Threshold Spectroscopy

Anyons in Action

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$N = 0$ supersymmetry

U(1) gauge symmetry

$D = 2 + 1$

not (2,1)

$1 << N$ *underlying* particles

emergent “particle” degrees of freedom

Anyons

In 2+1 dimensions, quantum kinematics allows new possibilities for quantum statistics, besides bosons and fermions. There are several informative perspectives on this:

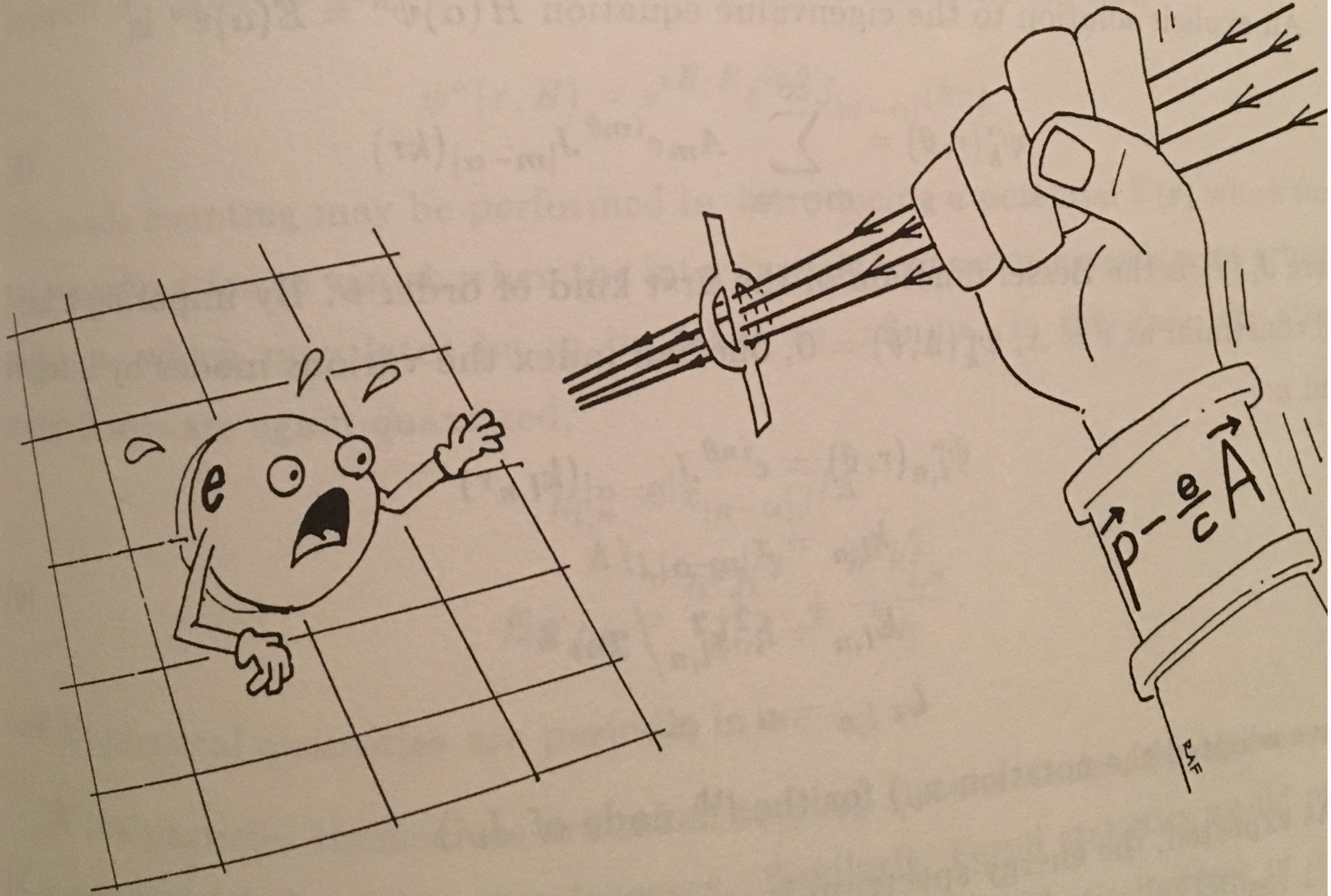
a. The topology of braids goes beyond permutations.

b. Abelian angular momentum allows fractional offset in the quantization condition.

c. Flux “tubes” are point-like, so one can have Aharonov-Bohm type effects for *particles* carrying effective charges and fluxes. This construction supports:

Abelian or non-abelian interactions

Mutual statistics for non-identical particles



osite.

QHE Complex

Anyons, but Not Much Action

Quasiparticles in many FQHE states are firmly predicted to embody these new possibilities.

Sadly, direct experimental tests have proved difficult. There is a simple and profound reason for this: electric charge is a daunting complication. Since the anyons are usually electrically charged:

they get pinned to impurities

they are highly unfree, in large background B field

they are subject to mutual long-range forces (1/r) that dominate the statistical interaction (1/r^2)

There may be ways to work around those difficulties:

in appropriate FQHE states, change quasiparticle statistics without changing their charge

wait for SFQHE, where the quasiparticles are often electrically neutral anyons

But here I will briefly discuss another, radically different possibility.

Spin Liquids

It is widely expected that several 2D materials should exhibit “spin liquid” phases.

The precise definition of “spin liquid” is somewhat vague.   Qualitative features include formation of a gap*, high degree of entanglement, and absence of a local order parameter.

Several candidate spin liquids include quasiparticles with unusual quantum statistics. That provides, in principle, precise characterizations and signatures for those universality classes.

A simple, exactly soluble model leading to a phase of this kind is the “toric code” model. It appears to be representative of a large universality class, plausibly realized already (see below).

In this universality class, there are two kinds of quasiparticles: “magnetic” M and “electric” E. Both, of course, are *electrically* neutral.

M and E, separately, are both bosons. But there is non-trivial mutual statistics. When M goes around E, there is a - sign. As a consequence, the relative orbital angular momentum is half an odd integer.

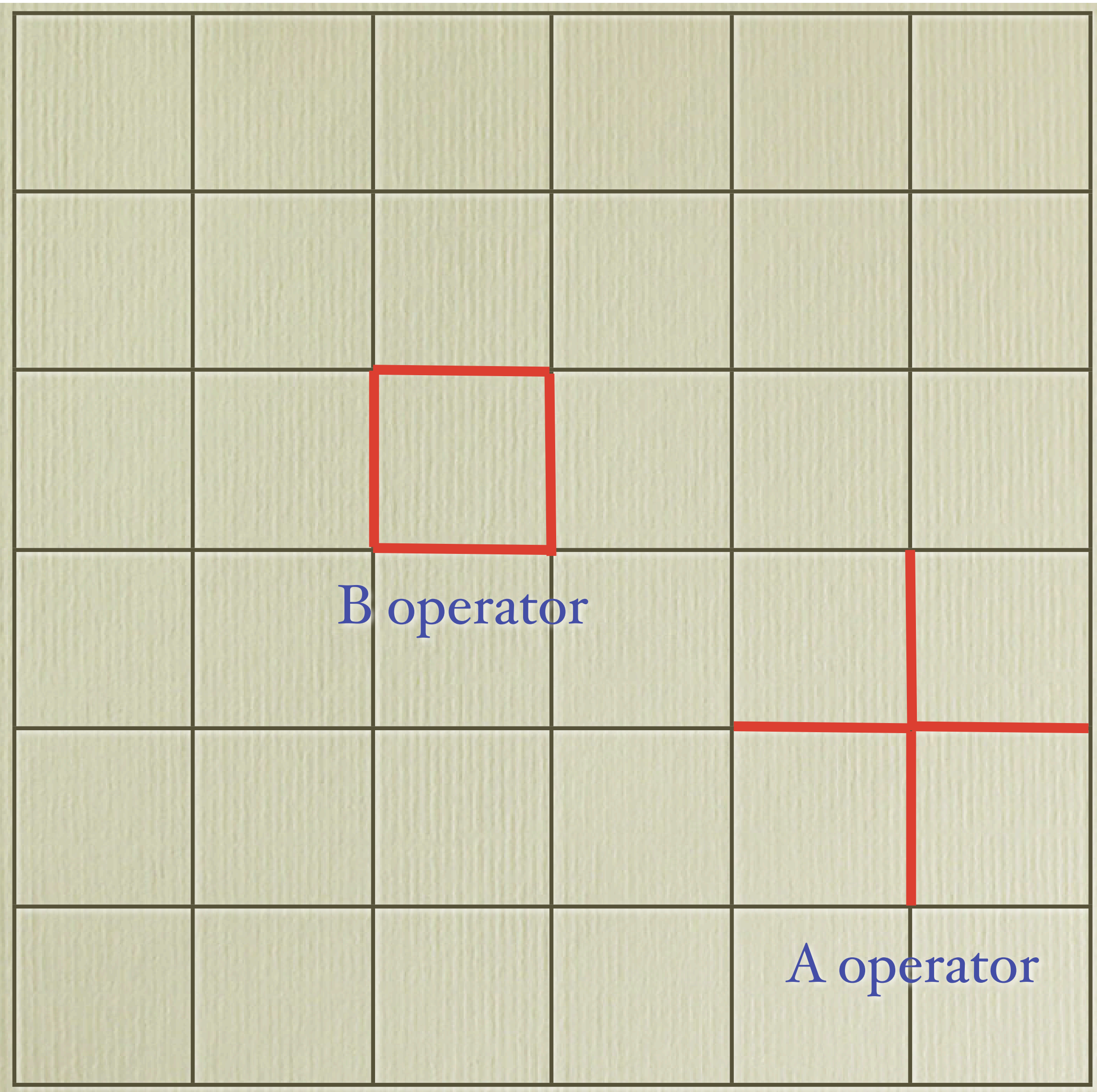
Here I’ve inserted a review of topic code basics:

Defining the Model

- Consider a kxk square lattice with spins 1/2 ("qubits") on each link - $2k^2$ qubits.
- A complete set of states can be labeled by the eigenvalues of the $\sigma^z{}_l$. If we use |1> for spin up, and |0> for spin down, they become binary arrays.
- Define operators A_s associated with sites and B_p associated with plaquettes:

$$A_j = \prod_{l \,\epsilon\, \mathrm{star}(j)} \sigma_l^x$$

$$B_k = \prod_{l \,\epsilon\, k} \sigma_l^z$$



Lattice with periodic boundary conditions (= torus)

- These operators all commute, because As always meet Bs at an even number of links. Their eignvalues are ±1.

- Define the "protected" states to be those left invariant (eigenvalue +1) by all the As and Bs. These define the allowed words of a simple code with remarkable properties, as we'll see.

- The protected states are also precisely the ground states of the **gapped** Hamiltonian

$$H = -\sum_{\text{sites } s} A_s - \sum_{\text{plaquettes } p} B_p$$

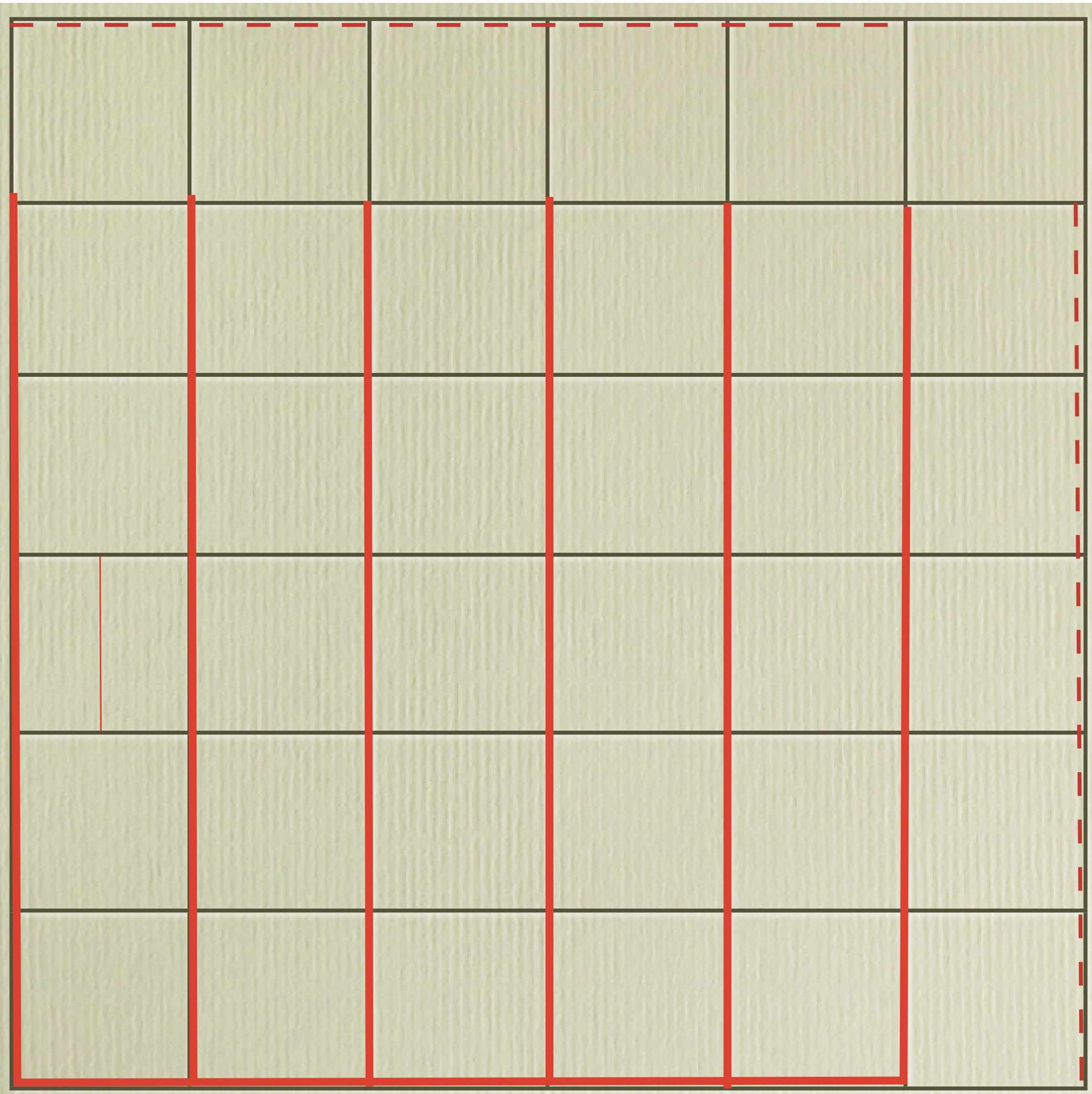
Finding the Protected States

- The B = 1 equations say that the number of down (or up) spins on the links of any plaquette must be even. That condition is of course obeyed by many spin configurations.
- The A = 1 equations will set the coefficients of many such configurations equal. To see how powerful those equations are, we use them to boil down the independent coefficients, by mapping to a few canonical spin configurations, whose coefficients will therefore determine the others.

- A maximal tree is a set of links that contains no loops, but that you can't add to without creating a loop.

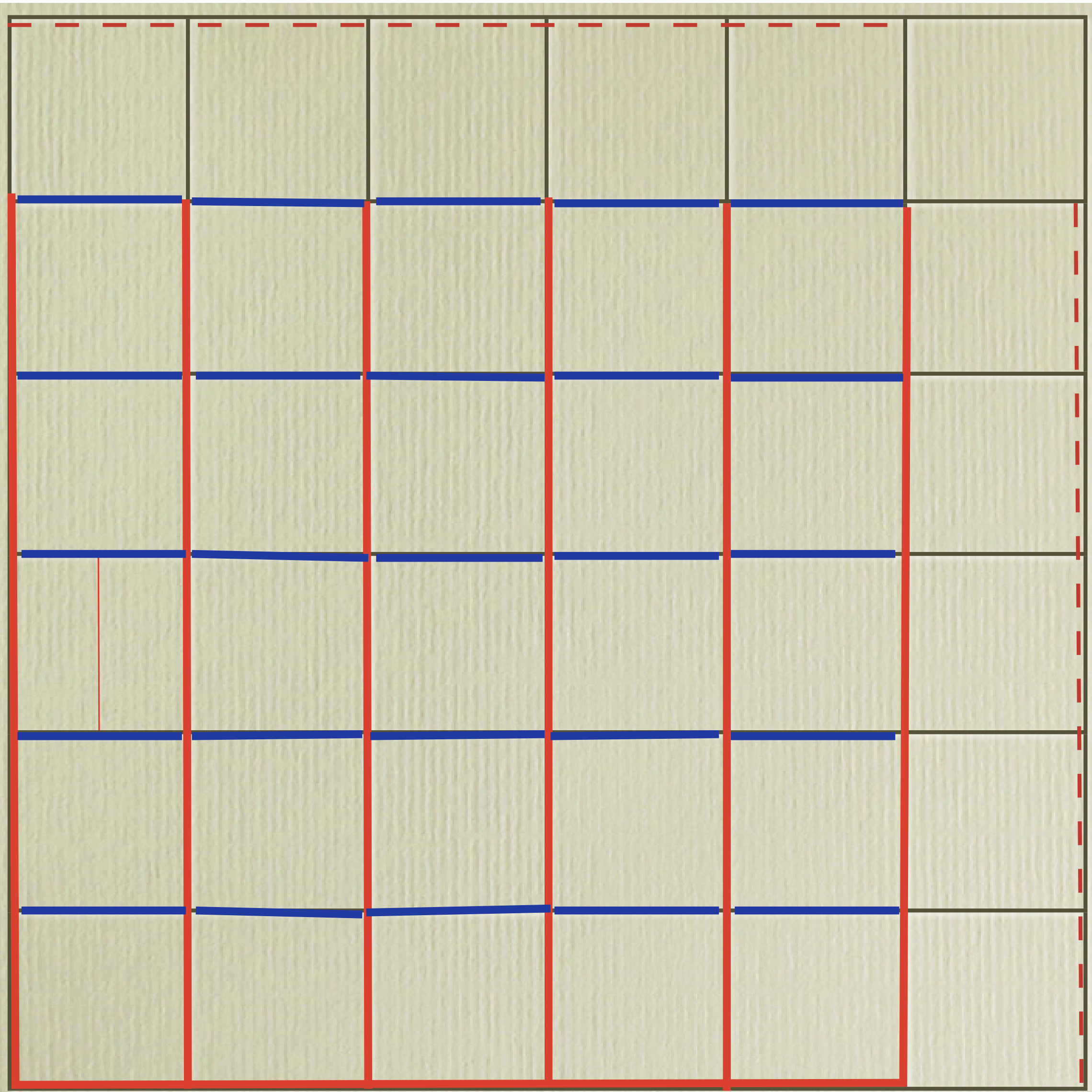
- By applying A operators to a state for which all the $B_k = 1$, you can set all the spins on the links of a maximal tree = |0> (i.e., down). This is similar to gauge fixing in a gauge theory.

