Evaluating Four-Loop Three-Point Feynman Integrals by Differential Equations

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In collaboration with Burkhard Eden, Johannes Henn, Alexander Smirnov and Matthias Steinhauser

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- Conclusion

[A.V. Kotikov'91, E. Remiddi'97, T. Gehrmann & E. Remiddi'00, J. Henn'13]

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A lot of applications [J.M. Henn, A.V. Smirnov, V.A. Smirnov, K. Melnikov, F. Caola, R. Bonciani, V. Del Duca, H. Frellesvig, F. Moriello, M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert, L. Tancredi, T. Gehrmann, A. von Manteuffel, E. Weihs, F. Dulat, B. Mistlberger, R. N. Lee,...]

Evaluating a family of Feynman integrals associated with a given graph with general integer powers of the propagators (indices)

$$F_\Gamma(q_1,\ldots,q_n;d;a_1,\ldots,a_L) \\ = \int\ldots\int I(q_1,\ldots,q_n;k_1,\ldots,k_h;a_1,\ldots,a_L)\mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \mathrm{d}^d k_h$$

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$$I(q_1,\ldots,q_n;k_1,\ldots,k_h;a_1,\ldots,a_L) = \frac{1}{(p_1^2-m_1^2)^{a_1}(p_2^2-m_2^2)^{a_2}\ldots}$$

$$d = 4-2\varepsilon$$

The old straightforward analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

The standard modern strategy:

to derive, without calculation, and then apply IBP identities between the given family of Feynman integrals as recurrence relations.

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The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

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- Solve DE

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DE:

$$\partial_i f(\epsilon, x) = A_i(\epsilon, x) f(\epsilon, x)\,,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and each A_i is an $N \times N$ matrix.

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In the differential form,

$$d\, f(\epsilon, x) = \epsilon\, (d\, \tilde{A}(x))\, f(x, \epsilon)\,,$$

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$$\tilde{A} = \sum_k \tilde{A}_k\, \log(\alpha_k)\,.$$

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Let us call it *epsilon form*.

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where $x^{(k)}$ is the set of singular points of the DE and $N \times N$ matrices a_k are independent of x and ϵ.

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For example, if $x_k = 0, -1, 1$ then results for elements of such a basis are expressed in terms of HPL.

$$\int_{0\leq\tau_1\leq\ldots\tau_k\leq x} d\tilde{A}(\tau_k)\ldots\cdot\ldots d\tilde{A}(\tau_1)$$

$\rightarrow$ a linear combination of integrals

$$\int_{0\leq\tau_1\leq\ldots\tau_k\leq x} \frac{d\tau_k}{\tau_k+a_k}\ldots\frac{d\tau_1}{\tau_1+a_1}$$

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HPLs

$$H(a_1,a_2,\ldots,a_n;x) = \int_0^x f(a_1;t)H(a_2,\ldots,a_n;t)dt\,,$$

where $f(\pm 1;t) = 1/(1\mp t),\;\; f(0;t) = 1/t,$

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- Use Feynman parametrization

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  B. Eden: a code in a special case.

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Let Γ_q^μ be the photon-quark vertex function.

The scalar form factor is

$$F_q(q^2) \;=\; -\frac{1}{4(1-\epsilon)q^2} \mathrm{Tr}\left(\not{p}_2\, \Gamma_q^\mu \not{p}_1\, \gamma_\mu\right) \;,$$

where $D = 4 - 2\epsilon$, $q = p_1 + p_2$ and p_1 (p_2) is the incoming (anti-)quark momentum.

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The large-N_c asymptotics of $F_q(q^2) \to$ planar Feynman diagrams.

Three-loop results
[P. A. Baikov, K. G. Chetyrkin, A. V. Smirnov, V. A. Smirnov and M. Steinhauser'09,
T. Gehrmann, E. W. N. Glover, T. Huber, N. Ikizlerli, and C. Studerus'10]

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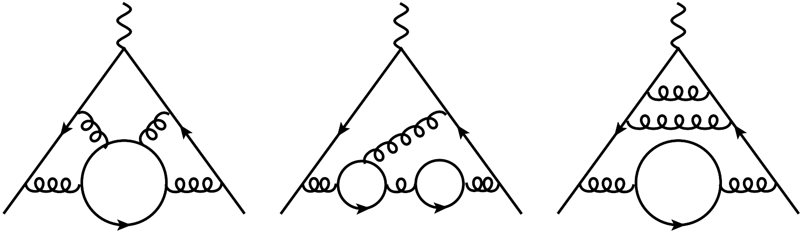
Analytic results for the three-loop master integrals up to weight 8
[R. N. Lee and V. A. Smirnov'10]
motivated by a future four-loop calculation.

