Simplifying Color and Kinematic Structures of QCD

based on work with Henrik Johansson
arXiv:1507.00332 [hep-ph]

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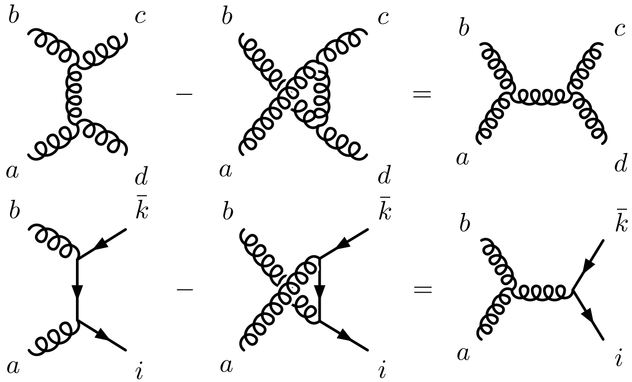
Color-kinematics duality for adjoint & fund. particles

Color algebra of Yang-Mills coupled to fund. matter

$$\tilde{f}^{dae}\tilde{f}^{ebc} - \tilde{f}^{dbe}\tilde{f}^{eac} = \tilde{f}^{abe}\tilde{f}^{dec}$$

$$T^{a}_{i\bar{\jmath}}T^{b}_{j\bar{k}} - T^{b}_{i\bar{\jmath}}T^{a}_{j\bar{k}} = \tilde{f}^{abe}T^{e}_{i\bar{k}}$$

Representation in terms of color graphs:

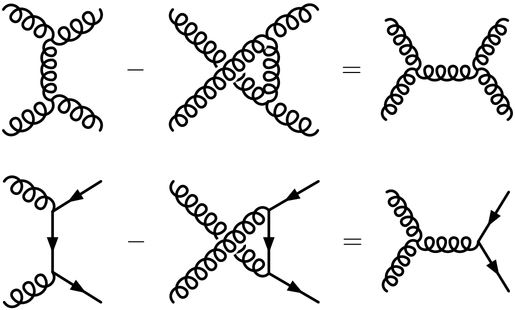


Color-kinematics duality

Bern, Carrasco, Johansson (2008,10)

Johansson, AO (2014,15)

$$\mathcal{A}_4^{\text{tree}} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$



- absorb quartic interactions into trivalent graphs
- take algebraic identities true for color factors
- check/impose on kinematic numerators $\Leftrightarrow$ <u>C-K duality</u>

Double copy construction of gravity amplitudes

Bern, Carrasco, Johansson (2008,10)

C-K duality for YM:

$$\begin{cases} \mathcal{A}_m^{L\text{-loop}} = i^L \sum_{i \in \text{cubic}} \int \frac{d^{LD}\ell}{(2\pi)^{DL}} \frac{1}{S_i} \frac{n_i c_i}{D_i} \\ c_i - c_j = c_k \quad \Leftrightarrow \quad n_i - n_j = n_k \\ c_i \to -c_i \quad \Leftrightarrow \quad n_i \to -n_i \end{cases}$$

Double copy for gravity:

$$\mathcal{M}_m^{L\text{-loop}} = i^{L+1} \sum_{i \in \text{cubic}} \int \frac{d^{LD}\ell}{(2\pi)^{DL}} \frac{1}{S_i} \frac{n_i n_i'}{D_i}$$

Double copy of fund./chiral matter

Bern, Carrasco, Johansson (2008,10)

Johansson, AO (2014)

C-K duality for YM + fund. matter:

$$\begin{cases} \mathcal{A}_m^{L\text{-loop}} = i^L \sum_{i \in \text{cubic}} \int \frac{d^{LD}\ell}{(2\pi)^{DL}} \frac{1}{S_i} \frac{n_i c_i}{D_i} \\ c_i - c_j = c_k \quad \Leftrightarrow \quad n_i - n_j = n_k \\ c_i \to -c_i \quad \Leftrightarrow \quad n_i \to -n_i \end{cases}$$

Double copy for gravity + abelian matter:

$$\mathcal{M}_m^{L\text{-loop}} = i^{L+1} \sum_{i \in \text{cubic}} \int \frac{d^{LD}\ell}{(2\pi)^{DL}} \frac{N_{\mathrm{M}}^{|i|}}{S_i} \frac{n_i \overline{n}'_i}{D_i}$$

$N_{\mathrm{M}} + 1$ = number of complex matter multiplets
$|i|$ = number of fundamental loops in i-th diagram
bar denotes flipping fundamental arrows
and their helicities (massless case)

Gauge theories with color-kinematics duality

- Pure $\mathcal{N} = 0, 1, 2, 4$ SYM — Bern, Carrasco, Johansson (2008,10)
  Bjerrum-Bohr, Damgaard, Vanhove; Mafra, Schlotterer, Stieberger; Feng, Huang, etc.
- Self-dual Yang-Mills — Monteiro, O'Connell (2011)
- Yang-Mills + F^3 theory — Broedel, Dixon (2012)
- Heterotic string theory — AO, Tourkine (2013)
  Stieberger, Taylor (2014)
- (S)QCD — Johansson, AO (2015)
- (S)YM coupled to matter in generic representation
  Johansson, AO (2014)
  Chiodaroli, Gunaydin, Johansson, Roiban (2014)
- (S)YM coupled to scalar φ^3 theory
  Chiodaroli, Gunaydin, Johansson, Roiban (2014)
- Spontaneously broken (S)YM
  Chiodaroli, Gunaydin, Johansson, Roiban (2015)
- Bagger-Lambert-Gustavsson theory in $D = 3$
  Bargheer, He, McLoughlin (2012)
  Huang, Johansson, Lee (2013)
- Nonlinear sigma model — Chen, Du (2013)