- In the following maximal tree, you can run along the top row, then act with As "from the bottom up" one column at a time setting spins = |0>. None of these actions interferes with the previous ones.



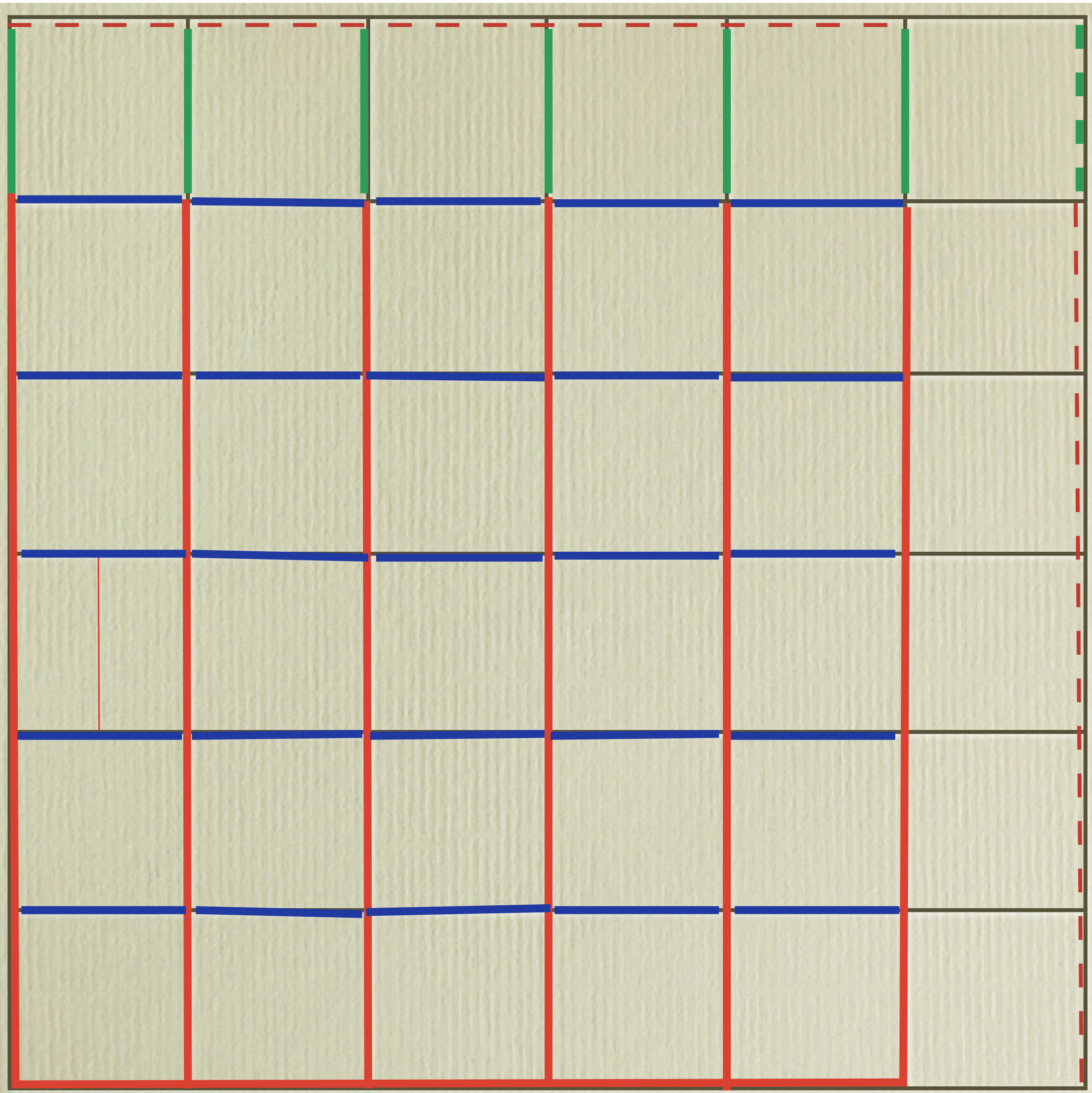
A maximal tree; dotted line is periodic reflection.

- The B = 1 condition then enforces many additional spins down:



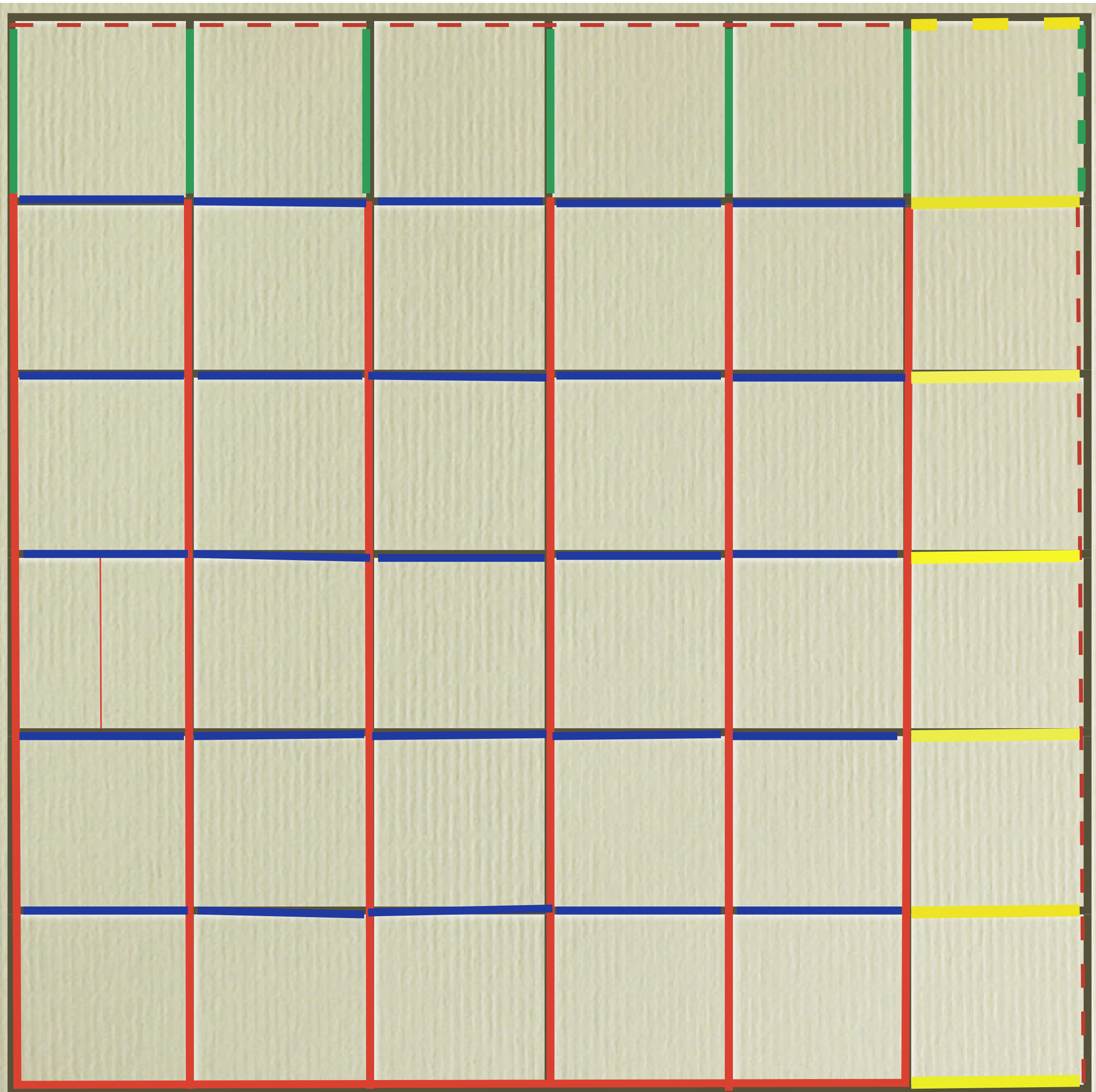
Spins down red links force (B=1) spins down on the blue links too.