The fermionic corrections ($\sim n_f$) to F_q in the large-N_c limit, to the four-loop order.



Numerical four-loop calculations [R. H. Boels, B. A. Kniehl, O. V. Tarasov & G. Yang'13,16]

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Results for some individual integrals in an analytical form [A. von Manteuffel, E. Panzer & R. M. Schabinger'15]

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Our result is the fermionic contribution to $F_q^{(4)}$ in the large-N_c limit.

$$\frac{1}{\epsilon^7}\left[\frac{1}{12}N_c^3 n_f\right] + \frac{1}{\epsilon^6}\left[\frac{41}{648}N_c^2 n_f^2 - \frac{37}{648}N_c^3 n_f\right] + \frac{1}{\epsilon^5}\left[\frac{1}{54}N_c n_f^3 + \frac{277}{972}N_c^2 n_f^2\right.$$

$$\left. + \left(\frac{41\pi^2}{648} - \frac{6431}{3888}\right) N_c^3 n_f\right] + \frac{1}{\epsilon^4}\left[\left(\frac{215\zeta_3}{108} - \frac{72953}{7776} - \frac{227\pi^2}{972}\right) N_c^3 n_f\right.$$

$$\left. + \frac{11}{54}N_c n_f^3 + \left(\frac{5}{24} + \frac{127\pi^2}{1944}\right) N_c^2 n_f^2\right] + \frac{1}{\epsilon^3}\left[\left(\frac{229\zeta_3}{486} - \frac{630593}{69984} + \frac{293\pi^2}{2916}\right) N_c^2 n_f^2\right.$$

$$\left. + \left(\frac{2411\zeta_3}{243} - \frac{1074359}{69984} - \frac{2125\pi^2}{1296} + \frac{413\pi^4}{3888}\right) N_c^3 n_f + \left(\frac{127}{81} + \frac{5\pi^2}{162}\right) N_c n_f^3\right]$$

$$+ \frac{1}{\epsilon^2}\left[\left(-\frac{41\zeta_3}{81} + \frac{29023}{2916} + \frac{55\pi^2}{162}\right) N_c n_f^3 + \left(\frac{11684\zeta_3}{729} - \frac{41264407}{419904} - \frac{155\pi^2}{72}\right.\right.$$

$$\left. + \frac{2623\pi^4}{29160}\right) N_c^2 n_f^2 + \left(-\frac{537625\zeta_3}{11664} - \frac{599\pi^2\zeta_3}{486} + \frac{12853\zeta_5}{180} + \frac{155932291}{839808}\right.$$

$$\left.\left. - \frac{27377\pi^2}{69984} - \frac{1309\pi^4}{7290}\right) N_c^3 n_f\right] + \ldots$$

The cusp and collinear anomalous dimensions

$$
\begin{aligned}
\gamma^0_{\text{cusp}} &= 4\,,\\
\gamma^1_{\text{cusp}} &= \left(-\frac{4\pi^2}{3}+\frac{268}{9}\right)N_c-\frac{40n_f}{9}\,,\\
\gamma^2_{\text{cusp}} &= \left(\frac{44\pi^4}{45}+\frac{88\zeta_3}{3}-\frac{536\pi^2}{27}+\frac{490}{3}\right)N_c^2\\
&\quad+\left(-\frac{64\zeta_3}{3}+\frac{80\pi^2}{27}-\frac{1331}{27}\right)N_c n_f-\frac{16n_f^2}{27}\,,\\
\gamma^3_{\text{cusp}} &= \left(-\frac{32\pi^4}{135}+\frac{1280\zeta_3}{27}-\frac{304\pi^2}{243}+\frac{3463}{81}\right)N_c n_f^2+\left(\frac{128\pi^2\zeta_3}{9}+224\zeta_5\right.\\
&\quad\left.-\frac{44\pi^4}{27}-\frac{16252\zeta_3}{27}+\frac{13346\pi^2}{243}-\frac{60391}{81}\right)N_c^2 n_f+\left(\frac{64\zeta_3}{27}-\frac{32}{81}\right)n_f^3+\ldots
\end{aligned}
$$