Gravity theories as double copies

- Pure $\mathcal{N} = 4, 5, 6, 8$ supergravity
  Bern, Carrasco, Johansson (2008,10)
- Self-dual gravity — Monteiro, O'Connell (2011)
- Einstein + R^3 theory — Broedel, Dixon (2012)
- Closed string theories — Mafra, Schlotterer, Stieberger (2011)
  Stieberger, Taylor (2014)
  Huang, Schlotterer, Wen (2016)
- Abelian matter coupled to (super)gravity — Johansson, AO (2014)
- (Super)gravity coupled to (S)YM
  Chiodaroli, Gunaydin, Johansson, Roiban (2014)
- Spontaneously broken YM-Einstein gravity
  Chiodaroli, Gunaydin, Johansson, Roiban (2015)
- Homogeneous and magical N=2 Maxwell-Einstein supergravities — Chiodaroli, Gunaydin, Johansson, Roiban (2015)
- $\mathcal{N} = 16$ supergravity in $D = 3$ — Bargheer, He, McLoughlin (2012)
  Huang, Johansson, Lee (2013)
- Born-Infeld, Dirac-Born-Infeld, Galileon theories
  Cachazo, He, Yuan (2014)

more in Chiodaroli's talk

This talk: applications to QCD

QCD $\equiv$ SU(N_c) YM + N_f massive quarks
also true for G_c YM + N_f massive fermions/scalars
in arbitrary representation of G_c
and with/without supersymmetry
in D dimensions

- Both adj./real & fund./complex representations
- Color-kinematics duality for numerators
  $\Rightarrow$ BCJ relations for color-ordered amplitudes
- Color structure also nontrivial
  $\Rightarrow$ New color decomposition for color-dressed amplitudes

New color decomposition for QCD tree amplitudes

Kleiss-Kuijf relations

Color ordering $\Rightarrow$ $(n-1)!$ gluonic primitive amplitudes:

$$\mathcal{A}_{n,0}^{\text{tree}} = \sum_{\sigma \in S_{n-1}(\{2,\ldots,n\})} \text{Tr}\Big(T^{a_1} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(n)}}\Big) A(1, \sigma(2), \ldots, \sigma(n))$$

KK relations: Kleiss, Kuijf (1988)

$$A(1, \beta, 2, \alpha) = (-1)^{|\beta|} \sum_{\sigma \in \alpha \sqcup\!\sqcup \beta^T} A(1, 2, \sigma)$$

$\Rightarrow$ KK basis of $(n-2)!$ primitives:

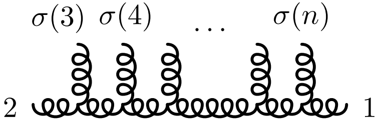
$$\Big\{ A(1, 2, \sigma) \;\Big|\; \sigma \in S_{n-2}(\{3, \ldots, n\}) \Big\}$$

$\Rightarrow$ DDM decomposition: Del Duca, Dixon, Maltoni (1999)

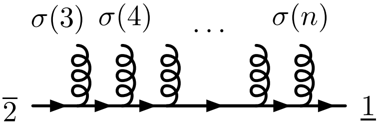
$$\mathcal{A}_{n,0}^{\text{tree}} = \sum_{\sigma \in S_{n-2}(\{3,\ldots,n\})} \tilde{f}^{\,a_2 a_{\sigma(3)} b_1} \tilde{f}^{\,b_1 a_{\sigma(4)} b_2} \ldots \tilde{f}^{\,b_{n-3} a_{\sigma(n)} a_1}$$
$$\times A(1, 2, \sigma(3), \ldots, \sigma(n))$$

DDM color decomposition

Del Duca, Dixon, Maltoni (1999)



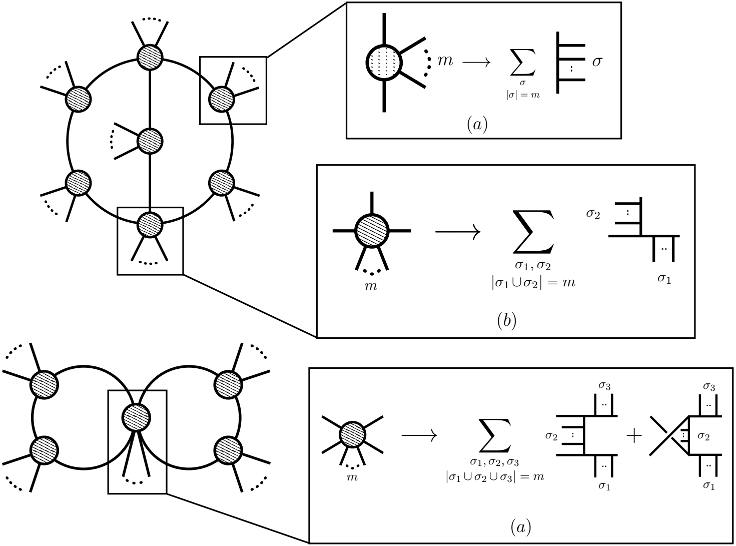
$$\mathcal{A}_{n,0}^{\text{tree}} = \sum_{\sigma \in S_{n-2}(\{3,\dots,n\})} \tilde{f}^{\,a_2 a_{\sigma(3)} b_1} \tilde{f}^{\,b_1 a_{\sigma(4)} b_2} \dots \tilde{f}^{\,b_{n-3} a_{\sigma(n)} a_1} \times A(1, 2, \sigma(3), \dots, \sigma(n))$$



$$\mathcal{A}_{n,1}^{\text{tree}} = \sum_{\sigma \in S_{n-2}(\{3,\dots,n\})} \left(T^{a_{\sigma(3)}} \dots T^{a_{\sigma(n)}}\right)_{\bar{\jmath}_2 i_1} A(\underline{1}, \overline{2}, \sigma(3), \dots, \sigma(n))$$

Aside: 2-loop color dressing in pure YM

Badger, Mogull, AO, O'Connell (2015)