- The value of the spin on the northwest link is enforced at other links, again through B = 1:



The green links can be up or down, but they must all be equal.

- Similarly, the value at the southeast link is enforced elsewhere ...



The yellow links can be up or down, but they must all be equal.

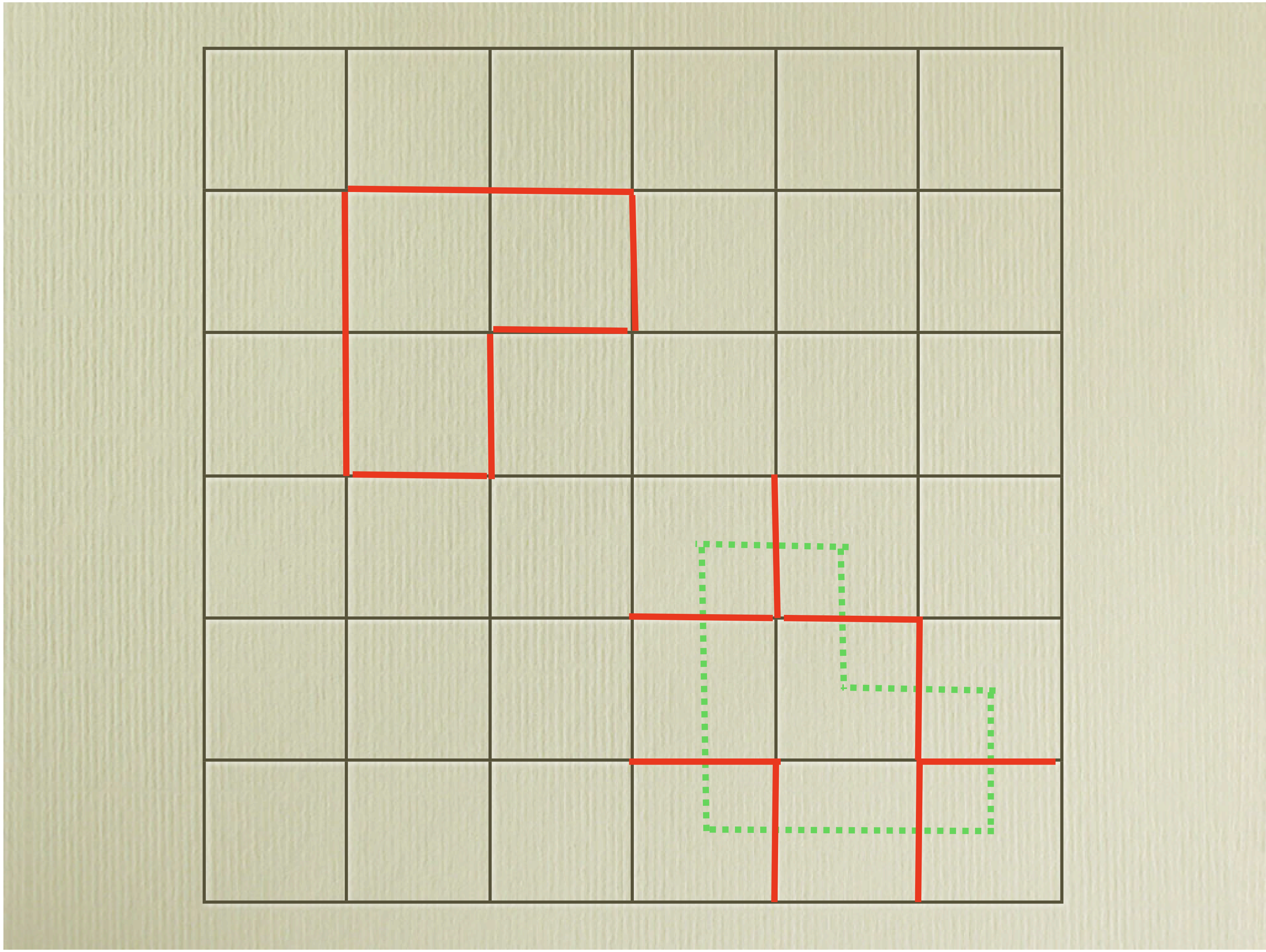
- The sum (mod 2) of spins along a vertical or horizontal loop cannot be altered by acting with As, so there is no further reduction.

- Thus there are four degenerate ground states. Each consists of a superposition of 1/4 of all possible solutions of the B = 0 equations, each taken with equal weight. The 4 different classes are characterized by the sum of spins along vertical and horizontal loops (mod 2, of course) = (0,0), (0,1), (1,0), or (1,1).

- A more profound view is based on finding the operators that commute with the As and Bs (a.k.a. the centralizer). There are two classes:

- Z-type operators, based on taking products of $\sigma^z{}_1$ around closed loops.

- X-type operators, based on taking products of $\sigma^x{}_1$ around links intersected by closed loops in the dual lattice. A picture is worth ~1000 words here:



- Z operators for contractible loops are products of Bs. (Just plaster the inside with plaquettes.)

- X operators for contractible loops are products of As. (Plaster the inside with plaquettes of the dual lattice, and use sites at the centers of those plaquettes.)

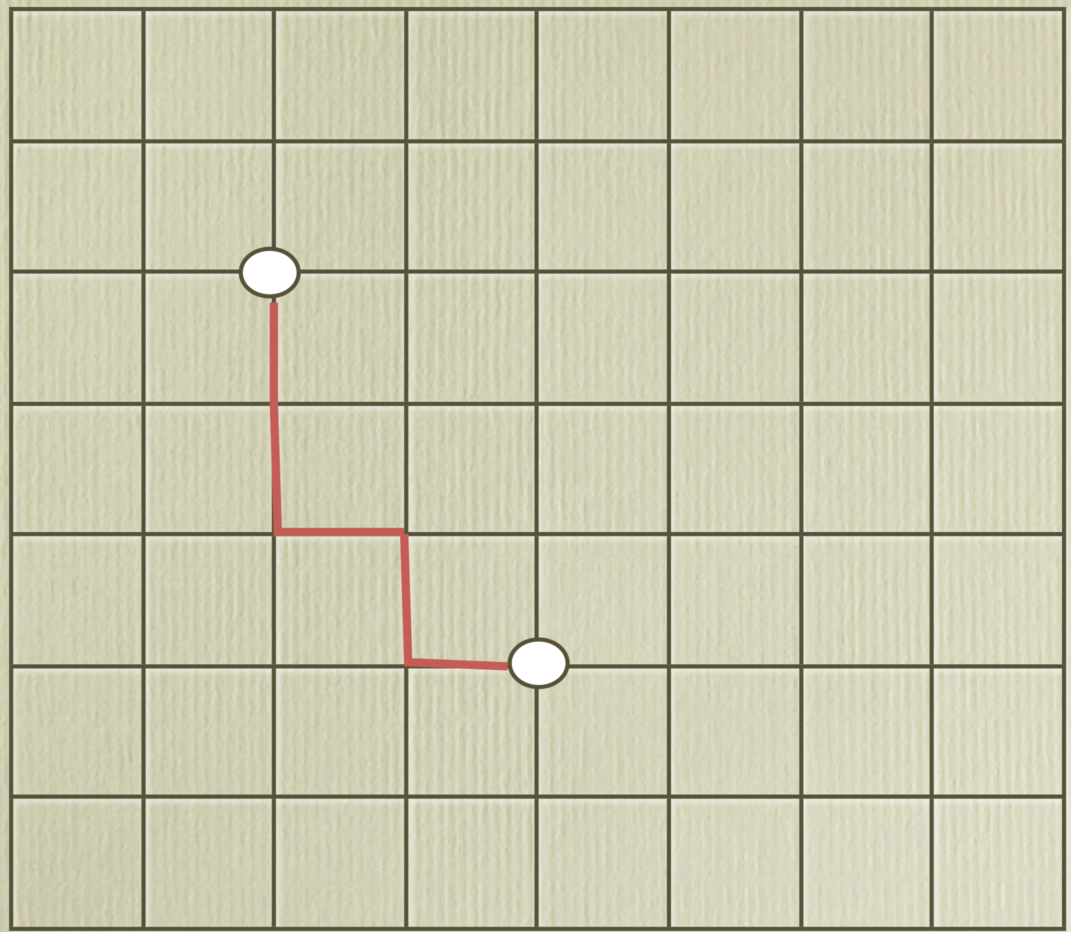
Electric and Magnetic Excitations

- Since the As and Bs commute with the Hamiltonian, and each other, we can diagonalize the Hamiltonian using their eigenstates.
- The excitations of our model Hamiltonian have energy measured by the number of stars and plaquettes they bring from 1 to -1.
- Due to the global constraints on the As and Bs, it is impossible to frustrate just one star or just one plaquette. The minimum is two.

