mathsf{s}}{\mathsf{u}} = -(\cos^2\frac{\theta}{2})^{-1}$
arguments of delta-functions never vanish for real θ

$$A_{\Phi\Phi\to\Phi\Phi} = 0$$

- $\Phi\,\Phi^* \;\to\; \Phi\,\Phi^*$:

$$A_{\Phi\Phi^*\to\Phi\Phi^*} = \frac{\kappa^{-1}}{2}\left[\,\delta(\tfrac{\mathsf{u}}{\mathsf{t}}) + \delta(\tfrac{\mathsf{u}}{\mathsf{s}})\,\right] = \frac{\kappa^{-1}}{2}\left[\,\delta(\cot^2\tfrac{\theta}{2}) - \delta(\cos^2\tfrac{\theta}{2})\,\right]$$

t-channel and s-channel contributions cancel each other

$$A_{\Phi\Phi^*\to\Phi\Phi^*} = 0$$

• individual spin s exchange contributions are nontrivial
but total amplitude =0 in particular summation prescription
• large underlying symmetry constrains the S-matrix:
$A_{4\Phi} = 0$ is implied by the global symmetry of CHS theory
(cf. integrability / hidden conserved charges in 2d theories)

Global CHS symmetry :

• global part of CHS gauge symmetry:
symmetry of scalar Laplace eq (conformal Killing tensors)
conformal generators plus other higher spin generators
• in particular, "hyper-translations"

$$\delta\Phi = \varepsilon_r \cdot P_r \Phi = \varepsilon^{\mu_1 \ldots \mu_r} \partial_{\mu_1} \ldots \partial_{\mu_r} \Phi$$

• this fixes amplitude to be

$$A_{\Phi\Phi\to\Phi\Phi}(\mathsf{s},\mathsf{t},\mathsf{u}) = k_1(\mathsf{t},\mathsf{u})\,\delta(\mathsf{s}) + k_2(\mathsf{s},\mathsf{u})\,\delta(\mathsf{t}) + k_3(\mathsf{t},\mathsf{s})\,\delta(\mathsf{u})$$

• use also invariance under dilatations $p \to \gamma p$

$$A_{\Phi\Phi\to\Phi\Phi}(\gamma^2\,\mathsf{s}, \gamma^2\,\mathsf{t}, \gamma^2\,\mathsf{u}) = A_{\Phi\Phi\to\Phi\Phi}(\mathsf{s}, \mathsf{t}, \mathsf{u})$$

• solution consistent with crossing and scaling symmetry

$$A_{\Phi\Phi\to\Phi\Phi}(\mathsf{s}, \mathsf{t}, \mathsf{u}) = 0$$

• regularization of the sum over s in which
tree-level scalar amplitude vanishes
is thus consistent with underlying CHS symmetry

CHS tree level scattering [Beccaria, Nakach, AT]

1. first find 2-, 3- and 4-point vertices in CHS action
from UV pole part of scalar loop integrals with J_s insertions
2. compute resulting CHS scatt amps 1-1-1-1, 2-2-2-2, etc.

coupling of external CHS fields to complex scalar

$$L = -\partial_\mu \Phi^* \, \partial^\mu \Phi + \sum_{s=0}^{\infty} J_{\mu(s)} \, h^{\mu(s)} \,, \qquad J_{\mu(s)} \equiv J_{\mu_1 \ldots \mu_s}$$

$$J_{\mu(s)}(x) = \frac{i^s \, 2^s}{(2s)!} \sum_{k=0}^{s} \binom{s}{k} \binom{\frac{s+k-1}{2}}{s} G^{(k)}_{\mu(s)}(x)$$

$$G^{(k)}_{\mu(s)}(x) = (\partial - \partial')^{\mu(k)} (\partial + \partial')^{\mu(s-k)} \Phi(x) \, \Phi^*(x') \Big|_{x=x'}$$

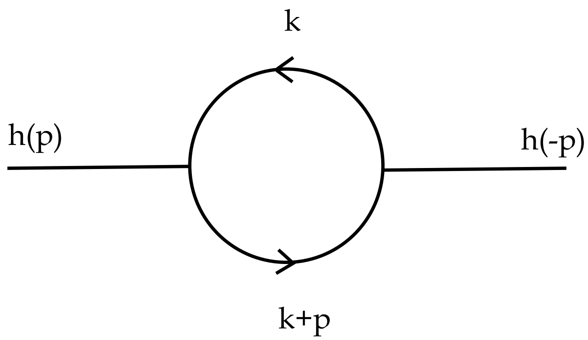
$$J = \Phi \, \Phi^*, \qquad J_\mu = \frac{i}{2} \, (\partial_\mu \Phi \, \Phi^* - \Phi \, \partial_\mu \Phi^*),$$

$$J_{\mu\nu} = \frac{1}{12} \Big[- \partial_\mu \partial_\nu \Phi \, \Phi^* - \Phi \, \partial_\mu \partial_\nu \Phi^* + 2 \, (\partial_\mu \Phi \, \partial_\nu \Phi^* + \partial_\nu \Phi \, \partial_\mu \Phi^*) \Big]$$

Induced CHS action

$$S = \int d^4x \Big(\sum_s h_s \partial^{2s} h_s + \sum_{s_i} \partial^{s_1+s_2+s_3-2} h_{s_1} h_{s_2} h_{s_3} + \sum_{s_i} \partial^{s_1+s_2+s_3+s_4-4} h_{s_1} h_{s_2} h_{s_3} h_{s_4} + \ldots \Big)$$

- kinetic term:

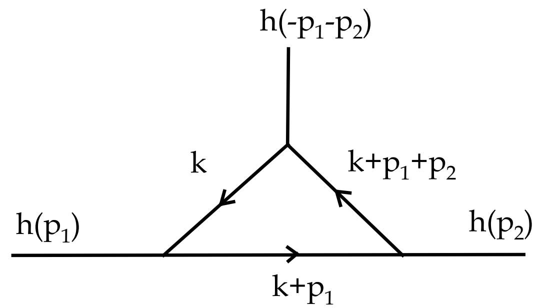


$$= \int \frac{d^d k}{(2\pi)^d} \frac{N(k,p)\ldots}{k^2\,(k+p)^2}$$

$\frac{1}{\varepsilon} = \frac{1}{d-4}$ UV pole part (for TT field h_s):

$$S_2 = \frac{1}{2^s\,(2s+1)!} \int d^4x \, h_{\mu(s)} \, \Box^s \, h^{\mu(s)}$$

• cubic vertex: from pole part of



for example: 1-1-s

$$V_{\mu\nu\rho(s)} = \int \left(\frac{dk}{2\pi}\right)^d \frac{k_\mu (k+p_1)_\nu (k+p_1+p_2)_{\rho(s)}}{k^2 (k+p_1)^2 (k+p_1+p_2)^2}\Bigg|_{\frac{1}{\varepsilon}\text{ part}}$$