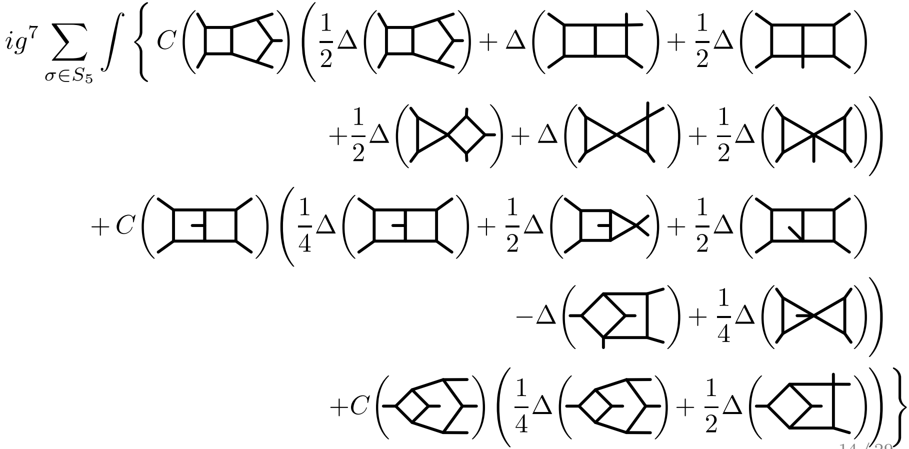


Aside: a 2-loop full-color amplitude

Badger, Mogull, AO, O'Connell (2015)

leading-color/planar amplitude $\underbrace{\Rightarrow}_{\text{BCJ relations on cuts}}$ full-color/nonplanar amplitude

$$\mathcal{A}_5^{\text{2-loop}}(1^+, 2^+, 3^+, 4^+, 5^+) =$$



Melia basis of QCD color-ordered amplitudes

Goal:
Color decomposition for $2k$ quarks and $(n-2k)$ gluons?
Basis w.r.t. KK relations found by Melia (2013)

Melia basis $\equiv$
$$\left\{A(\underline{1},\overline{2},\sigma)\;\middle|\;\sigma\in\{\text{quark brackets}\}_{k-1}\times\{\text{gluon insertions}\}_{n-2k}\right\}$$

$n=5,\ k=2$:

$$\{3\,4\}\Leftrightarrow A(\underline{1},\overline{2},5,\underline{3},\overline{4})\,,A(\underline{1},\overline{2},\underline{3},5,\overline{4})\,,A(\underline{1},\overline{2},\underline{3},\overline{4},5)$$

$n=6,\ k=3$:

$$\{3\,4\}\{5\,6\},\ \{5\,6\}\{3\,4\}\Leftrightarrow A(\underline{1},\overline{2},\underline{3},\overline{4},\underline{5},\overline{6}),\ A(\underline{1},\overline{2},\underline{5},\overline{6},\underline{3},\overline{4})$$
$$\big\{3\{5\,6\}4\big\},\ \big\{5\{3\,4\}6\big\}\Leftrightarrow A(\underline{1},\overline{2},\underline{3},\underline{5},\overline{6},\overline{4}),\ A(\underline{1},\overline{2},\underline{5},\underline{3},\overline{4},\overline{6})$$

New color decomposition

Johansson, AO (2015)

$$\mathcal{A}^{\text{tree}}_{n,k} = \sum_{\sigma \in \text{Melia basis}} C(\underline{1}, \overline{2}, \sigma)\, A(\underline{1}, \overline{2}, \sigma)\,,$$

$$C(\underline{1}, \overline{2}, \sigma) = (-1)^{k-1}\, \{2|\sigma|1\}\Big|_{\substack{\underline{q} \;\to\; \{q|\, T^b \otimes\, \Xi^b_{l-1} \\ \overline{q} \;\to\; |q\} \\ g \;\to\; \Xi^{a_g}_l}}$$

$$\Xi^a_l = \sum_{s=1}^{l} \underbrace{1 \otimes \cdots \otimes 1 \otimes \overbrace{T^a \otimes 1 \otimes \cdots \otimes 1 \otimes \overline{1}}^{s}}_{l}$$

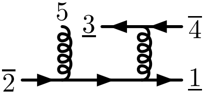
- checked analytically up to 8 points
- proven by induction by Melia (2015)

5-point color example

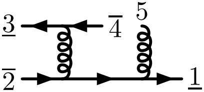
$$\mathcal{A}^{\text{tree}}_{5,2} = C_{\underline{1}\overline{2}5\underline{3}\overline{4}}\, A_{\underline{1}\overline{2}5\underline{3}\overline{4}} + C_{\underline{1}\overline{2}\underline{3}\overline{4}5}\, A_{\underline{1}\overline{2}\underline{3}\overline{4}5} + C_{\underline{1}\overline{2}\underline{3}5\overline{4}}\, A_{\underline{1}\overline{2}\underline{3}5\overline{4}}$$

Reminder: $$C(\underline{1},\overline{2},\sigma) = (-1)^{k-1}\,\{2|\sigma|1\}\Big|\begin{array}{lll} q & \to & \{q|\,T^b \otimes \Xi^b_{l-1} \\ \overline{q} & \to & |q\} \\ g & \to & \Xi^{a_g}_l \end{array}$$

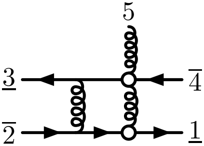
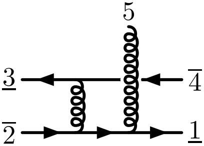
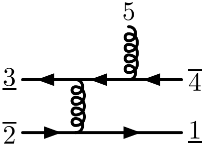
$$C_{\underline{1}\overline{2}5\underline{3}\overline{4}} = -\,\{2|\Xi^{a_5}_1\{3|T^b \otimes \Xi^b_1|4\}|1\} = -\,(\overline{T}^{a_5}\overline{T}^{b})_{\overline{\imath}_2 i_1} T^b_{i_3\overline{\imath}_4} =$$



$$C_{\underline{1}\overline{2}\underline{3}\overline{4}5} = -\,\{2|\{3|T^b \otimes \Xi^b_1|4\}\Xi^{a_5}_1|1\} = -\,(\overline{T}^{b}\overline{T}^{a_5})_{\overline{\imath}_2 i_1} T^b_{i_3\overline{\imath}_4} =$$