- Electric pairs can be created by **open** Z-type operators.

- No plaquettes are excited. The excited stars occur for sites at the end of the string. We say electric particles are at these sites.

- Since B operators move the string around, the connecting string can be jiggled around without changing the state. Its topology does matter, however. (Strings that wrap around cycles change the state, as we've seen with Z_1 and Z_2.)

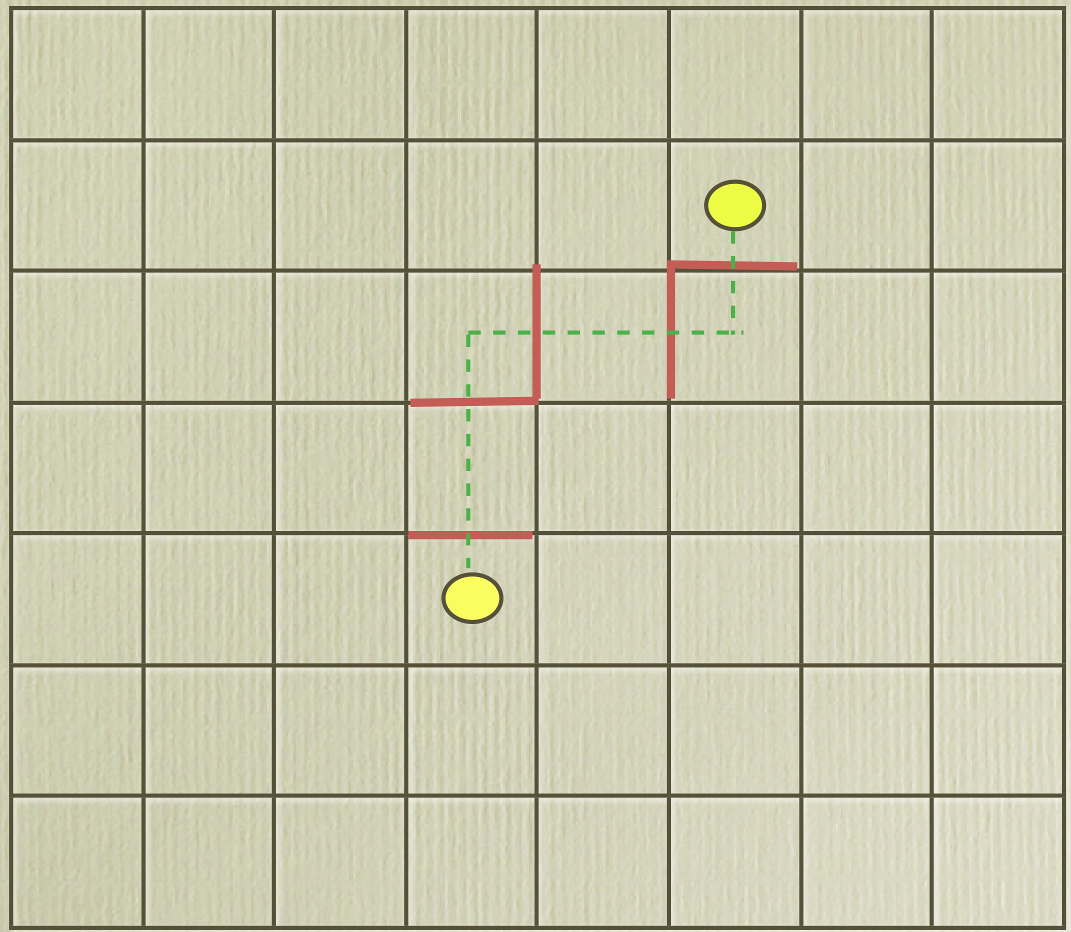


Electric pair with string

- Magnetic pairs can be created by **open** X-type operators.

- No stars are excited. The excited plaquettes occur around the (dual) sites at the end of the string. We say magnetic particles are at these sites.

- Since A operators move the string around, the connecting string can be jiggled around without changing the state. Its topology does matter, however. (Strings that wrap around cycles change the state, as we've seen with X_1 and X_2.)



Magnetic pair with string

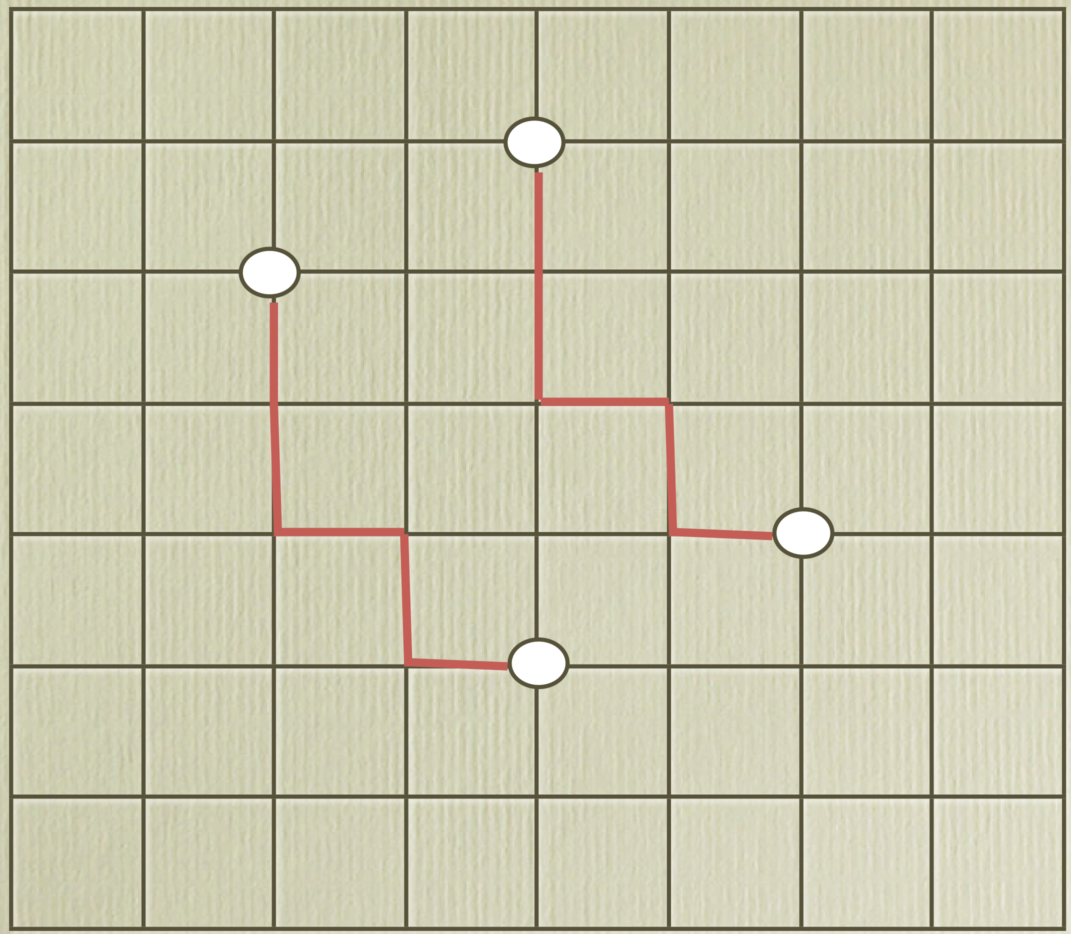
Mutual Anyonic Statistics

- Many-particle eigenstates, containing both electric and magnetic particles, can be constructed by plunking down multiple (non-crossing) open strings. The energies just add. Electric charges interact only by annihilating, as do magnetic charges.