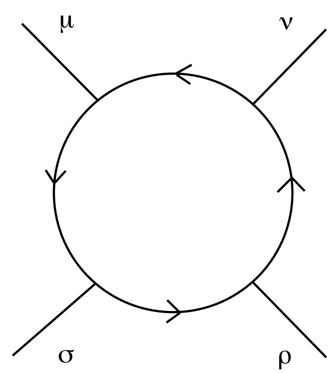
$$\begin{aligned} S_3(1,1,s) = \tfrac{i^s}{(s+2)!} \int d^4x \Big[& \partial^{\rho(s)} h_\mu h^\mu h_{\rho(s)} - 2 h_\mu \, \partial^\mu \, \partial_{\rho(s-1)} \, h_\nu \, h^{\nu\rho(s-1)} \\ & - \tfrac{s}{2} \partial^{\rho(s-2)} \Box h^\mu h^\nu h_{\mu\nu\rho(s-2)} - \tfrac{s}{2} \partial^{\rho(s-2)} h^\mu \Box h^\nu h_{\mu\nu\rho(s-2)} \\ & - \partial_\lambda \partial^{\rho(s-2)} h^\mu \partial^\lambda h^\nu h_{\mu\nu\rho(s-2)} \Big] \end{aligned}$$

e.g. 1-1-2 is same as in Maxwell $\int d^4x\, \sqrt{g}\, g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\mu\rho}$

$$S_3(1,1,2) = \tfrac{1}{24} \int d^4x \Big[\partial_\rho h_\mu\, \partial_\sigma h^\mu h^{\rho\sigma} - 2\partial_\rho h_\mu\, \partial^\mu\, h_\nu\, h^{\nu\rho} \\ + 2\, h^\mu\, \Box h^\nu h_{\mu\nu} + \partial_\lambda h^\mu \partial^\lambda h^\nu h_{\mu\nu} \Big]$$

• quartic vertex:

e.g. 4-vector contact term from pole part of diagram



$\frac{1}{16} \int d^4x (h_\mu h^\mu)^2$ combining into $\int d^4x (h_0 - \frac{1}{4} h_\mu h^\mu)^2$
contribution to 1-1-1-1 scattering cancels against h_0 exchange

• similar (more complicated) for 2-2-s and 2-2-2-2 vertices, etc.

CHS "S-matrix"

• $s = 1$ case is standard vector but for $s \geqslant 2$
higher-derivative ∂^{2s} kinetic term: non-unitary theory

• formal definition of "S-matrix": amputated Green's functions
with special asymptotic states attached

equivalent to: $\quad S = S(h_{\rm class}(h_{\rm in})) = A_3 h_{\rm in}^3 + A_4 h_{\rm in}^4 + \ldots$

$\frac{\delta S}{\delta h}\Big|_{h_{\rm class}} = 0, \quad h_{\rm class} = h_{\rm in} + O(h_{\rm in}^2), \qquad \partial^{2s} h_{\rm in} = 0$

• $s = 2$: ∂^4 Weyl graviton with 6 d.o.f.

$\frac{1}{p^4} \to \frac{1}{\epsilon}\left[\frac{1}{p^2} - \frac{1}{p^2+\epsilon}\right]_{\epsilon\to 0}$

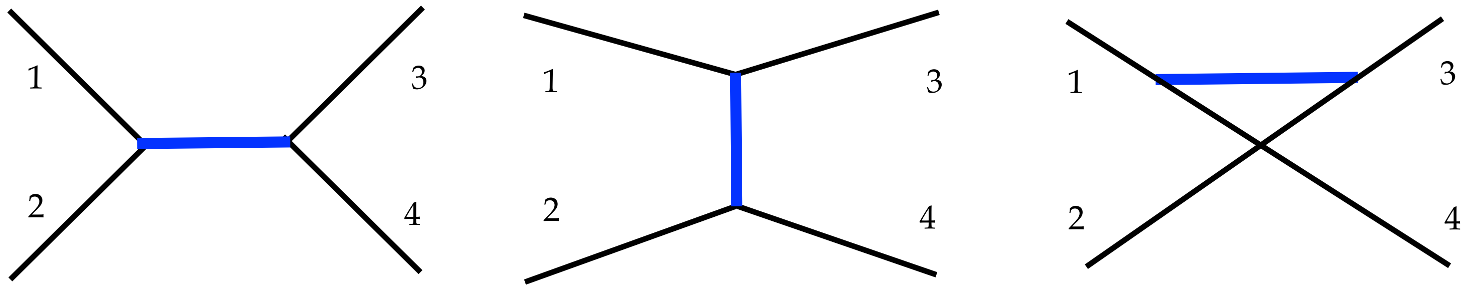
linearized Bach eqs $\partial_m \partial_k R_{kn} + \ldots = 0$ solved in particular
by $R_{mn} = 0$: choose standard helicity ± 2 graviton
as special asymptotic states

• same for $s > 2$: use CHS vertices and internal propagators
but standard massless spin s polarizations as asymptotic states

CHS 4-particle tree level amplitude

helicities $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and s, t, u $\quad (p_i^2 = 0$ for ext legs)

exchange diagrams



4-vector scattering

spin s exchange: two 1-1-s vertices and TT spin s propagator

$$V_{\alpha\beta\rho(s)}(p,q) = \frac{1}{(s+2)!}\Big\{\eta_{\alpha\beta}\left[\tfrac{1}{2}\,p_{\rho(s)} + \tfrac{1}{2}\,q_{\rho(s)}\right]$$

$$-\tfrac{1}{2}\eta_{\alpha\rho_1}p_\beta p_{\rho_2}\dots p_{\rho_s} + \tfrac{1}{2}\eta_{\beta\rho_1}q_\alpha p_{\rho_2}\dots p_{\rho_s} - \tfrac{1}{2}\eta_{\beta\rho_1}q_\alpha q_{\rho_2}\dots q_{\rho_s} + \tfrac{1}{2}\eta_{\alpha\rho_1}p_\beta q_{\rho_2}\dots q_{\rho_s}$$

$$-\tfrac{1}{2}\,\eta_{\alpha\rho_1}\eta_{\beta\rho_2}\,p_{\rho_3}\dots p_{\rho_s}\,p\cdot q - \tfrac{1}{2}\eta_{\alpha\rho_1}\eta_{\beta\rho_2}\,q_{\rho_3}\dots q_{\rho_s}\,p\cdot q\Big\}$$