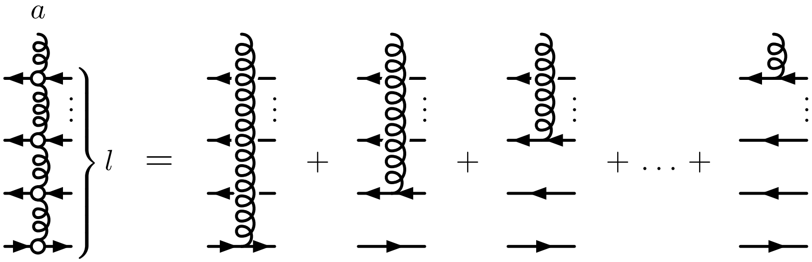


$$C_{\underline{1}\overline{2}\underline{3}5\overline{4}} = -\,\{2|\{3|(T^b \otimes \Xi^b_1)\,\Xi^{a_5}_2|4\}|1\} = -\,(\overline{T}^{b}\overline{T}^{a_5})_{\overline{\imath}_2 i_1} T^b_{i_3\overline{\imath}_4} - \overline{T}^b_{\overline{\imath}_2 i_1}(T^b T^{a_5})_{i_3\overline{\imath}_4}$$

$=$  $=$  $+$ 

Tensor color representation

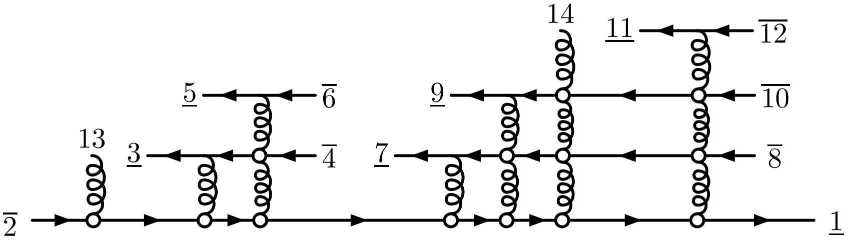
$$\Xi_l^a = \sum_{s=1}^{l} \underbrace{1 \otimes \cdots \otimes 1 \otimes \overbrace{T^a \otimes 1 \otimes \cdots \otimes 1 \otimes \overline{1}}^{s}}_{l} \,.$$



$$\left[\Xi_l^a, \Xi_l^b\right] = \tilde{f}^{abc}\, \Xi_l^c \,.$$

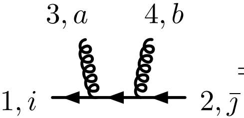
Higher-point color example

Color coefficient of $A(\underline{1}, \overline{2}, 13, \underline{3}, \underline{5}, \overline{6}, \overline{4}, \underline{7}, \underline{9}, 14, \underline{11}, \overline{12}, \overline{10}, \overline{8})$ in $\mathcal{A}^{\text{tree}}_{14,6}$:

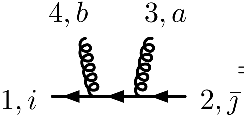


BCJ relations for QCD tree amplitudes

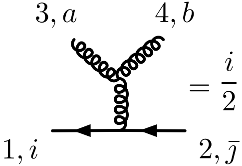
Color-kinematics at 4 points



$$= -\frac{i}{2}\,\frac{T^a_{i\bar{k}}T^b_{k\bar{\jmath}}}{s_{13}-m^2}\,(\bar{u}_1\not{\varepsilon}_3(\not{k}_{1,3}+m)\not{\varepsilon}_4 v_2) = \frac{c_1 n_1}{D_1}$$



$$= -\frac{i}{2}\,\frac{T^b_{i\bar{k}}T^a_{k\bar{\jmath}}}{s_{14}-m^2}\,(\bar{u}_1\not{\varepsilon}_4(\not{k}_{1,4}+m)\not{\varepsilon}_3 v_2) = \frac{c_2 n_2}{D_2}$$



$$= \frac{i}{2}\,\frac{\tilde{f}^{abc}T^c_{i\bar{\jmath}}}{s_{12}}\Big(2(k_4\cdot\varepsilon_3)(\bar{u}_1\not{\varepsilon}_4 v_2) - 2(k_3\cdot\varepsilon_4)(\bar{u}_1\not{\varepsilon}_3 v_2) + (\varepsilon_3\cdot\varepsilon_4)(\bar{u}_1(\not{k}_3-\not{k}_4)v_2)\Big) = \frac{c_3 n_3}{D_3}$$

$c_1 - c_2 = c_3$ — commutation relation

$$n_1 - n_2 - n_3 \propto \bar{u}_1\not{k}_1\not{\varepsilon}_3\not{\varepsilon}_4 v_2 + \bar{u}_1\not{\varepsilon}_3\not{\varepsilon}_4\not{k}_2 v_2 - (\varepsilon_3\cdot\varepsilon_4)(\bar{u}_1(\not{k}_1+\not{k}_2)v_2) = 0$$

BCJ relation at 4 points

Start with DDM color decomposition:

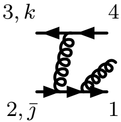
$$\mathcal{A}_{4,1}^{\text{tree}} = \frac{c_1 n_1}{D_1} + \frac{c_2 n_2}{D_2} + \frac{c_3 n_3}{D_3} = c_2 A_{\underline{1}\overline{2}34} + c_1 A_{\underline{1}\overline{2}43}$$

$$\begin{cases} A_{\underline{1}\overline{2}34} = \dfrac{n_2}{D_2} - \dfrac{n_3}{D_3} \\ A_{\underline{1}\overline{2}43} = \dfrac{n_1}{D_1} + \dfrac{n_3}{D_3} \\ n_1 - n_2 = n_3 \end{cases} \quad \Rightarrow \quad \boxed{(s_{14} - m^2) A_{\underline{1}\overline{2}34} = (s_{13} - m^2) A_{\underline{1}\overline{2}43}}$$

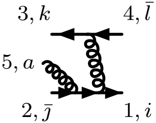
Minimal decomposition:

$$\mathcal{A}_{4,1}^{\text{tree}} = \left(T^{a_3}_{\bar{\imath}_2 j} T^{a_4}_{\bar{\jmath} i_1} + T^{a_4}_{\bar{\imath}_2 j} T^{a_3}_{\bar{\jmath} i_1} \frac{s_{14} - m^2}{s_{13} - m^2} \right) A_{\underline{1}\overline{2}34}$$