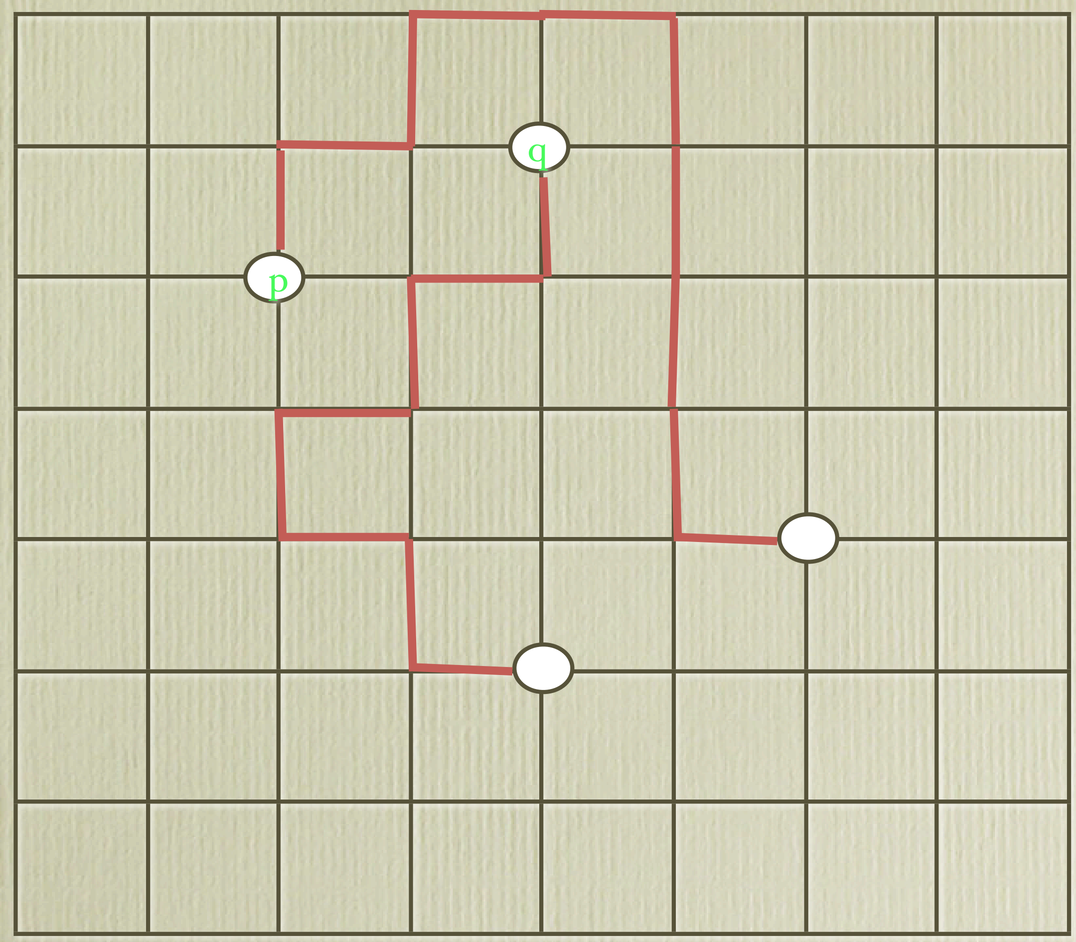
- The topology of the string network matters, however. Different topologies may define different states, or the same state with a different phase.

- Electric strings can be pulled through one another (since all the σ^z commute) as can magnetic strings (since all the σ^x commute).

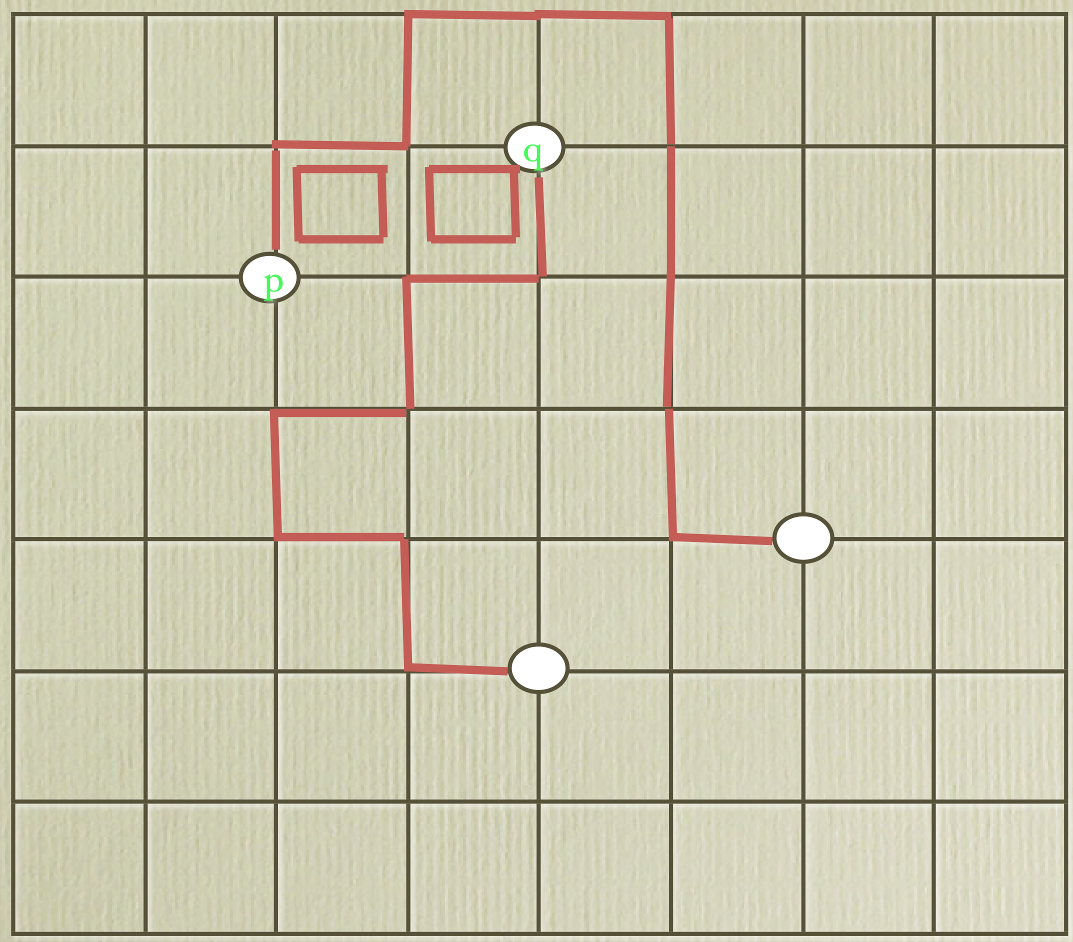
- Interchange of electric - or magnetic - particles gives back the same state. (See following Figures.) Thus, taken separately, they are bosons.



