Soft Dilatons

Universality in the soft behavior of dilatons in string and field theories

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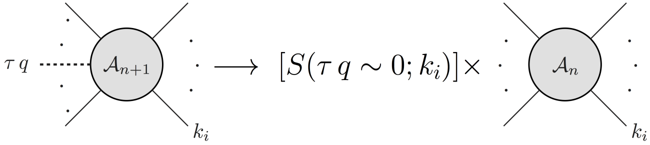
Broken Symmetries in Observables

Soft emission theorems: universal low-energy properties of amplitudes



Factorization only symmetry-dependent (universal)

$$S(\tau\, q; k_i) = \tau^{-1}\, S_{\mathrm{L}} + \tau^{0}\, S_{\mathrm{sL}} + \tau^{1}\, S_{\mathrm{ssL}} + \cdots + \mathcal{O}(\tau^{p})$$

Famous Examples:

‘58	Low’s soft photon theorem - gauge invariance	$S_{\mathrm{L}},\ S_{\mathrm{sL}}^{\mathrm{tree}}$
‘64	Weinberg’s soft graviton theorem - gauge invariance	$S_{\mathrm{L}},\ S_{\mathrm{sL}}^{\mathrm{tree}},\ S_{\mathrm{ssL}}^{\mathrm{tree}}$
‘65	Adler’s pion zero condition - shift symmetry	$S_{\mathrm{L}},\ S_{\mathrm{sL}}$
‘66	Double-soft pion theorem - coset symmetry	$S_{\mathrm{L}},\ S_{\mathrm{sL}}$

Recent developments (*) categorized reference list in [arxiv:1604.03355, Di Vecchia, Marotta, M.]

- Renewal of interest following work by Strominger et al. '13, '14, '15
  Gauge theory soft theorems as Ward identities of asymptotic symmetries in GR
- New graviton ssL soft theorem (tree-level) - '14
  Spinor-helicity & BCFW: [Cachazo, Strominger]
  CHY formalism: [Afkhami-Jeddi], [Schwab, Volovich]
  Gauge invariance: [Broedel, De Leeuw, Plefka, Rosso], [Bern, Davies, Di Vecchia, Nohle]
- New (double) soft theorems in gauge, string, supersymmetric and effective field theories* (hidden symmetries?)

$* * *$

- New uses of soft theorems: (see Y. Wang, C. Wen, and C. Cheung's talks tomorrow)
  Effective field theories from soft theorems [Weinberg '67]
  [Cheung, Kampf, Novotny, Trnka '14], [Low '14], [Huang, Wen '15], [Bianchi, Guerrieri, Huang, Lee, Wen '16]

  Soft BCFW construction [Cheung, Kampf, Novotny, Shen, Trnka '15], [Luo, Wen '15]

  Extended theories from soft limits in CHY [Cachazo, Cha, Mizera '16]

$$A_n^{\text{theory}_1} \xrightarrow{\text{soft limit}} \tau^p \, A_{n-1}^{\text{theory}_1 \oplus \text{theory}_2} + \mathcal{O}(\tau^{p+1}), \quad p > 0.$$

This Talk: Soft behavior of dilatons in string and field theory

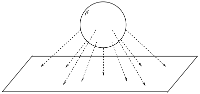
- Soft emission of gravitons and dilatons in string and field theory
  Work with Paolo Di Vecchia and Raffaele Marotta:
  I. **JHEP 1505** [arXiv:1502.05258], II. **JHEP 1606** [arXiv:1604.03355], III. [arXiv:16xx.xxxxx]

  Two surprises in new dilaton soft theorems

- Soft emission of a dilaton in conformal field theories (and in YM^2)
  Work with Paolo Di Vecchia, Raffaele Marotta and Josh Nohle:
  IV. **Phys.Rev. D93** [arXiv:1512.03316]

  New (exact) theorem in spontaneously broken CFTs

String Amplitudes and their soft limits



Tree-level string amplitude (cf. talk by M. B. Green)

$$\mathcal{M}_n(k_1,\ldots,k_n) \sim \int \frac{\prod_{i=1}^n dz_i}{d\Omega_M} \langle V_1(z_1,k_1)\cdots V_n(z_n,k_n)\rangle[\otimes c.c.]_{\text{closed}}$$