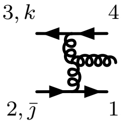
Color-kinematics at 5 points



$$5, a = \frac{i}{2\sqrt{2}} \frac{1}{(s_{15} - m_1^2) s_{34}} T^a_{i\bar{m}} T^b_{m\bar{j}} T^b_{k\bar{l}} (\bar{u}_1 \not{\varepsilon}_5 (\not{k}_{1,5} + m_1) \gamma^\mu v_2)(\bar{u}_3 \gamma_\mu v_4) = \frac{c_1 n_1}{D_1}$$



$$= \frac{-i}{2\sqrt{2}} \frac{1}{(s_{25} - m_2^2) s_{34}} T^b_{i\bar{m}} T^a_{m\bar{j}} T^b_{k\bar{l}} (\bar{u}_1 \gamma^\mu (\not{k}_{2,5} - m_2) \not{\varepsilon}_5 v_2)(\bar{u}_3 \gamma_\mu v_4) = \frac{c_2 n_2}{D_2}$$



$$5, a = \frac{i}{\sqrt{2}} \frac{1}{s_{12} s_{34}} \tilde{f}^{abc} T^c_{i\bar{j}} T^b_{k\bar{l}} \Big((\bar{u}_1 \not{\varepsilon}_5 v_2)(\bar{u}_3 \not{k}_5 v_4) - (\bar{u}_1 \not{k}_5 v_2)(\bar{u}_3 \not{\varepsilon}_5 v_4) - (\bar{u}_1 \gamma^\mu v_2)(\bar{u}_3 \gamma_\mu v_4)(k_{12} \cdot \varepsilon_5) \Big) = \frac{c_5 n_5}{D_5}$$

$$c_1 - c_2 = c_5 \qquad c_3 - c_4 = -c_5$$

$$n_1 - n_2 = n_5 \qquad n_3 - n_4 = -n_5$$

$$\Rightarrow \quad (s_{25} - m_2^2) A_{\underline{1}\bar{2}5\underline{3}\bar{4}} + (s_{14} - s_{23}) A_{\underline{1}\bar{2}\underline{3}5\bar{4}} - (s_{15} - m_1^2) A_{\underline{1}\bar{2}\underline{3}\bar{4}5} = 0$$

BCJ relations for QCD

number of independent BCJ relations:

$k \setminus n$	3	4	5	6	7	8
0	0	1	4	18	96	600
1	0	1	4	18	96	600
2	-	0	1	6	36	240
3	-	-	-	0	4	40
4	-	-	-	-	-	0

purely gluonic: $$\sum_{i=2}^{n-1} \left(\sum_{j=2}^{i} s_{jn} \right) A(1, 2, \ldots i, n, i+1, \ldots, n-1) = 0$$

cannot all stay true

in fact $$\sum_{i=2}^{n-1} \left(\sum_{j=2}^{i} s_{jn} - m_j^2 \right) A(1, 2, \ldots i, n, i+1, \ldots, n-1) = 0$$

where n is a gluon

proven via BCFW by De la Cruz, Kniss, Weinzierl (2015)

Solution to BCJ relations for QCD

General BCJ relations:

$$A(\underline{1},\overline{2},\alpha,\underline{q},\beta) = \sum_{\sigma\in S(\alpha)\sqcup\beta} A(\underline{1},\overline{2},\underline{q},\sigma)\prod_{i=1}^{|\alpha|}\frac{\mathcal{F}(q,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i}-m_2^2},$$

where α is purely gluonic

Melia basis of $(n-2)!/k!$ primitives

$$\left\{A(\underline{1},\overline{2},\sigma)\,\middle|\,\sigma\in\{\text{quark brackets}\}_{k-1}\times\{\text{gluon insertions}\}_{n-2k}\right\}$$

$\Rightarrow$ new BCJ basis of $(n-3)!(2k-2)/k!$ primitives

$$\left\{A(\underline{1},\overline{2},\underline{q},\sigma)\,\middle|\,\{\underline{q},\sigma\}\in\{\text{quark brackets}\}_{k-1}\times\{\text{gluon insertions in }\sigma\}_{n-2k}\right\}$$

Minimal amplitude decomposition

Full color-dressed amplutude in terms of only
$(n-3)!(2k-2)/k!$ color-ordered primitive amplitudes:

$$\mathcal{A}^{\text{tree}}_{n,k\geq 2} = \sum_{(\underline{q},\sigma)\in \text{BCJ basis}} A(\underline{1},\overline{2},\underline{q},\sigma)$$

$$\times\left\{ C(\underline{1},\overline{2},\underline{q},\sigma) + \sum_{\substack{\beta\subset\sigma \\ \sigma\setminus\beta \text{ gluonic}}} \sum_{\alpha\in S(\sigma\setminus\beta)} C(\underline{1},\overline{2},\alpha,\underline{q},\beta) \prod_{i=1}^{|\alpha|} \frac{\mathcal{F}(q,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i}-m_2^2} \right\}$$

$$\mathcal{A}^{\text{tree}}_{n,k\leq 1} = \sum_{\alpha\in S_{n-3}(\{4,\ldots,n\})} A(1,2,3,\sigma)$$

$$\times\left\{ C(1,2,3,\sigma) + \sum_{\beta\subset\sigma} \sum_{\alpha\in S(\sigma\setminus\beta)} C(1,2,\alpha,3,\beta) \prod_{i=1}^{|\alpha|} \frac{\mathcal{F}(3,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i}-m_2^2} \right\}$$

or $(n-3)!$ primitives for $k=0,1$

Conclusions

Summary

- Color-kinematics duality for QCD with or without masses

$$\mathcal{A} = \sum_i C_i \, A_i$$

n particles:	color	kinematics
n gluons only	KK relations $\Rightarrow$ KK basis, $(n-2)!$ $\Rightarrow$ DDM decomposition	BCJ relations $\Rightarrow$ BCJ basis, $(n-3)!$
$(n-2k)$ gluons k quark pairs	KK relations $\Rightarrow$ Melia basis, $(n-2)!/k!$ $\Rightarrow$ new color decomposition	new BCJ relations reduced BCJ basis, $(n-3)!(2k-2)/k!$

Outlook

- Color decomposition for general QCD tree amplitudes
  $\Rightarrow$ systematic full color at loop level
- BCJ relations with distinct massive particles
  $\Rightarrow$ applications to loop calculations in QCD
- Double copy with quark/scalar masses
  $\Rightarrow$ identify resulting theories
- Prescription for amplitude integrands in pure gravities
  $\Rightarrow$ revisit the UV divergence structure of pure gravities,
  expected subtlety: chiral fermions in D dimensions,
  relation to evanescent effects?