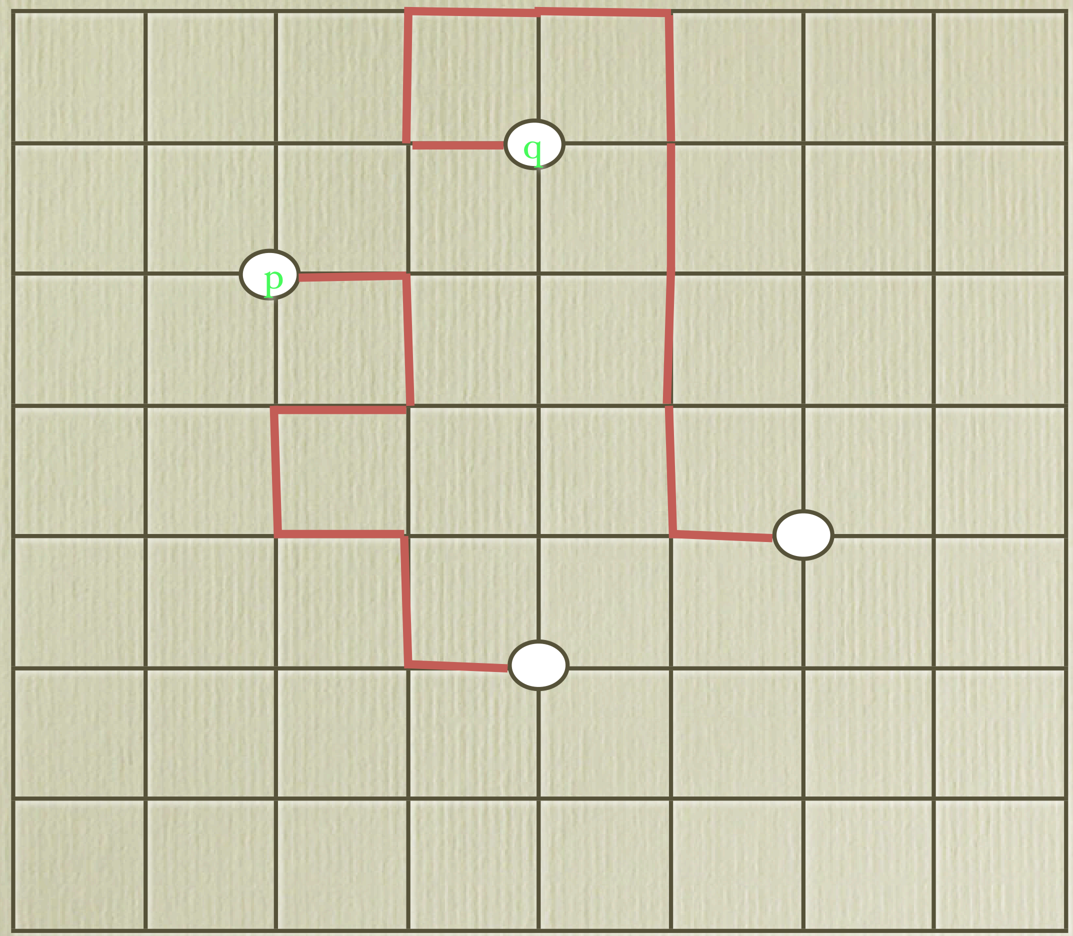
Two electric pairs with strings



Interchange, with p going over q

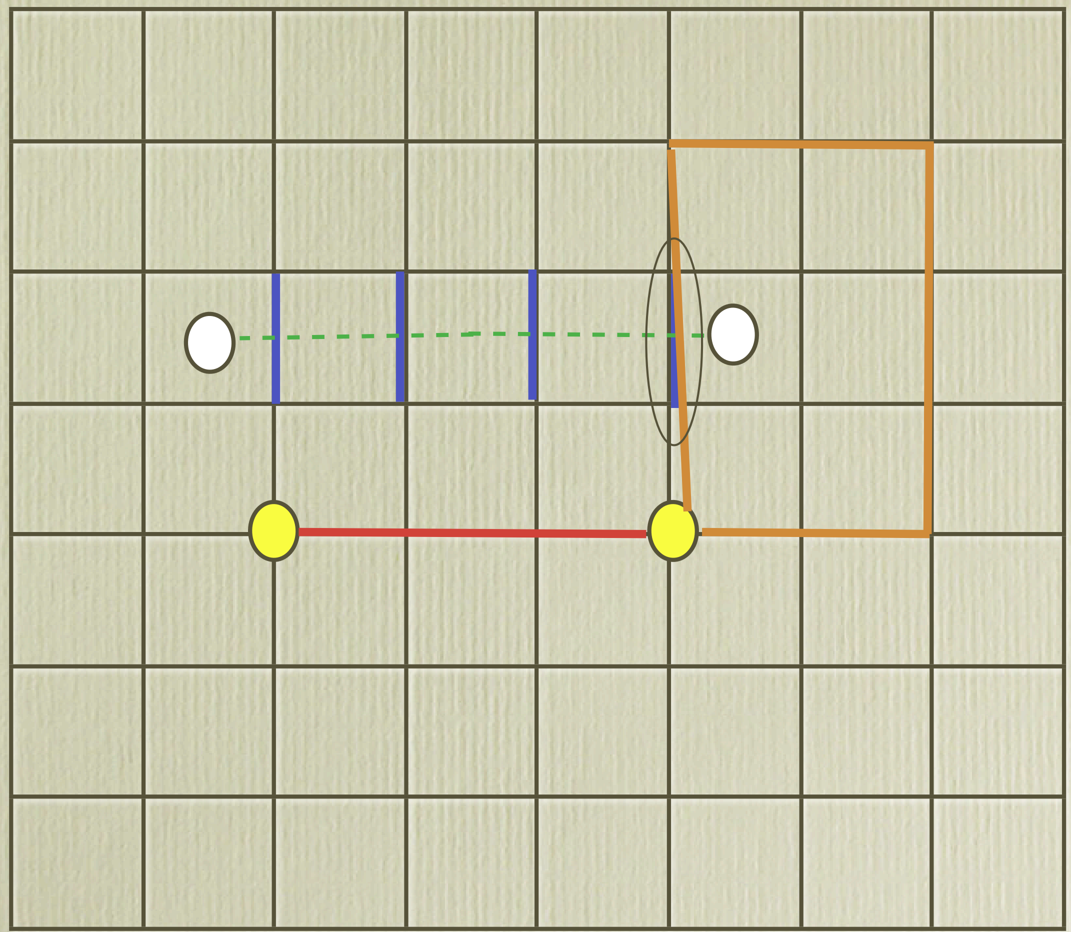


Getting back, with B operators



Reconnecting, with B operators

- But pulling an electric particle around a magnetic particle gives a minus sign, as a σ^z gets pulled through a σ^x:



The extra (orange) loop induces a factor -1

More formally, in operator language:

In the absence of the magnetic pair, our closed-loop Z type operator leaves the (that is, any of the 4!) ground state invariant:

$$Z_{\text{loop}}|0\rangle = |0\rangle$$

But in the presence of the X string there is a nontrivial commutator, and we get a factor -1:

$$Z_{\text{loop}}(X_{\text{string}}|0\rangle) = Z_{\text{loop}}X_{\text{string}}|0\rangle = -X_{\text{string}}Z_{\text{loop}}|0\rangle = -(X_{\text{string}}|0\rangle)$$

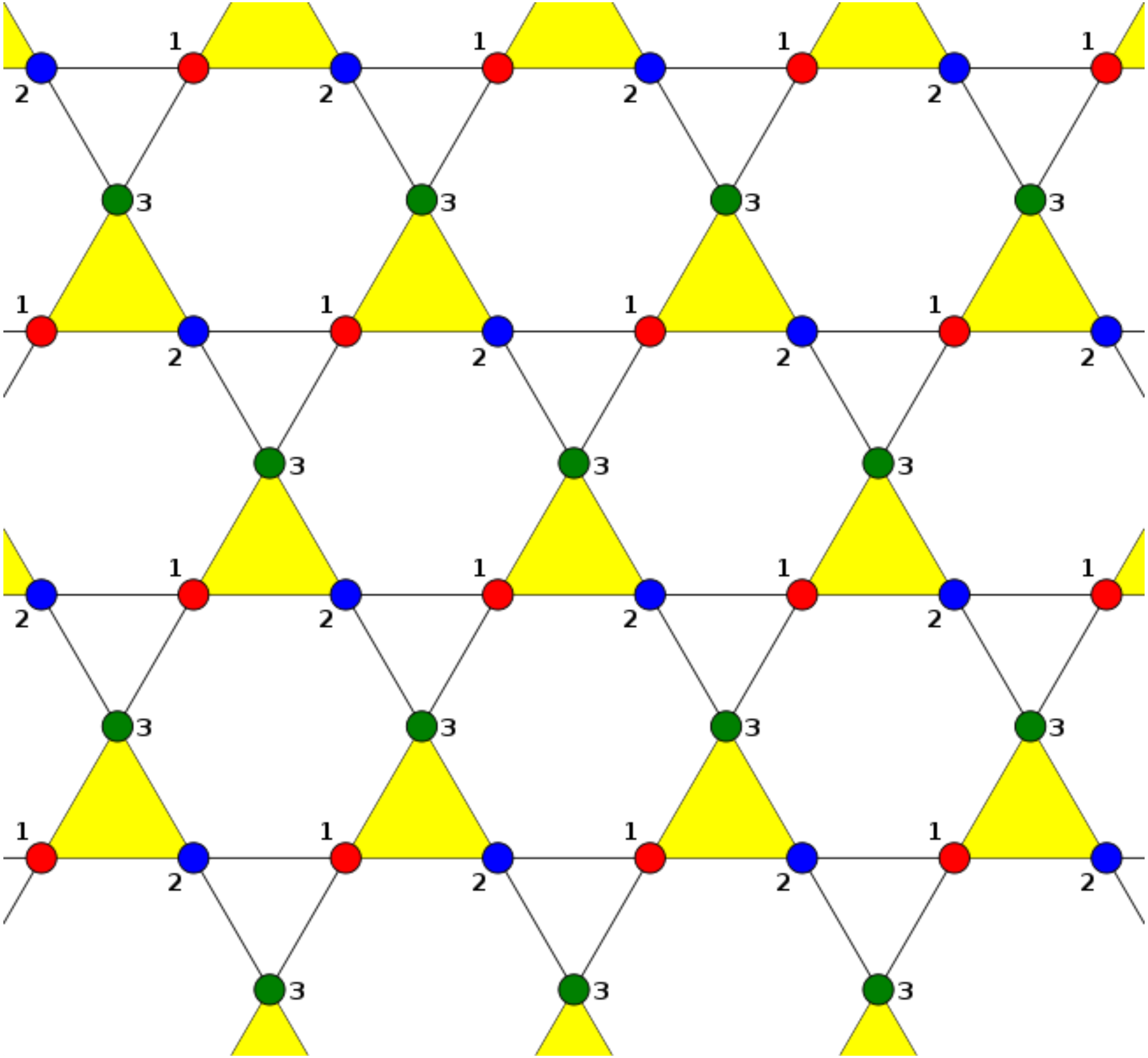
- This behavior is unlike conventional quantum particle behavior. The electric and magnetic particles have *mutual* anyon statistics.