Suitable for studying soft limits

$$\mathcal{M}_{n+1}(q,k_1,\ldots,k_n) \sim \underbrace{\int \frac{\prod_{i=1}^n dz_i}{d\Omega_M} \langle V_1(z_1,k_1)\cdots V_n(z_n,k_n)\rangle}_{\mathcal{M}_n(k_1,\ldots,k_n)} \underbrace{\int dz \prod_{j=1}^n \langle V_q(z,q) V_j(z_j,k_j)\rangle}_{S_q(q,\{k_i,z_i\})}$$

Existence of a soft theorem for $q \ll k_i$ implies

$$\mathcal{M}_n * S_q(q,\{k_i,z_i\}) = \hat{S}(q,k_i)\mathcal{M}_n + \mathcal{O}(q^p)$$

Straightforward extension to multi-soft expansions

For double-soft gluons and scalars, see [1507.00938, Di Vecchia, Marotta, M.]

A Soft Massless Closed String State graviton, dilaton, NS-NS B-field (Kalb-Ramond)

Polarization stripped amplitude contains triple information

$$\mathcal{M}_{n+1}(q,k_i) = \epsilon_{q,\mu}\bar{\epsilon}_{q,\nu}\mathcal{M}^{\mu\nu}_{n+1}(q,k_i)$$

Simplest case: Bosonic string scattering on n closed tachyons

$$\mathcal{M}^{\mu\nu}_{n+1}(q,k_i) \sim \underbrace{\int \frac{\prod_i d^2 z_i}{d\Omega_{\text{M}}} \prod_{i<j}^{n} |z_i - z_j|^{\alpha' k_i \cdot k_j}}_{\mathcal{M}_n(k_1,\ldots,k_n)} \underbrace{\int d^2 z \prod_{l=1}^{n} |z - z_l|^{\alpha' q \cdot k_l} \sum_{i,j=1}^{n} \frac{k_i^\mu k_j^\nu}{(z-z_i)(\bar{z}-\bar{z}_j)}}_{S_q^{\mu\nu}(q,\{k_i,z_i\}) \equiv \sum_{i,j} k_i^\mu k_j^\nu \, \mathcal{I}_i^j}$$

Soft-expansion of $\mathcal{I}_i^j$ (master-integral)

$$\mathcal{I}_i^i \sim \frac{2}{\alpha' q\cdot k_i}\left(1 + \alpha' \sum_{j\neq i}(k_j q)\log|z_i - z_j| + \frac{(\alpha')^2}{2}\sum_{j\neq i}\sum_{k\neq i}(k_j q)(k_k q)\log|z_i - z_j|\log|z_i - z_k|\right)$$
$$+ (\alpha')^2 \sum_{j\neq i}(k_j q)\log^2|z_i - z_j| + \log\Lambda^2 + \mathcal{O}(q^2)$$

$$\mathcal{I}_i^j \sim \sum_{m\neq i,j}\frac{\alpha' q\cdot k_m}{2}\left(\text{Li}_2\left(\frac{\bar{z}_i - \bar{z}_m}{\bar{z}_i - \bar{z}_j}\right) - \text{Li}_2\left(\frac{z_i - z_m}{z_i - z_j}\right) - 2\log\frac{\bar{z}_m - \bar{z}_j}{\bar{z}_i - \bar{z}_j}\log\frac{|z_i - z_j|}{|z_i - z_m|}\right)$$
$$- \log|z_i - z_j|^2 + \log\Lambda^2 + \mathcal{O}(q^2)$$

For $\mathcal{I}_i^j + \mathcal{I}_j^i$: Dilogs vanish. For $\mathcal{I}_i^j - \mathcal{I}_j^i$: Bloch-Wigner Dilog.

Soft graviton or dilaton emission from n-tachyon interaction

$$\mathcal{M}_{n+1}^{(\mu\nu)} = \mathcal{M}_n * S_q^{(\mu\nu)}(q, \{k_i, z_i\}) \stackrel{?}{=} \hat{S}^{(\mu\nu)}(q, k_i)\mathcal{M}_n + \mathcal{O}(q^2)$$

$?$ = Yes: (I, IV)

$$\hat{S}^{(\mu\nu)}(q, k_i) = \sum_{i=1}^{n} \left[\frac{k_i^\mu k_i^\nu}{q \cdot k_i} - i \frac{k_i^\mu q_\rho J_i^{\nu\rho}}{q \cdot k_i} - \frac{1}{2} \frac{q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{q \cdot k_i} + \frac{q_\alpha q_\beta}{q \cdot k_i} \{\eta^{\mu\nu}\}_{\rho\sigma}^{\alpha\beta} k_i^\rho \frac{\partial}{\partial k_{i\sigma}} \right]$$

No α'-term!