• $s = 2$ exchange (Weyl graviton)

same as 4-vector amplitude in conformal sugra $F^2 + C^2 + \ldots$

only MHV non-zero (++++, +++-,... =0)

λ	$A_{\rm s}^{(2)}$	$A_{\rm t}^{(2)}$	$A_{\rm u}^{(2)}$
$\pm\pm\mp\mp$	0	$\frac{5}{48}\,\frac{\rm s^2}{\rm t^2}$	$\frac{5}{48}\,\frac{\rm s^2}{\rm u^2}$
$\pm\mp\mp\pm$	$\frac{5}{48}\,\frac{\rm u^2}{\rm s^2}$	$\frac{5}{48}\,\frac{\rm u^2}{\rm t^2}$	0

• $s = 4$ exchange:

propagator ($P^\mu_\nu = \delta^\mu_\nu - \frac{\partial_\mu\partial_\nu}{\partial^2}$)

$$D^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\beta_1\beta_2\beta_3\beta_4}(p) = \frac{2^{s-1}(2s+1)!}{(p^2)^s}\Big[P^{(\alpha_1}_{(\beta_1}P^{\alpha_2}_{\beta_2}P^{\alpha_3}_{\beta_3}P^{\alpha_4)}_{\beta_4)}$$
$$-\tfrac{6}{7}\,P^{(\alpha_1\alpha_2}P_{(\beta_1\beta_2}P^{\alpha_3}_{\beta_3}P^{\alpha_4)}_{\beta_4)} + \tfrac{3}{35}\,P^{(\alpha_1\alpha_2}P^{\alpha_3\alpha_4)}P_{(\beta_1\beta_2}P_{\beta_3\beta_4)}\Big]$$

again only MHV are non-zero:

λ	$A_{\mathrm{s}}^{(4)}$	$A_{\mathrm{t}}^{(4)}$	$A_{\mathrm{u}}^{(4)}$
$\pm\pm\mp\mp$	0	$\frac{\mathrm{s}^2\,(28\,\mathrm{s}^2+42\,\mathrm{s}\,\mathrm{t}+15\,\mathrm{t}^2)}{80\,\mathrm{t}^4}$	$\frac{\mathrm{s}^2\,(28\,\mathrm{s}^2+42\,\mathrm{s}\,\mathrm{u}+15\,\mathrm{u}^2)}{80\,\mathrm{u}^4}$
$\pm\mp\mp\pm$	$\frac{\mathrm{u}^2\,(28\,\mathrm{u}^2+42\,\mathrm{s}\,\mathrm{u}+15\,\mathrm{s}^2)}{80\,\mathrm{s}^4}$	$\frac{\mathrm{u}^2\,(28\,\mathrm{u}^2+42\,\mathrm{t}\,\mathrm{u}+15\,\mathrm{u}^2)}{80\,\mathrm{t}^4}$	0

- General structure of spin s exchange 1111 amplitudes ($\neq 0$)

$$A_{\mathrm{t}}^{(s)}(\pm\pm\mp\mp) = \mathrm{c}_s\Big(\frac{\mathrm{s}}{\mathrm{t}}\Big)^s\,\mathrm{P}_s\Big(\frac{\mathrm{t}}{\mathrm{s}}\Big), \qquad A_{\mathrm{u}}^{(s)}(\pm\pm\mp\mp) = \mathrm{c}_s\Big(\frac{\mathrm{s}}{\mathrm{u}}\Big)^s\,\mathrm{P}_s\Big(\frac{\mathrm{u}}{\mathrm{s}}\Big),$$

$$A_{\mathrm{s}}^{(s)}(\pm\mp\mp\pm) = \mathrm{c}_s\Big(\frac{\mathrm{u}}{\mathrm{s}}\Big)^s\,\mathrm{P}_s\Big(\frac{\mathrm{s}}{\mathrm{u}}\Big), \qquad A_{\mathrm{t}}^{(s)}(\pm\mp\mp\pm) = \mathrm{c}_s\Big(\frac{\mathrm{u}}{\mathrm{t}}\Big)^s\,\mathrm{P}_s\Big(\frac{\mathrm{t}}{\mathrm{u}}\Big)$$

$$\mathrm{c}_s = \frac{2\,s+1}{2\,(s-1)\,s\,(s+1)\,(s+2)}$$

$$\mathrm{P}_s(x) = x^{s-2}\,P_{s-2}^{(4,0)}\Big(\frac{x+2}{x}\Big), \quad s-2 \text{ order} \qquad\qquad s = 2, 4, 6, \ldots$$

$P_n^{(a,b)}(x)$ are Jacobi polynomials, i.e.

$$\mathrm{P}_s(x) = \sum_{j=2}^{s} \frac{1}{(j-2)!\,(j+2)!}\,\frac{(s+j)!}{(s-j)!}\,x^{s-j} \sim\ x^{s-2}\,{}_2F_1\Big(2-s, s+3, 5; -\frac{1}{x}\Big)$$

Sum over spins

total $++--$ amplitude: t- plus u-channel

$$A^{(s)} = \mathrm{c}_s \left[\left(\tfrac{\mathrm{s}}{\mathrm{t}}\right)^s \mathrm{P}_s\left(\tfrac{\mathrm{t}}{\mathrm{s}}\right) + \left(\tfrac{\mathrm{s}}{\mathrm{u}}\right)^s \mathrm{P}_s\left(\tfrac{\mathrm{u}}{\mathrm{s}}\right)\right]$$

define $x = \mathrm{t}/\mathrm{s}$

$$A^{(s)}(x) = \sigma_s(x) + \sigma_s(-1-x), \qquad \sigma_s(x) = \mathrm{c}_s\, x^{-s}\, \mathrm{P}_s(x)$$

use generating function for Jacobi polynomials $P_{s-2}^{(4,0)}$

$$\sum_{s=2}^{\infty} x^{-s}\, \mathrm{P}_s(x)\, z^{s-2} = \frac{1}{x^2} \frac{16}{\sqrt{z^2 - \frac{2z(x+2)}{x} + 1}\left(\sqrt{z^2 - \frac{2z(z+2)}{z} + 1} - z + 1\right)^4}$$

$$\sigma(x) = \sum_{s=2,4,6,\ldots}^{\infty} \sigma_s(x) = \lim_{z\to 1} \sum_{s=2,4,6,\ldots}^{\infty} \mathrm{c}_s\, x^{-s}\, \mathrm{P}_s(x)\, z^{s-2}$$
$$= \frac{1}{8}\left[-2\,x + 2\,(x+1)\,x\, \log\left(\frac{1}{x} + 1\right) - 1\right].$$

summed over s amplitude is zero as in scalar scattering case

$$A(x) = \sum_{s=2,4,6,\ldots}^{\infty} A^{(s)}(x) = \sigma(x) + \sigma(-1-x) = 0$$