$$\gamma_q^0 = -\frac{3N_c}{2}, \quad \gamma_q^1 = \left(\frac{\pi^2}{6} + \frac{65}{54}\right) N_c n_f + \left(7\zeta_3 - \frac{5\pi^2}{12} - \frac{2003}{216}\right) N_c^2 ,$$

$$\gamma_q^2 = \left(-\frac{\pi^4}{135} - \frac{290\zeta_3}{27} + \frac{2243\pi^2}{972} + \frac{45095}{5832}\right) N_c^2 n_f + \left(-\frac{4\zeta_3}{27} - \frac{5\pi^2}{27} + \frac{2417}{1458}\right) + N_c^3 \left(-68\zeta_5 - \frac{22\pi^2\zeta_3}{9} - \frac{11\pi^4}{54} + \frac{2107\zeta_3}{18} - \frac{3985\pi^2}{1944} - \frac{204955}{5832}\right),$$

$$\gamma_q^3 = N_c^3 \left[\left(-\frac{680\zeta_3^2}{9} - \frac{1567\pi^6}{20412} + \frac{83\pi^2\zeta_3}{9} + \frac{557\zeta_5}{9} + \frac{3557\pi^4}{19440} - \frac{94807\zeta_3}{972} + \frac{354343\pi^2}{17496} + \frac{145651}{1728}\right) n_f\right] + \left(-\frac{8\pi^4}{1215} - \frac{356\zeta_3}{243} - \frac{2\pi^2}{81} + \frac{18691}{13122}\right) N_c n_f^3 + \left(-\frac{2}{3}\pi^2\zeta_3 + \frac{166\zeta_5}{9} + \frac{331\pi^4}{2430} - \frac{2131\zeta_3}{243} - \frac{68201\pi^2}{17496} - \frac{82181}{69984}\right) N_c^2 n_f^2 + \ldots$$

We reproduce results up to three loops
[A. Vogt'01; C.F. Berger'02; S. Moch, J.A.M. Vermaseren & A. Vogt'04,05; P.A. Baikov, K.G. Chetyrkin, A.V. Smirnov, V.A. Smirnov & M. Steinhauser'09; T. Becher & M. Neubert'09; T. Gehrmann, E.W.N. Glover, T. Huber, N. Ikizlerli & C. Studerus'10]

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Agreement of the n_f^2 term with
[Davies, B. Ruijl, T.Ueda, J. Vermaseren & A. Vogt'16]

We reproduce results up to three loops
[A. Vogt'01; C.F. Berger'02; S. Moch, J.A.M. Vermaseren & A. Vogt'04,05; P.A. Baikov, K.G. Chetyrkin, A.V. Smirnov, V.A. Smirnov & M. Steinhauser'09; T. Becher & M. Neubert'09; T. Gehrmann, E.W.N. Glover, T. Huber, N. Ikizlerli & C. Studerus'10]

The $N_c^3 n_f^3$ term of γ_{cusp}^3 agrees with
[M. Beneke & V.M. Braun'94]

Agreement of the n_f^2 term with
[Davies, B. Ruijl, T.Ueda, J. Vermaseren & A. Vogt'16]

All the other four-loop terms in γ_{cusp}^3 and γ_q^3 are new.

All planar four-loop on-shell form-factor integrals with $p_1^2 = p_2^2 = 0$, with $q^2 \equiv p_3^2 = (p_1 + p_2)^2$

$$
\begin{aligned}
Fa_1, \ldots, a_{18} \quad = \quad & \int \ldots \int \frac{\mathrm{d}^D k_1 \ldots \mathrm{d}^D k_4}{(-(k_1+p_1)^2)^{a_1}(-(k_2+p_1)^2)^{a_2}(-(k_3+p_1)^2)^{a_3}} \\
& \times \frac{1}{(-(k_4+p_1)^2)^{a_4}(-(k_1-p_2)^2)^{a_5}(-(k_2-p_2)^2)^{a_6}(-(k_3-p_2)^2)^{a_7}} \\
& \times \frac{1}{(-(k_4-p_2)^2)^{a_8}(-k_1^2)^{a_9}(-k_2^2)^{a_{10}}(-k_3^2)^{a_{11}}(-k_4^2)^{a_{12}}} \\
& \times \frac{1}{(-(k_1-k_2)^2)^{-a_{13}}(-(k_1-k_3)^2)^{-a_{14}}(-(k_1-k_4)^2)^{-a_{15}}} \\
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\end{aligned}
$$

At most 12 indices can be positive.