In the field theory limit $\alpha' \to 0$, $\mathcal{M}_n$ becomes the amplitude of n massive ϕ^3 scalars [Scherk '71].

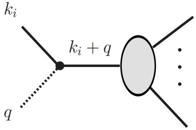
Soft dilaton ($k_i^2 = -m_i^2 = \frac{4}{\alpha'}$): [Ademollo, D'Adda, D'Auria, Gliozzi, Napolitano, Sciuto, Di Vecchia '75]

$$\epsilon_d^{\mu\nu} \hat{S}_{\mu\nu} = 2 - \sum_{i=1}^{n} \mathcal{D}_i + \frac{q_\rho}{2} \sum_{i=1}^{n} \mathcal{K}_i^\rho - \sum_{i=1}^{n} \frac{m_i^2}{q \cdot k_i} \left(1 + q^\rho \frac{\partial}{\partial k_i^\rho} + \frac{1}{2} q^\rho q^\sigma \frac{\partial^2}{\partial k_i^\rho \partial k_i^\sigma} \right)$$

Non-singular terms are conformal transformations:

$$\mathcal{D}_i = k_i^\rho \frac{\partial}{\partial k_i^\rho} \quad \text{and} \quad \mathcal{K}_i^\rho = k_i^\rho \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} - 2k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_{i\rho}} + i\mathcal{S}_i^{\mu\rho} \frac{\partial}{\partial k_i^\mu}$$

Singular mass-terms:



One soft + n hard massless closed strings

Soft part of $\mathcal{M}_{n+1}^{\mu\nu} = \mathcal{M}_n * S^{\mu\nu}$

$$S^{\mu\nu}(q,\{k_i,z_i\}) = \int d^2z \sum_{i=1}^{n}\left(\frac{\theta_i\epsilon_i^{\mu}}{(z-z_i)^2} + \sqrt{\frac{\alpha'}{2}}\frac{k_i^{\mu}}{z-z_i}\right)\sum_{j=1}^{n}\left(\frac{\bar{\theta}_j\bar{\epsilon}_j^{\nu}}{(\bar{z}-\bar{z}_j)^2} + \sqrt{\frac{\alpha'}{2}}\frac{k_i^{\nu}}{\bar{z}-\bar{z}_i}\right)$$
$$\times \exp\left[-\sqrt{\frac{\alpha'}{2}}\sum_{i=1}^{n}\frac{\theta_i\epsilon_i\cdot q}{z-z_i}\right]\exp\left[-\sqrt{\frac{\alpha'}{2}}\sum_{i=1}^{n}\frac{\bar{\theta}_i\bar{\epsilon}_i\cdot q}{\bar{z}-\bar{z}_i}\right]\prod_{i=1}^{n}|z-z_i|^{\alpha' q\cdot k_i}$$

All integrals involved (after partial expansion in q)

$$\mathcal{I}_{i_1 i_2\cdots}^{j_1 j_2\cdots} = \int d^2z \frac{\prod_{l=1}^{n}|z-z_l|^{\alpha' q\cdot k_l}}{(z-z_{i_1})(z-z_{i_2})\cdots(\bar{z}-\bar{z}_{j_1})(\bar{z}-\bar{z}_{j_2})\cdots}$$

All can be related to the master integral by simple tricks as:

$$\mathcal{I}_{ii}^{j} = \frac{1}{1-\frac{\alpha' q\cdot k_i}{2}}\frac{\partial}{\partial z_i}\mathcal{I}_i^j \quad \text{and} \quad \mathcal{I}_i^{ij} = \frac{\mathcal{I}_i^i - \mathcal{I}_i^j}{\bar{z}_i - \bar{z}_j}$$