Generalize to $s > 1$ external states

Why Jacobi polynomials? cf. partial wave expansion in terms of intermediate angular mom J states [Jacob, Wick 1959]

$$A_{\lambda_1,\lambda_2;\lambda_3,\lambda_4}(\mathrm{s},\theta) = f_{\{\lambda_i\}}(\theta) \sum_J (J+\tfrac{1}{2})\mathrm{F}^{(J)}_{\{\lambda_i\}}(\mathrm{s})\ P^{(|\lambda+\mu|,|\lambda-\mu|)}_{J-M}(\cos\theta)$$

$$\lambda = \lambda_1 - \lambda_2, \quad \mu = \lambda_3 - \lambda_4, \quad M = \max(|\lambda|, |\mu|)$$

$$f_{\{\lambda_i\}}(\theta) = \left(\cos\tfrac{\theta}{2}\right)^{|\lambda+\mu|} \left(\sin\tfrac{\theta}{2}\right)^{|\lambda-\mu|} = \left(-\tfrac{\mathrm{u}}{\mathrm{s}}\right)^{\frac{1}{2}|\lambda+\mu|} \left(-\tfrac{\mathrm{t}}{\mathrm{s}}\right)^{\frac{1}{2}|\lambda-\mu|}$$

• identification of J-th partial wave with contribution of exchange of intermediate spin J field (Lorentz invariance)

• scale invariance controls how F depends on s

e.g., for dim 1 external particles $\mathrm{F}^{(J)}_{\{\lambda_i\}}(\mathrm{s}) = \mathrm{const}$

• general prediction for Jacob-Wick coefficient for scattering of CHS fields of dim $\Delta_i = 2 - |\lambda_i|$ (no dim$\neq 0$ parameters!)

$$\mathrm{F}^{(J)}_{\{\lambda_i\}}(\mathrm{s}) = k_{\lambda,\mu} \, \tfrac{[J-\max(|\lambda|,|\mu|)]!}{[J+\min(|\lambda|,|\mu|)]!} \, \mathrm{s}^r \ , \qquad r = 2 - \tfrac{1}{2}\sum_{i=1}^{4} \Delta_i$$

Special cases $(J = s)$:

• External scalar scattering $\Phi\Phi^* \to \Phi\Phi^*$: $\lambda_i = 0, \ \ \Delta_i = 1$

$$A_{0,0;0,0}(\mathrm{s},\theta) = \sum_{s=0,2,\ldots} (s+\tfrac{1}{2})\mathrm{F}^{(s)}_0 \, P^{(0,0)}_s(\cos\theta)$$

same as s-channel exchange from Lagrangian with $\mathrm{F}^{(s)}_0 = \mathrm{const}$

- $1_+1_+ \to 1_+1_+$

t-channel $\quad (\cos\theta = -1 - 2\,\frac{\mathrm{s}}{\mathrm{t}})$

$$A_{++;++}(\theta) = \,(\sin\tfrac{\theta}{2})^{-4} \sum_{s=2,4,\ldots} (s+\tfrac{1}{2})\mathrm{F}^{(s)}_{++;++}\, P^{(4,0)}_{s-2}(\cos\theta)$$

agrees with Lagrangian result and $\quad \mathrm{F}^{(s)}_{++;++} = \frac{1}{(s-1)\,s\,(s+1)\,(s+2)}$

- $2_+2_+ \to 2_+2_+$

t-channel $++ \to ++$ or $++--$ MHV (s-channel vanishes)

$$A_{++;++}(\mathrm{t},\theta) = \,\frac{\mathrm{s}^4}{\mathrm{t}^4} \sum_{s=4,6,\ldots} (s+\tfrac{1}{2})\mathrm{F}^{(s)}\mathrm{t}^2\, P^{(8,0)}_{s-4}(\cos\theta)$$

explicit computation gives for full (t- plus u- channel) amplitude

$$A^{(s)} = \mathrm{c}_s\,\mathrm{s}^2 \left[\, \left(\tfrac{\mathrm{s}}{\mathrm{t}}\right)^{s-2}\, \mathrm{P}_s\!\left(\tfrac{\mathrm{t}}{\mathrm{s}}\right) + \left(\tfrac{\mathrm{s}}{\mathrm{u}}\right)^{s-2}\, \mathrm{P}_s\!\left(\tfrac{\mathrm{u}}{\mathrm{s}}\right)\right]$$

$$\mathrm{P}_s(x) = x^{s-2}\,\mathrm{P}^{(8,0)}_{s-4}\!\left(\tfrac{x+2}{x}\right)\,, \qquad\qquad \mathrm{c}_s = \frac{9}{32}\,\frac{2s+1}{(s-3)(s-2)(s-1)s(s+1)(s+2)(s+3)(s+4)}$$