Bern, Cheung, Chi, Davies, Dixon, Nohle (2015)
more in Chiodaroli's, Bern's talks

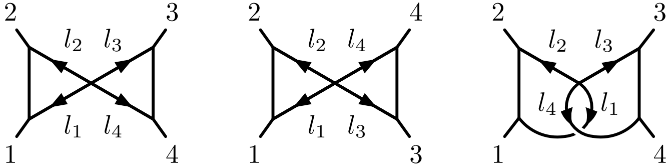
Thank you!

Backup slides

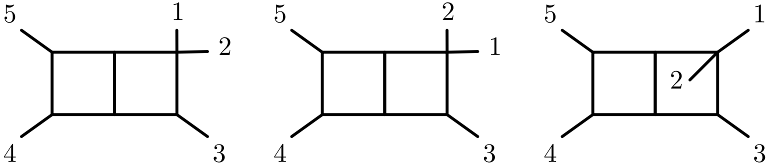
DDM decomposition applied to loops

$$A(1,2,3,4) + A(1,2,4,3) + A(1,4,2,3) = 0$$

4 points, 2 loops:



5 points, 2 loops:



BCJ relations

Bern, Carrasco, Johansson (2008)

$$\mathcal{A}_{n,0}^{\text{tree}} = \sum_{\text{cubic graphs } \Gamma_i} \frac{c_i n_i}{D_i}$$

$$c_i - c_j = c_k \quad \Leftrightarrow \quad n_i - n_j = n_k$$

$$\Rightarrow$$

$$\sum_{i=2}^{n-1} \Big(\sum_{j=2}^{i} s_{jn} \Big) A(1,2,\ldots,i,n,i+1,\ldots,n-1) = 0$$

Existence of such numerators is enough!

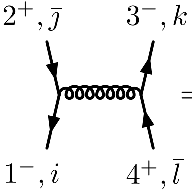
All other BCJ relations can be derived from relabelings thereof.

Feng, Huang, Jia (2010)

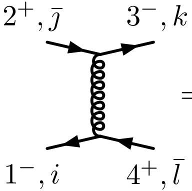
$$\Rightarrow$$

basis of $(n-3)!$ primitives

Trivial tree-level example



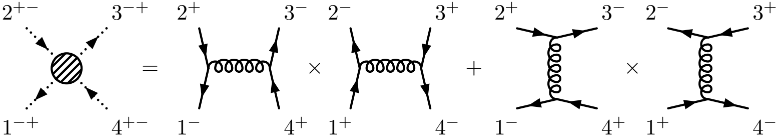
$$= -i\, T^a_{i\bar{\jmath}} T^a_{k\bar{l}} \frac{\langle 13\rangle [24]}{s} = \frac{c_1 n_1}{D_1}$$



$$= -i\, T^a_{i\bar{l}} T^a_{k\bar{\jmath}} \frac{\langle 13\rangle [24]}{t} = \frac{c_2 n_2}{D_2}$$

no algebraic relations among diagrams

Double-copy at 4 and 5 points



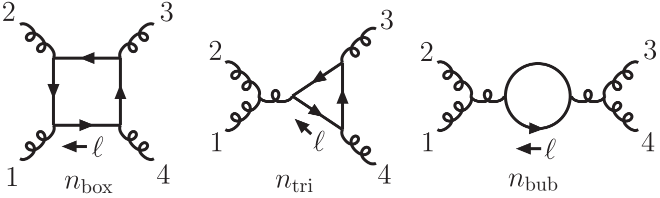
$$\mathcal{M}_4^{\text{tree}}(1_\phi^{-+}, 2_\phi^{+-}, 3_\phi^{-+}, 4_\phi^{+-}) = i\left\{\frac{n_1\overline{n}_1}{s} + \frac{n_2\overline{n}_2}{t}\right\} = \frac{iu^3}{st}$$

ϕ^{+-} and ϕ^{-+} are mixed dilaton-axion states

$$\mathcal{M}_5^{\text{tree}}(1_\phi^{-+}, 2_\phi^{+-}, 3_\phi^{-+}, 4_\phi^{+-}, 5_G^{++}) = i\sum_{i=1}^{10}\frac{n_i\overline{n}_i}{D_i}$$

- We get: $\psi^+ \otimes \psi^- \Rightarrow \phi^{+-}$ $\psi^- \otimes \psi^+ \Rightarrow \phi^{-+}$
- Usually: $A^+ \otimes A^- \Rightarrow \phi^{+-}$ $A^- \otimes A^+ \Rightarrow \phi^{-+}$

One loop calculations for gauge theory



Particle content:

helicity	1	1/2	0	−1/2	−1
$\mathcal{N}=1$ chiral (adjoint)	0	1	2	1	0
$\mathcal{N}=1$ chiral (fund.)	0	1	1	0	0
$\mathcal{N}=1$ chiral (antifund.)	0	0	1	1	0
fundamental fermion	0	1	0	0	0
antifundamental fermion	0	0	0	1	0
complex scalar	0	0	2	0	0

after integration analytic agreement with Bern, Morgan (1995)

Carrasco, Chiodaroli, Günaydin, Jin, Roiban (2012-13)

Bern, Davies, Dennen, Huang, Nohle (2013)