Is the soft theorem universal? (I,II)

$$\mathcal{M}_{n+1}^{(\mu\nu)}(q,k_i) = \mathcal{M}_n * S^{(\mu\nu)} = \left(\hat{S}_{\text{tachyon}}^{(\mu\nu)} + \hat{S}_{\text{spin}}^{(\mu\nu)}\right)\mathcal{M}_n + \mathcal{O}(q^2)$$

$$\hat{S}_{\text{spin}}^{(\mu\nu)} = \sum_{i=1}^{n}\left[\frac{q_\alpha q_\beta}{q\cdot k_i}\{\eta^{\mu\nu}\}_{\rho\sigma}^{\alpha\beta} + \alpha'\left(\frac{k_i^{\mu}k_i^{\nu}}{2}\frac{q_\rho q_\sigma}{q\cdot k_i} + q_\alpha\{K_i^{\mu\nu}\}_{\rho\sigma}^{\alpha}\right)\right]\left(\epsilon_i^{\rho}\frac{\partial}{\partial\epsilon_{i\sigma}} + \bar{\epsilon}_i^{\rho}\frac{\partial}{\partial\bar{\epsilon}_{i\sigma}}\right)$$

Antisymmetric part: Soft theorem through $\mathcal{O}(q^0)$ - problem of dilogs at $\mathcal{O}(q)$ ...

The Superstring Story and the Question of Universality

Soft part of integrand separates (III)

$$S^{\mu\nu}(q, \{k_i, z_i\}) = S^{\mu\nu}_{\text{bosonic}}(q, \{k_i, z_i\}) + S^{\mu\nu}_{\text{susy}}(q, \{k_i, z_i\})$$

No new integrals appear in S_{susy} through $\mathcal{O}(q^2)$!

Soft theorem

$$\mathcal{M}^{\mu\nu}_{n+1} = \left(\hat{S}^{\mu\nu}_{\text{bosonic}}\big|_{\alpha'=0}\right)\mathcal{M}_n + \mathcal{O}(q^2)$$

Soft graviton + n massless closed strings in the bosonic/super-string

$$\epsilon^g_{\mu\nu}\,\hat{S}^{\mu\nu}_{\text{bosonic}} = \hat{S}^{\text{graviton}}_{\text{field theory}} + \alpha'\epsilon^g_{\mu\nu}\sum_{i=1}^{n}\left[\frac{k_i^\mu k_i^\nu}{2}\frac{q_\rho q_\sigma}{q\cdot k_i} + q_\alpha\left\{K_i^{\mu\nu}\right\}^{\alpha}_{\rho\sigma}\right]\left(\epsilon^\rho_i\frac{\partial}{\partial\epsilon_{i\sigma}} + \bar{\epsilon}^\rho_i\frac{\partial}{\partial\bar{\epsilon}_{i\sigma}}\right)$$

Soft behavior in field theory = superstring/sugra $\neq$ bosonic string.

Soft dilaton + n massless closed strings in the bosonic/super-string

$$\epsilon^d_{\mu\nu}\,\hat{S}^{\mu\nu}_{\text{bosonic}} = 2 - \sum_{i=1}^{n}\mathcal{D}_i + \frac{q_\rho}{2}\sum_{i=1}^{n}\left[\mathcal{K}^\rho_i + \frac{q^\sigma}{q\cdot k_i}\left(\mathcal{S}_{i,\rho\mu}\eta^{\mu\nu}\mathcal{S}_{i\nu\sigma} + D\left(\epsilon^\rho_i\frac{\partial}{\partial\epsilon_{i\sigma}} + \bar{\epsilon}^\rho_i\frac{\partial}{\partial\bar{\epsilon}_{i\sigma}}\right)\right)\right]$$

α'-terms vanish!

divergent terms due to d-d-g and d-B-B interactions

Universal soft behavior in field theory = superstring/sugra = bosonic string

Three-point Amplitudes and Low-Energy Actions

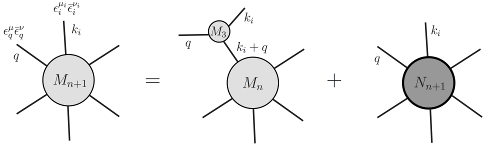
Bosonic, Heterotic and Type II Superstring low-energy actions ($\lambda_0 = \frac{1}{4}, \frac{1}{8}, 0$)
[Zwiebach, **PLB156**, '85], [Metsaev, Tseytlin, **NPB293**, '87]

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-G} \Big\{ R - G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} e^{-\frac{4}{\sqrt{D-2}}\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} + \alpha' \lambda_0 \, e^{-\frac{2}{\sqrt{D-2}}\phi} \left[R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 + \cdots \right] + \mathcal{O}(\alpha'^2) \Big\}$$

Amplitude of three massless closed bosonic or super strings ($\lambda_0 = \frac{1}{4}, 0$)

$$\mathcal{M}_3^{\mu_1\nu_1, \mu_2\nu_2, \mu_3\nu_3}(k_1, k_2, k_3) = 2\kappa_D \left(\eta^{\mu_1\mu_2} k_1^{\mu_3} + (\circlearrowleft 1, 2, 3) + 2\alpha' \lambda_0 k_1^{\mu_3} k_2^{\mu_1} k_3^{\mu_2} \right) \times \left(\mu_i \leftrightarrow \nu_i \right)$$

For the heterotic case take (bosonic)×(supersymmetric).