• sum over spins:

$$\sigma(x) = \sum_{s=4,6,8,\dots}^{\infty} \sigma_s(x) = \lim_{z\to 1} \sum_{s=4,6,8,\dots}^{\infty} \mathrm{c}_s\, x^{-(s-2)}\, \mathrm{P}_s(x)\, z^{s-4}$$

$$= \tfrac{1}{4320}\left[60\,(x+1)^3\,x^3\,\log\left(\tfrac{1}{x}+1\right) - 60\,x^5 - 150\,x^4 - 110\,x^3 - 15\,x^2 + 3\,x - 1\right]$$

total amplitude vanishes: t- and u- channels cancel

$$\sigma(x) + \sigma(-1-x) = 0$$

• still to add contribution of $s = 0, 2$ exchanges + 2222 vertex

$$A^{0,\mathrm{s}}_{++;++} = \frac{\mathrm{s}^2}{18432}, \quad A^{0,\mathrm{t}}_{++;++} = \frac{\mathrm{t}^2\,\mathrm{u}^4}{2048\,\mathrm{s}^4}, \quad A^{0,\mathrm{u}}_{++;++} = \frac{\mathrm{t}^4\,\mathrm{u}^2}{2048\,\mathrm{s}^4},$$

$$A^{2,\mathrm{s}}_{++;++} = \frac{\mathrm{s}^2+6\,\mathrm{s}\,\mathrm{t}+6\,\mathrm{t}^2}{92160}, \quad A^{2,\mathrm{t}}_{++;++} = \frac{\mathrm{u}^2\left(2\,\mathrm{s}^4-10\,\mathrm{s}^3\,\mathrm{t}+33\,\mathrm{s}^2\,\mathrm{t}^2-24\,\mathrm{s}\,\mathrm{t}^3+3\,\mathrm{t}^4\right)}{30720\,\mathrm{s}^4}$$

$$A^{2,\mathrm{u}}_{++;++} = \frac{\mathrm{t}^2\left(2\,\mathrm{s}^4-10\,\mathrm{s}^3\,\mathrm{u}+33\,\mathrm{s}^2\,\mathrm{u}^2-24\,\mathrm{s}\,\mathrm{u}^3+3\,\mathrm{u}^4\right)}{30720\,\mathrm{s}^4}$$

$$A^{\mathrm{contact}}_{++;++} = -\frac{\mathrm{s}^6-\mathrm{s}^5\,\mathrm{t}+26\,\mathrm{s}^4\,\mathrm{t}^2+63\,\mathrm{s}^3\,\mathrm{t}^3+54\,\mathrm{s}^2\,\mathrm{t}^4+27\,\mathrm{s}\,\mathrm{t}^5+9\,\mathrm{t}^6}{7680\,\mathrm{s}^4}$$

non-trivial cancellation (similarly for all other helicity choices)

$$A^{0,\mathrm{s}} + A^{0,\mathrm{t}} + A^{0,\mathrm{u}} + A^{2,\mathrm{s}} + A^{2,\mathrm{t}} + A^{2,\mathrm{u}} + A^{\mathrm{contact}} = 0$$

thus full 2222 amplitude vanishes as it did in 1111 case

• same cancellation checked for 1122 amplitude:
expressed in terms of $P_{s-4}^{(6,2)}(-1-2\frac{\mathrm{t}}{\mathrm{s}})$ in s-channel and $P_{s-4}^{(6,0)}(-1-2\frac{\mathrm{t}}{\mathrm{s}})$ in t-channel in agreement with J-W;
exchanges cancel against 1122 contact term

• similar considerations should apply for $ss \to ss$ amplitude

• conjecture: full CHS S-matrix is trivial
• this should follow from underlying global CHS symmetry as in external scalar scattering case

CHS symmetries

define $h(x,u) \equiv h_{\mu_1\ldots\mu_s} u^{\mu_1}\ldots u^{\mu_s}$

$$f(x,u) \star g(x,u) = f(x,u)\, e^{\frac{i}{2}(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_u - \overleftarrow{\partial}_u \cdot \overrightarrow{\partial}_x)}\, g(x,u)$$

$$[f(x,u), g(x,u)] = 2f(x,u)\cos[\tfrac{i}{2}(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_u - \overleftarrow{\partial}_u \cdot \overrightarrow{\partial}_x))]\, g(x,u)$$

$$\{f(x,u), g(x,u)\} = 2f(x,u)\sin[\tfrac{i}{2}(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_u - \overleftarrow{\partial}_u \cdot \overrightarrow{\partial}_x)]\, g(x,u)$$

diff and algebraic symm of scalar-CHS system [Segal 02]

$$\delta_\epsilon h(x,u) = (u \cdot \partial_x)\epsilon(x,u) - \tfrac{i}{2}\,[h(x,u), \epsilon(x,u)]$$

$$\delta_\alpha h(x,u) = \left(u^2 - \tfrac{1}{4}\partial_x^2\right)\alpha(x,u) - \tfrac{1}{2}\,\{h(x,u), \alpha(x,u)\}$$

$$\delta_{\epsilon+i\alpha}\Phi(x) = e^{-\frac{i}{2}\partial_{x'}\cdot\partial_u}\big(\epsilon(x,u) + i\alpha(x,u)\big)\Phi(x)\Big|_{x=x',\ u=0}$$

$\delta h = \delta^{[0]}h + \delta^{[1]}h$: $\quad \delta^{[0]}h_s \sim \partial\epsilon_{s-1} + \eta\alpha_{s-2}$ gauge symmetry

global symmetry from $\delta^{[1]}h \sim \epsilon\,\partial h + \partial\epsilon\; h + \ldots$ for special ϵ

spin s field transforms in terms of $s' < s$ fields

$$\delta_\epsilon^{[1]} h_0 \sim \sum_k \tfrac{1}{k!} \epsilon^{\mu(k)} \partial_{\mu(k)} h_0$$
$$\delta_\epsilon^{[1]} h^\rho \sim \sum_k \left[\tfrac{1}{(k+1)!} \epsilon^{\rho\mu(k)} \partial_{\mu(k)} h_0 + \tfrac{1}{k!} \epsilon^{\mu(k)} \partial_{\mu(k)} h^\rho \right]$$
$$\delta_\epsilon^{[1]} h^{\rho\sigma} \sim \sum_k \left[\tfrac{1}{(k+2)!} \epsilon^{\rho\sigma\mu(k)} \partial_{\mu(k)} h_0 + \tfrac{1}{(k+1)!} \epsilon^{\mu(k)(\rho} \partial_{\mu(k)} h^{\sigma)} + \tfrac{1}{2!k!} \epsilon^{\mu(k)} \partial_{\mu(k)} h^{\rho\sigma} \right]$$

special choice of global symmetry parameters:
constraints on amplitudes as in external scalar scattering case

higher spin global symmetries $\rightarrow$ higher spin conserved charges
$\rightarrow$ triviality of S-matrix (cf. Coleman-Mandula)

Massless HS theory in flat space

• 2-derivative unitary free theory is known but

is there a consistent (gauge-invariant, local) interacting theory?

• which is underlying symmetry?

• expect HS symmetry $\to \infty$ tower of HS conserved charges

hidden simplicity? fixing S-matrix uniquely?

S-matrix is "trivial"? non-trivial only for special momenta?

UV finiteness?