- It is a subtle long-range quantum interaction, in a system with an energy gap.

Scholium

- With the schematic $-\Sigma\,(A + B)$ Hamiltonian, the electric and magnetic particles at definite positions are exact eigenstates. There is no tendency to “hop”. With a more general Hamiltonian, of course, that wouldn’t necessarily be the case.

The toric code model appears to be representative of a robust universality class. There is numerical evidence that it is realized for the antiferromagnetic Heisenberg model on a Kagome lattice. There is circumstantial evidence that it is realized experimentally in the mineral herbertsmithite.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Threshold Production

The angular momentum associated with a newly opening channel has a direct, quantitative effect on the near-threshold behavior of the production cross section.

Centrifugal barriers thin out the wave function near the origin, for small momentum (k^L).

This can be deduced from elementary quantum mechanics.   We have also checked it, in the relevant cases, directly numerically.

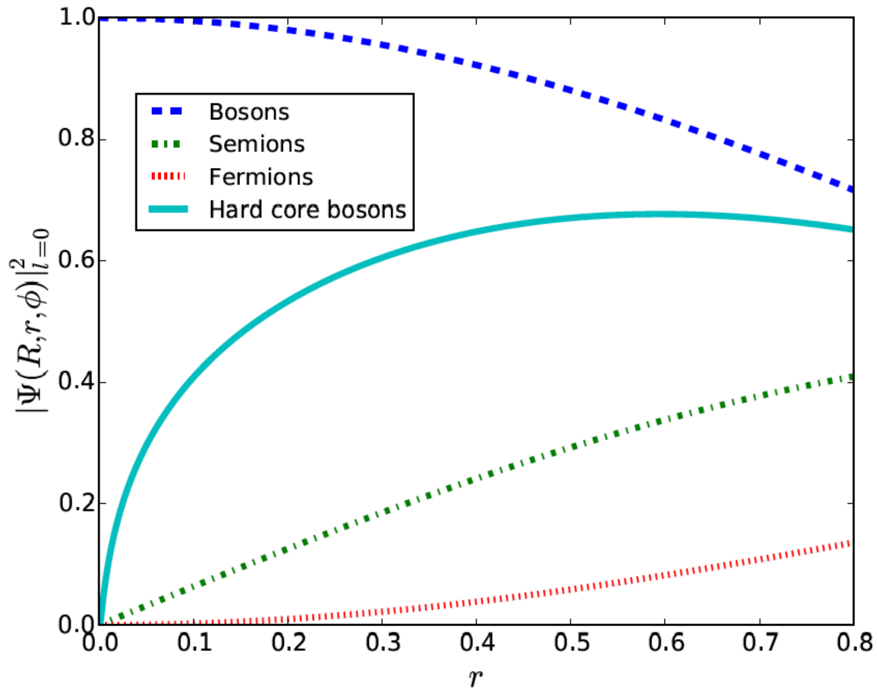

Figure 2. Probability densities for different two-anyon systems as a function of their relative distance r at a fixed k. The s-wave eigenstates ($l = 0$) will dominate the low-energy behaviour of the spectral function.

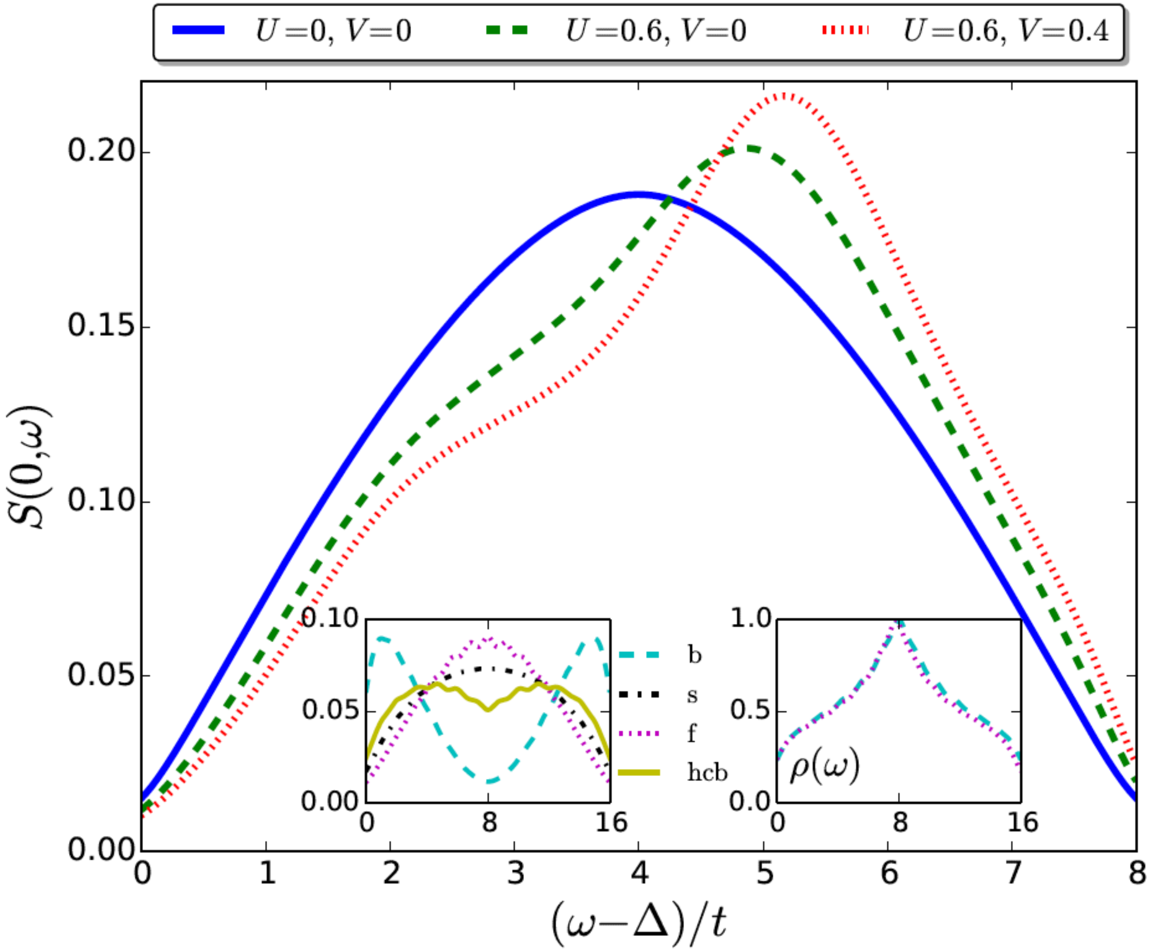

Figure 3. Two particle spectral function from exact diagonalization (ED) on a 100×100 square lattice for fermions with nearest neighbor repulsion U and next-nearest neighbor repulsion V. The high energy behavior is drastically affected by interactions, but the low-energy linear onset is unchanged. The on-set is not exactly zero because of the averaging involved in ED with ϵ=0.2. The left inset compares the spectral function for non-interacting bosons (b), semions (s), fermions (f), and hard-core bosons (hcb) on a 20 $\times$ 20 lattice. The right inset shows the corresponding density of states for bosons and fermions.

The angular momentum associated with a newly opening channel has a direct, quantitative effect on the near-threshold behavior of the production cross section. The cross-section sets in as k^{2L}.

One can aspire to see these effects in neutron scattering experiments.

Measurements of this kind diagnose the quantum statistics of anyon production channels, generally.

There are interesting complications when one produces three anyons; there one accesses more of the dynamics.

I expect that the threshold spectroscopy of quantum statistics will evolve rapidly from a demonstration into a diagnostic.