On-shell gauge invariance fixes soft theorems of the graviton and dilaton!

[Bern, Davies, Di Vecchia, Nohle '14], [Di Vecchia, Marotta, Nohle, M. '15], [Di Vecchia, Marotta, M. '16]

Remarks

- Studied soft behavior of $\mathcal{M}^{\mu\nu}_{n+1}$ generically in different string theories
- Found generic ssL soft theorem for $\mathcal{M}^{(\mu\nu)}_{n+1}$ (and sL soft theorem for $\mathcal{M}^{[\mu\nu]}_{n+1}$)
  α' appears at the ssL order - consequence of $\mathcal{M}_3$ and gauge invariance
- Gauss-Bonnet terms modify graviton soft theorem at ssL
  graviton soft theorem is different in bosonic/heterotic/superstring
- The dilaton soft theorem instead has two surprises:
  1. It is universal among all string and field theories!
  2. Contains the space-time generators of dilatation and special conformal transformation!
- Is there a symmetry reason behind the universality of the soft dilaton?

Dilaton as the Nambu-Goldstone boson of spontaneously broken conformal symmetry

- CFT with a Lorentz-scalar primary operator ξ with $\langle 0|\xi(0)|0\rangle \neq 0$
  $\mathcal{D}$ and $\mathcal{K}_\mu$ broken, $\mathcal{P}_\mu$ and $\mathcal{M}_{\mu\nu}$ (Poincaré) unbroken
- Only one Goldstone mode (dilaton) [Low, Manohar '01]

$$\langle 0|T_{\mu\nu}|\xi;q\rangle \sim q_\mu q_\nu \langle 0|\xi(0)|0\rangle \, , \quad \langle 0|T^\mu_\mu|\xi;q\rangle \sim q^2 = 0$$

- Field theory dilaton: Energy-momentum tensor and broken currents

$$\begin{aligned} T_{\mu\nu} &= -\frac{D-2}{2(D-1)}\langle\xi\rangle(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\xi(x) + \cdots \\ \partial_\mu j^\mu_{\mathcal{D}} &= T^\mu_\mu = v(-\partial^2\xi) \, , \qquad \partial_\mu j^\mu_{\mathcal{K},\rho} = 2x_\rho v(-\partial^2\xi) \end{aligned}$$

- Ward identity (WI) of $\langle j^\mu\phi\cdots\rangle_{n+1} \equiv T^*\langle 0|j^\mu(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$

$$-iq_\mu\langle\tilde{j}^\mu(q)\tilde{\phi}\cdots\rangle_{n+1} = \langle\partial_\mu\tilde{j}^\mu\tilde{\phi}\cdots\rangle_{n+1} + \sum_{i=1}^{n}\langle\cdots\delta\tilde{\phi}(k_i+q)\cdots\rangle_n$$

- Conformal field transformations of ϕ with scaling dimension d

$$\delta_{\mathcal{D}}\phi = [\mathcal{D},\phi] = i(d + x\cdot\partial)\phi \, , \quad \delta_\mu\phi = [\mathcal{K}_\mu,\phi] = i[(2x_\mu x_\nu + \eta_{\mu\nu}x^2)\partial^\nu + 2d\, x_\mu]\phi$$

Soft Theorem from Ward Identities

LSZ on WI of broken dilatation current [Callan '70, Boels&Wormsbecher '15], (IV)

$$vq^2 \langle \tilde{\xi}(q)\tilde{\phi}(k_1)\cdots\rangle_{n+1} = -i\sum_{i=1}^{n}\left(d - D - (k_i+q)\cdot\partial_{k_i}\right)\langle\cdots\tilde{\phi}(k_i+q)\cdots\rangle_n$$

$$\overset{\text{LSZ}}{\Longrightarrow}\quad v\mathcal{T}_{n+1}(q,k_i) = \left[D - nd - \sum_i k_i\cdot\partial_{k_i} - \sum_i \frac{m_i^2}{k_i\cdot q}\left(1 + q\cdot\partial_{k_i}\right)\right]\mathcal{T}_n(k_i) + \mathcal{O}(q)$$