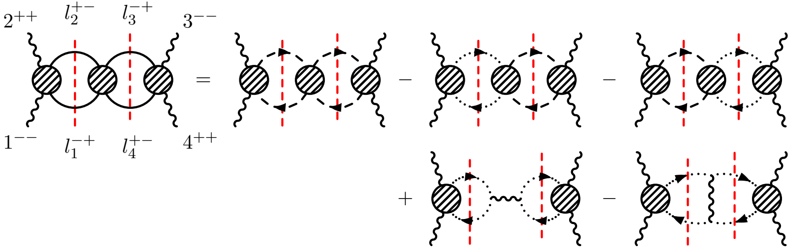
One-loop calculations for gravity

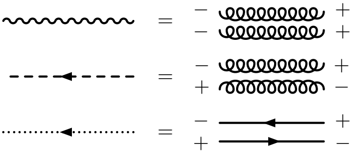
SUGRA	tensoring vector states	ghosts = matter $\otimes$ $\overline{\text{matter}}$
$\mathcal{N} = 0 + 0$	$A^\mu \otimes A^\nu \to h^{\mu\nu}, \phi, a$	$(\phi \otimes \overline{\phi}) \oplus (\overline{\phi} \otimes \phi) \to 2\phi$ or $(\psi^+ \otimes \psi^-) \oplus (\psi^- \otimes \psi^+) \to \phi, a$
$\mathcal{N} = 1 + 0$	$\mathcal{V}_{\mathcal{N}=1} \otimes A^\mu \to G_{\mathcal{N}=1}, \Phi_{\mathcal{N}=2}$	$(\Phi_{\mathcal{N}=1} \otimes \overline{\phi}) \oplus (\overline{\Phi}_{\mathcal{N}=1} \otimes \phi) \to \Phi'_{\mathcal{N}=2}$ or $(\Phi_{\mathcal{N}=1} \otimes \psi^-) \oplus (\overline{\Phi}_{\mathcal{N}=1} \otimes \psi^+) \to \Phi_{\mathcal{N}=2}$
$\mathcal{N} = 2 + 0$	$\mathcal{V}_{\mathcal{N}=2} \otimes A^\mu \to G_{\mathcal{N}=2}, \mathcal{V}_{\mathcal{N}=2}$	$(\Phi_{\mathcal{N}=2} \otimes \psi^-) \oplus (\overline{\Phi}_{\mathcal{N}=2} \otimes \psi^+) \to \mathcal{V}_{\mathcal{N}=2}$
$\mathcal{N} = 1 + 1$	$\mathcal{V}_{\mathcal{N}=1} \otimes \mathcal{V}_{\mathcal{N}=1} \to G_{\mathcal{N}=2}, 2\Phi_{\mathcal{N}=2}$	$(\Phi_{\mathcal{N}=1} \otimes \overline{\Phi}_{\mathcal{N}=1}) \oplus (\overline{\Phi}_{\mathcal{N}=1} \otimes \Phi_{\mathcal{N}=1}) \to 2\Phi_{\mathcal{N}=2}$
$\mathcal{N} = 2 + 1$	$\mathcal{V}_{\mathcal{N}=2} \otimes \mathcal{V}_{\mathcal{N}=1} \to G_{\mathcal{N}=3}, \mathcal{V}_{\mathcal{N}=4}$	$(\Phi_{\mathcal{N}=2} \otimes \overline{\Phi}_{\mathcal{N}=1}) \oplus (\overline{\Phi}_{\mathcal{N}=2} \otimes \Phi^+_{\mathcal{N}=1}) \to \mathcal{V}_{\mathcal{N}=4}$
$\mathcal{N} = 2 + 2$	$\mathcal{V}_{\mathcal{N}=2} \otimes \mathcal{V}_{\mathcal{N}=2} \to G_{\mathcal{N}=4}, 2\mathcal{V}_{\mathcal{N}=4}$	$(\Phi_{\mathcal{N}=2} \otimes \overline{\Phi}_{\mathcal{N}=2}) \oplus (\overline{\Phi}_{\mathcal{N}=2} \otimes \Phi_{\mathcal{N}=2}) \to 2\mathcal{V}_{\mathcal{N}=4}$

after integration analytic agreement with Dunbar, Norridge (1994)

Two-loop checks on cuts

Matter circulates in both loops:





Melia basis of primitive amplitudes

Melia (2013)

$$\left\{ A(\underline{1}, \overline{2}, \sigma) \;\middle|\; \sigma \in \mathrm{Dyck}_{k-1} \times \{\text{gluon insertions}\}_{n-2k} \right\}$$

Melia basis for $n = 5$, $k = 2$:

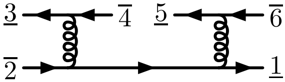
$$A(\underline{1}, \overline{2}, 5, \underline{3}, \overline{4}) \,, A(\underline{1}, \overline{2}, \underline{3}, 5, \overline{4}) \,, A(\underline{1}, \overline{2}, \underline{3}, \overline{4}, 5)$$

$$\varkappa(n,k) = \underbrace{\overbrace{\frac{(2k-2)!}{k!(k-1)!}}^{\text{empty brackets}} \times (k-1)!}_{\text{dressed quark brackets}} \times \underbrace{(2k-1)(2k)\ldots(n-2)}_{\text{insertions of } (n-2k) \text{ gluons}} = \frac{(n-2)!}{k!}$$

6-point color example

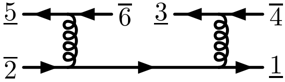
$$\mathcal{A}_{6,3}^{\text{tree}} = C_{\underline{1}\overline{2}\underline{3}\overline{4}\underline{5}\overline{6}}\, A_{\underline{1}\overline{2}\underline{3}\overline{4}\underline{5}\overline{6}} + C_{\underline{1}\overline{2}\underline{5}\overline{6}\underline{3}\overline{4}}\, A_{\underline{1}\overline{2}\underline{5}\overline{6}\underline{3}\overline{4}} + C_{\underline{1}\overline{2}\underline{35}\overline{64}}\, A_{\underline{1}\overline{2}\underline{35}\overline{64}} + C_{\underline{1}\overline{2}\underline{53}\overline{46}}\, A_{\underline{1}\overline{2}\underline{53}\overline{46}}$$