FIRE $\rightarrow$ 99 master integrals.

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504 master integrals

Derive (KZ) differential equations with respect to $x = p_2^2/p_3^2$

$$\partial_x f(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1-x} \right] f(x,\epsilon)$$

where a and b are x- and ϵ-independent 504×504 matrices.

Solving these equations in terms of HPL with letters 0 and 1.

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Asymptotic behaviour at the points $x = 0$ and $x = 1$

$$f(x,\epsilon) \overset{x\to 0}{\sim} x^{\epsilon a} f_0(\epsilon)\,,$$
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Natural boundary conditions at the point $x = 1$:
there is no singularity and it corresponds to propagator-type integrals which are known
[P. Baikov and K. Chetyrkin'10; R. Lee and V. Smirnov'11]

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To obtain analytical results for the 99 one-scale MI, we perform (with the help of the HPL package [D. Maître'06]) matching at the point $x = 0$.

Transporting boundary conditions at $x = 1$ to the point $x = 0$.

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Results for the 99 one-scale MI can be obtained from the 'naive' part of the asymptotic expansion at $x = 0$, i.e. from the part corresponding to the zero eigenvalue of the matrix a, i.e. to $j = 0$.

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Two alternative descriptions of the asymptotic expansion: with DE and with with expansion by regions [M. Beneke and V. Smirnov'98]

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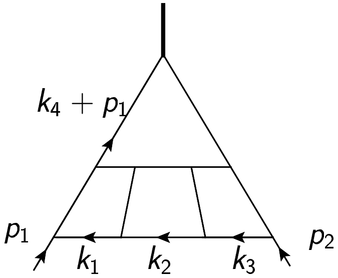
Two alternative descriptions of the asymptotic expansion: with DE and with with expansion by regions [M. Beneke and V. Smirnov'98]

$j = 0$ corresponds to the hard-...-hard region while terms with $j < 0$ to other regions (soft, collinear, ...). No positive j.

An example of our result



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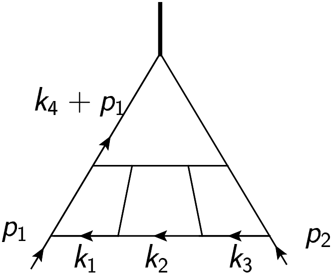

$$I_{12} = \int \cdots \int \prod_{j=1}^{4} \mathrm{d}^D k_j \, \frac{(k_4^2)^2}{k_1^2 k_2^2 k_3^2 (k_1 - k_2)^2 (k_2 - k_3)^2 (k_1 - k_4)^2}$$

$$\times \frac{1}{(k_2 - k_4)^2 (k_3 - k_4)^2 (k_1 + p_1)^2 (k_4 + p_1)^2 (k_4 - p_2)^2 (k_3 - p_2)^2}$$

$$
\begin{aligned}
=&\frac{1}{576\varepsilon^8}+\frac{1}{216}\pi^2\frac{1}{\epsilon^6}+\frac{151}{864}\zeta_3\frac{1}{\epsilon^5}+\frac{173}{10368}\pi^4\frac{1}{\epsilon^4}+\left[\frac{505}{1296}\pi^2\zeta_3+\frac{5503}{1440}\zeta_5\right]\frac{1}{\epsilon^3}+\\
&+\left[\frac{6317}{155520}\pi^6+\frac{9895}{2592}\zeta_3^2\right]\frac{1}{\epsilon^2}+\left[\frac{89593}{77760}\pi^4\zeta_3+\frac{3419}{270}\pi^2\zeta_5-\frac{169789}{4032}\zeta_7\right]\frac{1}{\epsilon}\\
&+\left[\frac{407}{15}s_{8a}+\frac{41820167}{653184000}\pi^8+\frac{41719}{972}\pi^2\zeta_3^2-\frac{263897}{2160}\zeta_3\zeta_5\right]+\mathcal{O}(\epsilon)\,,
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$$

where $s_{8a}=\zeta_8+\zeta_{5,3}, \zeta_i=\zeta(i), \zeta_{i,j}=\zeta(i,j)$

Correlation functions in $\mathcal{N} = 4$ SYM (in particular of the stress-tensor multiplet).