LSZ on WI of broken $\mathcal{K}_\mu$ currents (IV)

$$-2v\partial_q^\mu\mathcal{T}_{n+1}(q,k_i) = \sum_{i=1}^{n}\left[\mathcal{K}_i^\mu - 2d\partial_{k_i}^\mu + \frac{m_i^2}{k_i\cdot q}\left(\frac{k_i^\mu}{k_i\cdot q} - \partial_{k_i}^\mu\right)\left(1 + q\cdot\partial_{k_i} + \frac{q_\nu q_\rho}{2}\partial_{k_i}^\nu\partial_{k_i}^\rho\right)\right]\mathcal{T}_n(k_i) + \mathcal{O}(q)$$

$$\Longrightarrow\quad v\mathcal{T}_{n+1}(q,k_i)|_{\mathcal{O}(q)} = \left[\frac{q_\mu}{2}\sum_i\left(\mathcal{K}_i^\mu - 2d\partial_{k_i}^\mu\right) - \sum_i\frac{m_i^2}{k_i\cdot q}\left(\frac{q_\nu q_\rho}{2}\partial_{k_i}^\nu\partial_{k_i}^\rho\right)\right]\mathcal{T}_n(k_i)$$

Soft behavior completely fixed through ssL order

$$v\mathcal{T}_{n+1}(q;k_i) = \sum_{i=1}^{n}\left[\frac{D-nd}{n} - \mathcal{D}_i + \frac{q_\mu}{2}\left(\mathcal{K}_i^\mu - 2d\,\partial_{k_i}^\mu\right) - \frac{m_i^2}{k_i\cdot q}\left(1 + q_\mu\partial_{k_i}^\mu + \frac{q_\nu q_\rho}{2}\partial_{k_i}^\nu\partial_{k_i}^\rho\right)\right]\mathcal{T}_n(k_i) + \mathcal{O}(q^2)$$

Similar relations derived for scale-invariant primordial fluctuations in cosmology [Maldacena '02, Kundu et al. '15]

Comparison with the 'gravity' dilaton

$$\kappa_D^{-1}\mathcal{M}_{n+1}(q;k_i) \sim \sum_{i=1}^{n}\left[\frac{2}{n} - \mathcal{D}_i + \frac{q_\mu}{2}\mathcal{K}_i^\mu - \frac{m_i^2}{k_i\cdot q}\left(1 + q_\mu\partial_{k_i}^\mu + \frac{q_\nu q_\rho}{2}\partial_{k_i}^\nu\partial_{k_i}^\rho\right)\right]\mathcal{M}_n(k_i) + \mathcal{O}(q^2)$$

Summary and Conclusions

- String theory offers a convenient framework to study soft behaviors of amplitudes
  Natural factorization at integrand level. Few moduli space master integrals.
- Graviton: Soft-theorem receives corrections from Gauss-Bonnet term.
  Examples are the Bosonic and Heterotic Strings.
- NS-NS B-field: Discovered sL soft theorem - needs further studies.
- Dilaton: Discovered **universal** ssL soft theorem, containing dilatation and **special conformal transformations**. What happens at loop-level?
- All a consequence of the same tree-level on-shell gauge invariance.
- Generalized form of factorization appearing at sLL order (not discussed here).
  Cf. $R\phi^2$ and $R^2\phi$ interactions [Bianchi, He, Huang, Wen '14], [Di Vecchia, Marotta, M. '16]
- Ward identities of $\mathcal{D}$ and $\mathcal{K}_\mu$ control soft behavior of the Nambu-Goldstone dilaton of a spontaneously broken CFT through ssL order:
  Similar, but not exactly equal to the gravity dilaton soft theorem - **why?**
- Side-story: ssL soft theorem for NG dilaton new and exact.
  Uses: dilaton effective action, Coulomb-branch of $\mathcal{N} = 4$ sYM, soft-BCFW relations, ...
  [see C. Wen's talks]
  Applications: cosmology (inflation) and particle phenomenology [Boels, Wormsbecher '15]