$C_{\underline{1}\overline{2}\underline{3}\overline{4}\underline{5}\overline{6}} =$



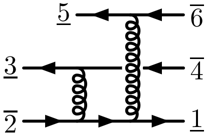
$= \{2|\{3|T^a \otimes \Xi_1^a|4\}\{5|T^b \otimes \Xi_1^b|6\}|1\}$

$C_{\underline{1}\overline{2}\underline{5}\overline{6}\underline{3}\overline{4}} =$

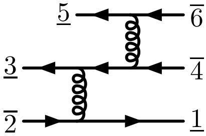


$= \{2|\{5|T^a \otimes \Xi_1^a|6\}\{3|T^b \otimes \Xi_1^b|4\}|1\}$

$C_{\underline{1}\overline{2}\underline{35}\overline{64}} =$

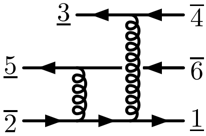


$+$

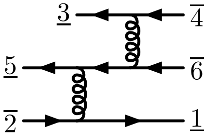


$= \{2|\{3|T^a \otimes \Xi_1^a\{5|T^b \otimes \Xi_2^b|6\}|4\}|1\}$

$C_{\underline{1}\overline{2}\underline{53}\overline{46}} =$



$+$



$= \{2|\{5|T^a \otimes \Xi_1^a\{3|T^b \otimes \Xi_2^b|4\}|6\}|1\}$

Theory multiplication table

Factorizable pure gravities:

$$(\mathcal{N}=4 \text{ SYM}) \times (\mathcal{N}=4 \text{ SYM}) = (\mathcal{N}=8 \text{ SUGRA})$$
$$(\mathcal{N}=4 \text{ SYM}) \times (\mathcal{N}=2 \text{ SYM}) = (\mathcal{N}=6 \text{ SUGRA})$$
$$(\mathcal{N}=4 \text{ SYM}) \times (\mathcal{N}=1 \text{ SYM}) = (\mathcal{N}=5 \text{ SUGRA})$$
$$(\mathcal{N}=4 \text{ SYM}) \times (\text{ Yang-Mills }) = (\mathcal{N}=4 \text{ SUGRA})$$

Factorizable non-pure gravities:

$$(\mathcal{N}=2 \text{ SYM}) \times (\mathcal{N}=2 \text{ SYM}) = (\mathcal{N}=4 \text{ SUGRA}) + 2 \times (\mathcal{N}=4 \text{ matter})$$
$$(\mathcal{N}=2 \text{ SYM}) \times (\mathcal{N}=1 \text{ SYM}) = (\mathcal{N}=3 \text{ SUGRA}) + (\mathcal{N}=4 \text{ matter})$$
$$(\mathcal{N}=2 \text{ SYM}) \times (\text{ Yang-Mills }) = (\mathcal{N}=2 \text{ SUGRA}) + (\mathcal{N}=2 \text{ matter})$$
$$(\mathcal{N}=1 \text{ SYM}) \times (\mathcal{N}=1 \text{ SYM}) = (\mathcal{N}=2 \text{ SUGRA}) + 2 \times (\mathcal{N}=2 \text{ matter})$$
$$(\mathcal{N}=1 \text{ SYM}) \times (\text{ Yang-Mills }) = (\mathcal{N}=1 \text{ SUGRA}) + (\mathcal{N}=2 \text{ matter})$$
$$(\text{ Yang-Mills }) \times (\text{ Yang-Mills }) = \text{Einstein gravity} + \text{dilaton} + \text{axion}$$

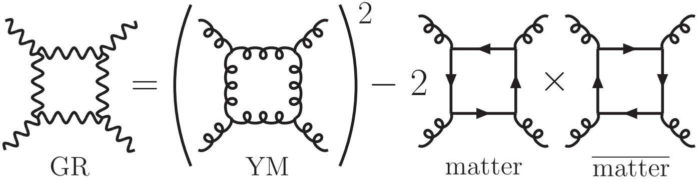
Schematic depiction of ghost prescription for amplitudes in pure gravity

Johansson, AO (2014)

$N_{\rm M} + 1 =$ number of complex matter multiplets

$N_{\rm M} = -1 \qquad \Rightarrow \qquad$ pure gravity

$$\mathcal{M}_m^{L\text{-loop}} = i^{L+1} \sum_{i \in \text{cubic}} \int \frac{d^{LD}\ell}{(2\pi)^{DL}} \frac{(-1)^{|i|}}{S_i} \frac{n_i \overline{n}_i'}{D_i}$$



Einstein gravity = (Yang-Mills)^2 − (dilaton + axion)

UV properties of gravity

Naively, SUSY allows R^4 counterterm — to be seen at 3 loops

- $\mathcal{N}=8$ supergravity: valid D^8R^4 counterterm at 7 loops

  Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger (2010)
  works by Kallosh, Ramond, Björnsson, Green, Bossard, Howe, Stelle, Vanhove,
  Lindström, Berkovits, Grisaru, Siegel, Russo, etc.

  state of the art: UV finite at 3 and 4 loops

  Bern, Carrasco, Dixon, Johansson, Kosower, Roiban (2007)
  Bern, Carrasco, Dixon, Johansson, Roiban (2009)

- $\mathcal{N}=4$ supergravity: valid R^4 counterterm at 3 loops

  Bossard, Howe, Stelle, Vanhove (2010-11)

  state of the art: UV finite at 3 loops, divergent at 4 loops

  Tourkine, Vanhove (2012)
  Bern, Davies, Dennen, Huang (2012)
  Bern, Davies, Dennen, Smirnov, Smirnov (2013)

- pure $\mathcal{N}<4$ supergravities: lower-loop divergences expected
  but less accessible by the BCJ approach

- Einstein gravity: UV finite at 1 loop, divergent at 2 loops

  Goroff, Sagnotti (1985)
  van de Ven (1991)
  Bern, Cheung, Chi, Davies, Dixon, Nohle (2015)

(pure YM) $\times$ (pure YM) = Einstein gravity + dilaton + axion