• "flat limit" of Vasiliev's theory in AdS?

leading Regge trajectory "truncation" of

$\alpha' \to \infty$ limit of flat-space string?

Interacting massless higher spins in flat $d \geqslant 4$ space:

- free theory $\int d^4x\, \partial\phi_s \partial\phi_s, \quad \delta\phi_s = \partial\epsilon_{s-1}$
- interacting theory? various $s > 2$ "no-go theorems"

no minimal interactions – no long-range forces [Weinberg]

- consistent theory may still exist if contains

(i) infinite tower of spins $s = 0, 1, 2, 3, ..., \infty$

(ii) higher derivative (non-minimal) cubic interactions

$$\partial^n \phi_{s_1}\phi_{s_2}\phi_{s_3}, \quad s_2 + s_3 - s_1 \leqslant n \leqslant s_2 + s_3 + s_1 \quad (s_1 \leqslant s_2 \leqslant s_3)$$

e.g. 2-2-2 vertex has ∂^2, ∂^6

[l.c: Bengtsson, Bengtsson, Brink; Metsaev;

cov: Manvelyan, Mkrtchyan, Ruhl; Sagnotti, Taronna; Joung]

- Noether procedure: deform $\delta\phi_s = \partial\epsilon_{s-1} + ...$, add 4-vertex,...

should fix 3-point coupling consts

l.c. gauge: [Metsaev] $\qquad g_{s_1 s_2 s_3} = g\,\frac{\ell^{s_1+s_2+s_3-1}}{(s_1+s_2+s_3-1)!}$

- two parameters: g= dimensionless and ℓ= length

$$\frac{1}{g^2}\int d^4x \Big[\sum_s \partial\phi_s\partial\phi_s + \sum \ell^{n-1}\partial^n\phi_{s_1}\phi_{s_2}\phi_{s_3} + \sum \ell^{k-2}\partial^k\phi^4 + \dots\Big]$$

– ϕ^3 terms: two covariant structures $\partial^{s_1+s_2+s_3}$ and $\partial^{s_2+s_3-s_1}$

– ϕ^4 remains to be fixed (local?)

– effectively non-local theory: no. ∂ grows with s and n of ϕ^n

- motivation to study:

possible relation to AdS theory (UV limit, loops, etc.)

- despite ∂^n vertices and scale ℓ theory may be UV finite

[in particular summation prescription; cf. string and CHS]

Free higher spin action

symmetric higher spin tensors

$$\phi_s(x,u) = \phi^{a_1 \dots a_s}(x)\, u_{a_1} \dots u_{a_s}$$

Fronsdal action

$$S^{(2)}[\phi_s] = \tfrac{1}{2} \int d^d x \big[\phi_s(x,\partial_u)\, \widehat{T} \widehat{\mathcal{F}}\, \phi_s(x,u)\big]_{u=0}$$

$$\widehat{T} = 1 - \tfrac{1}{4} u^2 \partial_u^2, \quad \widehat{\mathcal{F}} \equiv \partial_x^2 - (u \cdot \partial_x)\, \widehat{D}, \quad \widehat{D} \equiv (\partial_x \cdot \partial_u) - \tfrac{1}{2}(u \cdot \partial_x)\partial_u^2$$

off-shell field ϕ_s double-traceless

$$(\partial_u^2)^2 \phi_s(x,u) = 0$$

gauge transformations

$$\delta_s^{(0)} \phi_s(x,u) = (u \cdot \partial_x) \varepsilon_{s-1}(x,u)$$

de Donder gauge: $\partial^{a_1} \phi_{a_1 \dots a_s} + \dots = 0$

equations of motion $\Box \phi_s(x,u) = 0$

• scattering of spin 0 particles:

need cubic interaction vertices with $s_1 = 0, s_2, s_3$

traceless-transverse part of cubic vertex $\quad (\partial_{x_{ij}} \equiv \partial_{x_i} - \partial_{x_j})$

$$S^{(3)}[\phi_0, \phi_{s_2}, \phi_{s_3}] = g_{0 s_2 s_3} \int d^d x \Big[(\partial_{u_2} \cdot \partial_{x_{31}})^{s_2} (\partial_{u_3} \cdot \partial_{x_{12}})^{s_3}$$

$$\times \phi_0(x_1) \phi_{s_2}(x_2, u_2) \phi_{s_3}(x_3, u_3) \Big]_{\substack{u_i = 0 \\ x_i = x}}$$

propagator: $\mathcal{D}_s^d(u, u'; p) = -\frac{i}{p^2} \mathcal{P}_s^{d-2}(u, u')$ – traceless in $d - 2$

in $d = 4$: $\qquad \mathcal{P}_s^2(u, u') = \frac{1}{(s!)^2} \left(\sqrt{u^2 u'^2}\right)^s T_s\left(\frac{u \cdot u'}{\sqrt{u^2 u'^2}}\right)$

$$T_s(z) \equiv \frac{s}{2} \sum_{k=0}^{[s/2]} \frac{(-1)^k (s-k-1)!}{k!(s-2k)!} (2z)^{s-2k}$$

$$= \frac{1}{2} \left[\left(z + \sqrt{z^2 - 1}\right)^s + \left(z - \sqrt{z^2 - 1}\right)^s \right]$$

T_s = Chebyshev polynomial of first kind

Tree-level 4-scalar scattering amplitude   [Ponomarev, AT 16]

• exchange of tower of higher spin fields

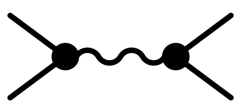
[Bekaert, Joung, Mourad 09]

here real scalar is $s = 0$ member of HS tower

(i) use of explicit values of coupling constants of HS theory

(ii) add contribution of contact 4-vertex

Exchange contribution: s-channel exchange of spin s field



$$\equiv \quad \mathcal{A}^s_{exch}(\mathrm{s}, \mathrm{t}, \mathrm{u})$$

Mandelstam variables $(p_i^2 = p_i'^2 = 0, \quad \mathrm{s} + \mathrm{t} + \mathrm{u} = 0)$

$$\mathcal{A}_{exch} = \sum_{s=0,2,4,\ldots}^{\infty} \mathcal{A}^s_{exch}, \quad \mathcal{A}^s_{exch}(\mathrm{s}, \mathrm{t}, \mathrm{u}) = -\frac{i g^2_{00s}}{\mathrm{s}} \big(\mathrm{t} + \mathrm{u}\big)^s \, T_s\big(\tfrac{\mathrm{t}-\mathrm{u}}{\mathrm{t}+\mathrm{u}}\big)$$