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Correlation functions $\rightarrow$ both scattering amplitudes and the dual polygonal Wilson loops [L. F. Alday, B. Eden, G. P. Korchemsky, J. Maldacena and E. Sokatchev'11, B. Eden, G. P. Korchemsky and E. Sokatchev]

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The complexity increases very much at higher loops.
An explicit result for the two-loop four-point stress-tensor correlator
[B. Eden, C. Schubert and E. Sokatchev'00, M. Bianchi, S. Kovacs, G. Rossi and Y. S. Stanev'00]

Three-loop calculations [B. Eden, P. Heslop, G. P. Korchemsky and E. Sokatchev'12, J. Drummond, C. Duhr, B. Eden, P. Heslop, J. Pennington & V. A. Smirnov'13, D. Chicherin, J. Drummond, P. Heslop and E. Sokatchev'15]

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In [J. Drummond, C. Duhr, B. Eden, P. Heslop, J. Pennington & V. A. Smirnov'13] also one four-loop integral was evaluated (with one external vertex is connected to the rest of this diagram only by a single line.)

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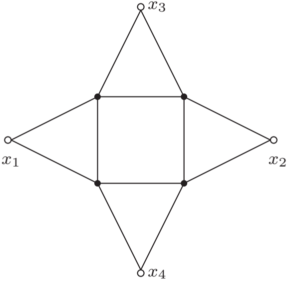
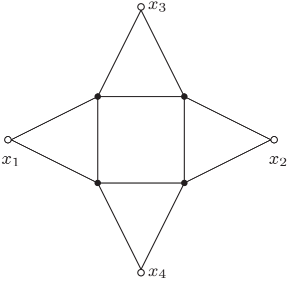
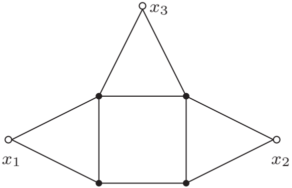
Four loops: 26 genuine four-loop integrals in the planar part of the correlator five of which can be related to the ladder with four rungs by flip identities on subintegrals.


x_3
x_1
x_2
x_4



$x_4 \to \infty;\ x_3 = 0$



Evaluating by DE:

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- Using 4-dimensional IBP relations?
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The goals:

- To evaluate this four-dimensional four-loop integral.

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B. Eden & V.S.: the D-dimensional way.
The goals:

- To evaluate this four-dimensional four-loop integral.
- To evaluate the whole set of the master integrals in D dimensions. (The first example of such a calculation.)

$$F_{a_1,\ldots,a_{18}} = \int\ldots\int \frac{\mathrm{d}^D x_5\,\mathrm{d}^D x_6\,\mathrm{d}^D x_7\,\mathrm{d}^D x_8}{[-x_5^2]^{a_1}[-x_6^2]^{a_2}[-(x_1-x_5)^2]^{a_3}[-(x_1-x_7)^2]^{a_4}}$$
$$\times\frac{[-(x_2-x_5)^2]^{-a_{11}}[-(x_1-x_6)^2]^{-a_{12}}[-(x_2-x_7)^2]^{-a_{13}}[-(x_6-x_7)^2]^{-a_{14}}}{[-(x_2-x_6)^2]^{a_5}[-(x_2-x_8)^2]^{a_6}[-(x_5-x_6)^2]^{a_7}[-(x_5-x_7)^2]^{a_8}}$$
$$\times\frac{[-x_7^2]^{-a_{15}}[-(x_1-x_8)^2]^{-a_{16}}[-(x_5-x_8)^2]^{-a_{17}}[-x_8^2]^{-a_{18}}}{[-(x_6-x_8)^2]^{a_9}[-(x_7-x_8)^2]^{a_{10}}}\ .$$