$$\mathcal{A}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) = -\frac{i}{\mathrm{s}}\Big[F\big(\sqrt{\mathrm{s}+\mathrm{t}}+\sqrt{\mathrm{t}}\,\big) + F\big(\sqrt{\mathrm{s}+\mathrm{t}}-\sqrt{\mathrm{t}}\,\big)\Big]$$

$$F(z) \equiv \sum_{s=0,2,4,\ldots}^{\infty} g^2_{00s}\,(\tfrac{z^2}{4})^s = \tfrac{1}{8}g^2\,(\ell z)^2\big[I_0(\ell z) - J_0(\ell z)\big]$$

sum over spins here is convergent:
non-trivial dependence on Mandelstam variables and ℓ

$$\widehat{\mathcal{A}}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) = \mathcal{A}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) + \mathcal{A}_{exch}(\mathrm{t},\mathrm{s},\mathrm{u}) + \mathcal{A}_{exch}(\mathrm{u},\mathrm{t},\mathrm{s})$$

- Regge limit: $\mathrm{t}\to\infty$, s=fixed

$$\widehat{\mathcal{A}}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) \sim -\frac{ig^2}{\mathrm{s}}\ell^2\mathrm{t}\,I_0(\ell\sqrt{8\mathrm{t}}) \sim -\frac{ig^2}{\mathrm{s}}\,(\ell^2\mathrm{t})^{3/4}\,e^{\ell\sqrt{8\mathrm{t}}}$$

- Fixed angle limit: $\mathrm{s},\mathrm{t},\mathrm{u}\to\infty$, $\frac{\mathrm{t}}{\mathrm{s}} = -\sin^2\frac{\theta}{2}$, $\frac{\mathrm{u}}{\mathrm{s}} = -\cos^2\frac{\theta}{2}$

$$\widehat{\mathcal{A}}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) \sim ig^2|\mathrm{s}|^{3/4}\,e^{\ell\sqrt{|\mathrm{s}|}\,f(\theta)} \to \infty\,, \qquad\qquad f(\theta) > 0$$

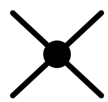
• cf. string theory: Shapiro-Virasoro amplitude is UV-soft

$$A_4 = g^2 \frac{\Gamma(-1-\frac{1}{4}\alpha'\mathrm{s})\Gamma(-1-\frac{1}{4}\alpha'\mathrm{s})\Gamma(-1-\frac{1}{4}\alpha'\mathrm{s})}{\Gamma(2+\frac{1}{4}\alpha'\mathrm{s})\Gamma(2+\frac{1}{4}\alpha'\mathrm{s})\Gamma(2+\frac{1}{4}\alpha'\mathrm{s})}$$

$$A_4 \to \quad g^2|\mathrm{s}|^{-6}(\sin\theta)^{-6}e^{-\alpha'|\mathrm{s}|\,h(\theta)} \to 0$$

$$h(\theta) = -\tfrac{1}{4}\Big(\sin^2\tfrac{\theta}{2}\log\sin^2\tfrac{\theta}{2} - \cos^2\tfrac{\theta}{2}\log\cos^2\tfrac{\theta}{2}\Big) > 0$$

4-scalar vertex contribution ?



• expected to be effectively “non-local” – infinite series in ∂^n:
may “soften” large p behaviour of exchange contribution
• guess 4-scalar vertex in flat-space HS action from
its form in AdS action reconstructed using AdS/CFT

[Bekaert, Erdmenger, Ponomarev, Sleight 2015]: $\nabla \to \partial$

$$S^{(4)}[\phi_0] = g^2 \int d^4x \Big[\sum_{s=0}^{\infty} f_{2s}(\Delta_{x_{34}}) \left(\partial_{x_{12}} \cdot \partial_{x_{34}}\right)^{2s} \times \phi_0(x_1)\phi_0(x_2)\phi_0(x_3)\phi_0(x_4)\Big]_{x_i=x}$$

$$\Delta_{x_{34}} \equiv (\partial_{x_3} + \partial_{x_4})^2, \qquad \partial_{x_{12}} \equiv \partial_{x_1} - \partial_{x_2}$$

$f_{2s}(z)$ = infinite series in z, regular at $z = 0$: no poles

$$z \to \infty : \qquad f_{2s}(z) \to c_{2s} \frac{\ell^{4s-2}}{z}\,, \qquad c_{2s} = \frac{1}{[(2s-1)!]^2}$$

then asymptotic contribution to 4-scalar amplitude is

$$\sum_{s=0}^{\infty} f_{2s}(\mathrm{s})\,(\mathrm{t}-\mathrm{u})^{2s} = \frac{2\mathrm{t}+\mathrm{s}}{2\mathrm{s}} \Big[I_0\big(2\ell\sqrt{2\mathrm{t}+\mathrm{s}}\big) - J_0\big(2\ell\sqrt{2\mathrm{t}+\mathrm{s}}\big)\Big]$$

UV as in exchange amplitude: possible cancellation?

- simplest self-energy 1-loop diagram is exp UV divergent

but may be made finite once 4-vertex tadole contribution is added?

0-0-0-s tree-level scattering amplitude

gauge-invariance constraints on higher-spin vertices:
impose linearized gauge invariance on on-shell amplitude
more efficient than off-shell Lagrangian approach

Conditions:

- linearized gauge invariance $\delta\phi_{m_1\ldots m_s} \sim \partial_{(m_1}\epsilon_{m_2\ldots m_s)}$
of full amplitude $\mathcal{A}_4 = \mathcal{A}_{\rm exch} + \mathcal{A}_{cont}$
- locality of 4-point vertex V_{000s} (no $1/p^2$ poles)

Strategy:

- solve non-trivial ("inhomogeneous") gauge-inv cond
- add solution of "homogeneous" eq.: invariant 4-vertex
- choose minimal solution consistent with locality of 4-vertex

Example: scalar electrodynamics

$L = \partial^m \phi^* \partial_m \phi + i A^m (\phi^* \partial_m \phi - \phi \partial_m \phi^*) + A^m A_m \phi^* \phi$

$\delta A_m = \partial_m \epsilon, \;\; \delta \phi = i \phi \epsilon$

$A(1)\phi(2)\phi(3)A(4)$ scattering amplitude:

$A_m \to \zeta_m(p) e^{ip\cdot x}, \;\; p \cdot \zeta = 0$

$$\mathcal{A}_{\rm exch} = \frac{1}{p_{12}^2} \zeta_1 \cdot p_2 \, \zeta_4 \cdot p_3 + \frac{1}{p_{13}^2} \zeta_1 \cdot p_3 \, \zeta_4 \cdot p_2$$

- gauge transformation in leg 1: $\delta \zeta_1 = p_1 \epsilon_1, \;\; \delta \phi = 0$

$\delta \mathcal{A}_{\rm exch} = (\zeta_4 \cdot p_3 + \zeta_4 \cdot p_2)\epsilon_1 = -\zeta_4 \cdot p_1 \, \epsilon_1$

- can be cancelled by adding contact $A^m A_m \phi^* \phi$ vertex

$\mathcal{A}_{\rm cont} = \zeta_1 \cdot \zeta_4 \to \;\; \delta \mathcal{A}_{\rm cont} = p_1 \cdot \zeta_4 \, \epsilon_1$

- thus 4-point vertex can be found from condition

of linearized gauge invariance of on-shell amplitude

0-0-0-s exchange amplitude: [Roiban, AT]

0-0-s' and 0-s'-s vertices in de Donder gauge: $\phi_s \to \zeta_s(p)\, e^{ip\cdot x}$

$\zeta_s(p,q^s) \equiv \zeta_{m_1\dots m_s}(p)\; q^{m_1}\dots q^{m_s}, \quad p_{ij}=p_i\cdot p_j, \quad p_i^2=0$

s-channel:

$$\mathcal{A}_{\rm exch} = -\frac{ig^2}{p_{12}^2}\sum_{s'} \frac{\ell^{2s'+s-2}}{(s'-1)!(s+s'-1)!}(p_{12}^2)^{s'} T_{s'}\big(\tfrac{p_{13}^2-p_{23}^2}{p_{12}^2}\big)\, \zeta_s(p_4,p_3^s)$$

$$T_s(z) = \tfrac{1}{2}\Big[(z+\sqrt{z^2-1})^s + (z-\sqrt{z^2-1})^s\Big]$$

$$\mathcal{A}_{\rm exch} = -\frac{2ig^2}{p_{12}^2}\Big[F_s(z_+)+F_s(z_-)\Big]\,\zeta_s(p_4,p_3^s)$$

$$F_s(z) = z^{2-s}\Big[I_s(z)-J_s(z)\Big]\,, \qquad z_\pm = \ell(\sqrt{p_{13}^2}\pm\sqrt{p_{12}^2+p_{13}^2})$$

add t and u channels, apply $\delta\zeta_{m_1\dots m_s}(p) = p_{(m_1}\epsilon_{m_2\dots m_s)}$

$1/p^2$ poles go away in the variation

$$\delta\mathcal{A}_{\text{exch}} = -2sg^2\Big[F_s(z_+) + F_s(z_-)\Big]\;\epsilon_{s-1}(p_4, p_3^{s-1}) + \ldots$$

cancel this against variation of contribution of 0-0-0-s vertex

$$\textstyle\sum_{k=0}^{s/2} V_{sk}(p_1,p_2,p_3)\;\phi_0(p_1)(p_2{\cdot}\partial_u)^k\phi_0(p_2)(p_3{\cdot}\partial_u)^{s-k}\phi_0(p_3)\;\zeta_s(p_4,u)$$

$$\delta\mathcal{A}_{\text{cont}} = sV_{s0}(p_1,p_2,p_3)\;p_{24}^2\,\zeta_{s-1}(p_4,p_2^{s-1}) + \ldots$$

gauge-invariance: relation of V_{sk} to Bessel functions in $\mathcal{A}_{\text{exch}}$

local solution for 4-vertex exists for $s=2$ and $s=4$

- $s = 2$:

$$V_{20} = \frac{g^2}{p_{12}^2}\Big[F_2(z_+) + F_2(z_-)$$
$$-\tfrac{1}{2}\left[p_{13}^2 R_2(p_{13}^2) + p_{23}^2 R_2(p_{23}^2) + p_{12}^2 R_2(p_{12}^2)\right]\Big]$$

$R_s(x) \equiv \frac{1}{2x}\left[I_s(\sqrt{-x}) - J_s(\sqrt{-x})\right]$
is $x \to 0$ residue of $F_2(x)$

V_{20} is regular in $p_{12}^2 \to 0$ limit
complete 0-0-0-2 amplitude: simpler than exchange one

$$\mathcal{A} = g^2\left[p_{13}^2 R_2(p_{13}^2) + p_{23}^2 R_2(p_{23}^2) + p_{12}^2 R_2(p_{12}^2)\right]$$
$$\times\Big(\frac{\zeta_2(p_4, p_3^2)}{p_{12}^2} + \frac{\zeta_2(p_4, p_2^2)}{p_{13}^2} + \frac{\zeta_2(p_4, p_1^2)}{p_{23}^2}\Big)$$

- $s = 4$:

4-vertex in terms of $R_4 \sim$ Bessels, regular at small p

complete 0-0-0-4 amplitude:

$$\mathcal{A} = U(p_1, p_2, p_3)\,\zeta_4\big(p_4, (p_{12}^2 p_2 - p_{13}^2 p_3)^4\big) - \frac{i p_{12}^2}{15 p_{13}^2}\,\zeta_4(p_4, p_2^4) + \ldots$$

$$U = \left(\frac{1}{p_{13}^2} + \frac{1}{p_{23}^2}\right) R_4(p_{12}^2) + \text{ cycle}$$

- $s > 4$: no local solution appears to exist [also: Taronna 11]
- related obstruction from Weinberg's soft theorem

starting with 5-point function

- similar conclusions from BCFW constructibility

[Benincasa, Cachazo; Benincasa, Conde; Dempster, Tsulaia]

- relax locality assumption?!

Conclusions / Open questions

- beginning to learn how to do quantum computations in theories with infinite number of massless higher spin fields: importance of defn of quantum theory consistent with symm
- remarkable simplifications due to large HS symmetry: 1-loop $Z = 1$, zero effective number of d.o.f.
- conformal HS theory:

– vanishing of one-loop conformal anomalies in bosonic theory

– vanishing of scattering amplitudes with HS exchange
  as required by conformal higher spin symmetry

- explore possibility of interacting HS theory in flat space: one motivation: simplified version of HS theory in AdS
  loop corrections should be simple or vanish?
- problems with gauge invariance starting with 4-vertex:
  relax locality assumption?