$$
\begin{aligned}
F_{a_1,\ldots,a_{18}} \quad = \quad & \int \ldots \int \frac{\mathrm{d}^D x_5 \, \mathrm{d}^D x_6 \, \mathrm{d}^D x_7 \, \mathrm{d}^D x_8}{[-x_5^2]^{a_1}[-x_6^2]^{a_2}[-(x_1-x_5)^2]^{a_3}[-(x_1-x_7)^2]^{a_4}} \\
& \times \frac{[-(x_2-x_5)^2]^{-a_{11}}[-(x_1-x_6)^2]^{-a_{12}}[-(x_2-x_7)^2]^{-a_{13}}[-(x_6-x_7)^2]^{-a_{14}}}{[-(x_2-x_6)^2]^{a_5}[-(x_2-x_8)^2]^{a_6}[-(x_5-x_6)^2]^{a_7}[-(x_5-x_7)^2]^{a_8}} \\
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\end{aligned}
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$a_i \leq 0$ for $i \geq 11$

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Three coordinate differences squared off the light cone
$x_1^2 = -z\bar{z}, \quad x_2^2 = -(1-z)(1-\bar{z}), \quad (x_1-x_2)^2 = -1$

$$F_{a_1,\ldots,a_{18}} = \int \ldots \int \frac{\mathrm{d}^D x_5\, \mathrm{d}^D x_6\, \mathrm{d}^D x_7\, \mathrm{d}^D x_8}{[-x_5^2]^{a_1}[-x_6^2]^{a_2}[-(x_1-x_5)^2]^{a_3}[-(x_1-x_7)^2]^{a_4}}$$
$$\times \frac{[-(x_2-x_5)^2]^{-a_{11}}[-(x_1-x_6)^2]^{-a_{12}}[-(x_2-x_7)^2]^{-a_{13}}[-(x_6-x_7)^2]^{-a_{14}}}{[-(x_2-x_6)^2]^{a_5}[-(x_2-x_8)^2]^{a_6}[-(x_5-x_6)^2]^{a_7}[-(x_5-x_7)^2]^{a_8}}$$
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$x_1^2 = -z\bar{z}, \quad x_2^2 = -(1-z)(1-\bar{z}), \quad (x_1-x_2)^2 = -1$

Our integral is $F_{1,1,1,1,1,1,1,1,1,1,0,\ldots}$

FIRE $\rightarrow$ 213 master integrals

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Two variables: $z_1 = z$ and $z_2 = \bar{z}$

FIRE → 213 master integrals

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DE

$$\begin{aligned} \frac{\partial}{\partial z_1} f &= A_1(z_1, z_2, \varepsilon) f \,, \\ \frac{\partial}{\partial z_2} f &= A_2(z_1, z_2, \varepsilon) f \,. \end{aligned}$$

Constructing a canonical basis

FIRE → 213 master integrals

Two variables: $z_1 = z$ and $z_2 = \bar{z}$

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Constructing a canonical basis
We used a code constructed by Burkhard Eden.

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Constructing a canonical basis
We used a code constructed by Burkhard Eden.
212 of 213 elements of a canonical basis were obtained with this code.

DE in our canonical basis

$$\frac{\partial}{\partial z_1} f = \varepsilon \bar{A}_1(z_1, z_2) f \,,$$
$$\frac{\partial}{\partial z_2} f = \varepsilon \bar{A}_2(z_1, z_2) f \,.$$

DE in our canonical basis

$$\begin{aligned} \frac{\partial}{\partial z_1} f &= \varepsilon \bar{A}_1(z_1, z_2) f \,, \\ \frac{\partial}{\partial z_2} f &= \varepsilon \bar{A}_2(z_1, z_2) f \,. \end{aligned}$$

$\bar{A}_i = \frac{\partial}{\partial z_i} \tilde{A}$
with

$$\tilde{A} = \sum_k \tilde{A}_k \log(\alpha_k) \,.$$

and letters taken from the alphabet
$\{z_1, 1-z_1, z_2, 1-z_2, -z_1+z_2, 1-z_1-z_2, 1-z_1 z_2, z_1+z_2-z_1 z_2\}$

Solve DE order in order in ε

$$f = \sum_{i=0}^{8} f^{(i)} \varepsilon^i$$

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[J. M. Henn, K. Melnikov and V. A. Smirnov'14]:

First, solve

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First, solve

$$\frac{\partial}{\partial z_1} f^{(i)} = \bar{A}_1(z_1, z_2) f^{(i-1)}$$

Solution

$$f^{(i)}(z_1, z_2) = \int_0^{z_1} d\bar{z}_1 \bar{A}_1(\bar{z}_1, z_2) f^{(i-1)}(\bar{z}_1, z_2) + h^{(i)}(z_2)$$

The result is a linear combination of multiple (Goncharov) polylogarithms (MPL) $G(a_1, a_2, \ldots, a_w; z_1)$
where $a_i \in \{0, 1, z_2, 1 - z_2, 1/z_2, -z_2/(1 - z_2)\}$.

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$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)$$

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$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)$$

Substitute solution into the second equation to obtain

$$\frac{\partial}{\partial z_2} h^{(i)}(z_2) = \bar{A}_2(z_1, z_2) h^{(i-1)}(z_2) + \bar{A}_2(z_1, z_2) \int_0^{z_1} \mathrm{d}\bar{z}_1 \bar{A}_1(\bar{z}_1, z_2) f^{(i-2)}(\bar{z}_1, z_2) - \frac{\partial}{\partial z_2} \int_0^{z_1} \mathrm{d}\bar{z}_1 \bar{A}_1(\bar{z}_1, z_2) f^{(i-1)}(\bar{z}_1, z_2)$$

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To fix these 213×9 unknown constants, we match our results in terms of multiple polylogarithms to the leading order asymptotic behaviour of the solution of DE in the limit $z, \bar{z} \to 0$ which corresponds to the Euclidean limit $x_1 \to 0$.

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If y is the expansion parameter in the limit $z_1, z_2 \to 0$ then we encounter the following power dependence
$y^0, y^{-\varepsilon}, y^{-2\varepsilon}, y^{-3\varepsilon}, y^{-4\varepsilon}$

We again use an interplay between expansion by regions and solving canonical DE in the given limit.

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One could consider also the short-distance limit $x_2 \to 0$ and the limits $z_1 \to 0, z_2 \to 1$ and $z_1 \to 1, z_2 \to 0$ which are light-cone limits $x_1^2 \to 0$ and $x_2^2 \to 0$.

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It turns out that the information about the the Euclidean limit $x_1 \to 0$ gives sufficient information.

The result for our integral is $(z-\bar{z})^{-2}$ times a linear combination of single valued multiple polylogarithms [F. C. S. Brown'04]

$$\mathcal{L}_{\{a_1,\ldots,a_8\}} = (-1)^{\sum a_i} G(a_1,\ldots a_8;z) + \sum c_{ij}\, G(\underline{a}_i;z)\, G(\underline{a}_j;\bar{z})$$

where $\underline{a}_i \cup \underline{a}_j$ has length 8 and $\underline{a}_j$ is never the empty word.

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where $\underline{a}_i \cup \underline{a}_j$ has length 8 and $\underline{a}_j$ is never the empty word.

The coefficients c_{ij} are polynomials of multiple zeta values such that all branch cuts cancel. The entries in the weight vectors are in the set $\{0,1\}$ and the "condensed notation" $\ldots 0,0,0,1\ldots = \ldots 4\ldots$ etc. is used

After flipping points $x_2 \leftrightarrow x_3$ (i.e. $z \to 1/z$, $\bar{z} \to 1/\bar{z}$) followed by $x_1 \leftrightarrow x_2$ (which implies $z \to 1-z$, $\bar{z} \to 1-\bar{z}$): this function takes the form

$$\begin{aligned} - & \mathcal{L}_{\{3,5\}} + \mathcal{L}_{\{5,3\}} + \mathcal{L}_{\{2,5,0\}} - \mathcal{L}_{\{4,3,0\}} - \mathcal{L}_{\{1,5,0,0\}} + \mathcal{L}_{\{3,3,0,0\}} \\ - & \mathcal{L}_{\{2,3,0,0,0\}} + \mathcal{L}_{\{1,3,0,0,0,0\}} \end{aligned}$$

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This result as well as those for some other elements in the basis were checked by a numerical calculation with FIESTA [A.V. Smirnov'15]

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Agreement with independent calculations by O. Schnetz and E. Panzer.

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- More new results will be obtained with this method in the future.