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THE 3-LOOP SOFT ANOMALOUS DIMENSION IN MULTI-LEG AMPLITUDES

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The 3-loop soft anomalous dimension in multi-leg scattering

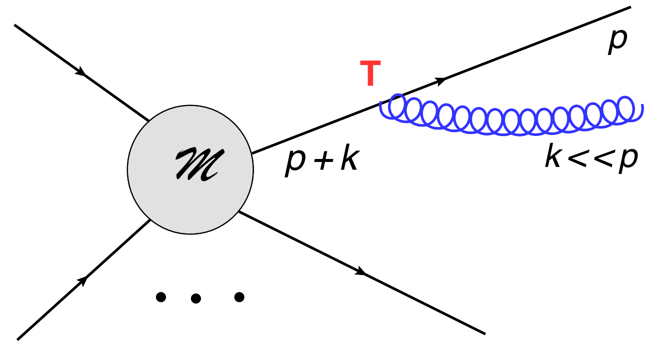
Plan of the talk

- Soft singularities from Wilson lines: fixed-angle factorization and rescaling symmetry.
- The soft anomalous dimension for massless partons: *the dipole formula*.
- The complete 3-loop soft anomalous dimension.
- Calculation of connected webs in near light-like kinematics.
- Colour conservation.
- Special kinematics: collinear limit, Regge limit.

The soft (eikonal) approximation and rescaling symmetry

Eikonal Feynman rules:

Assuming $k \ll p$
(all components of k are small):



$$\bar{u}(p)\left(-\mathrm{i}g_s T^{(a)}\gamma^\mu\right)\frac{\mathrm{i}(\not{p}+\not{k}+m)}{(p+k)^2-m^2+\mathrm{i}\varepsilon} \longrightarrow \bar{u}(p)g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon}$$

Rescaling invariance: soft gluon emission only depends on the direction and colour charge of the emitter.

$$g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon} = g_s T^{(a)}\frac{\beta^\mu}{\beta\cdot k+\mathrm{i}\varepsilon}$$

equivalent to emission from a Wilson line:

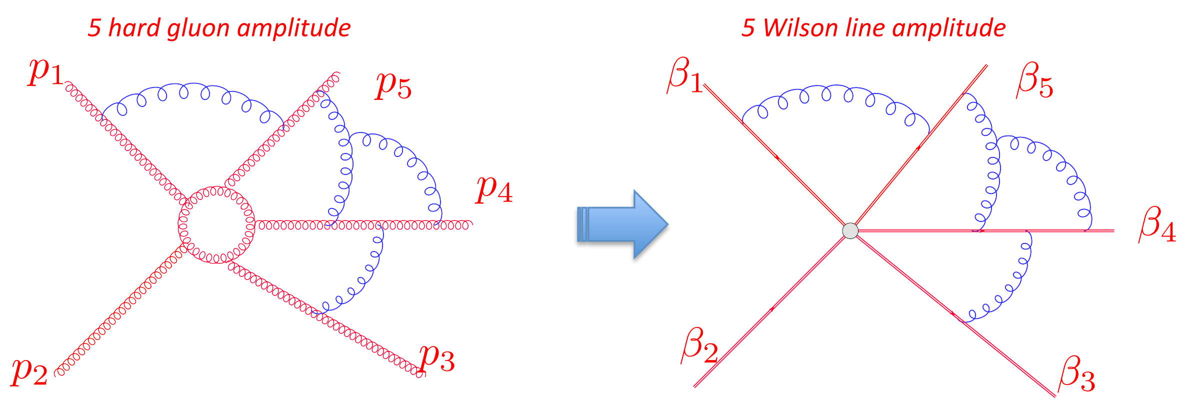
$$\Phi_{\beta_i}(\infty,0) \equiv P\exp\left\{\mathrm{i}g_s\int_0^\infty d\lambda\beta\cdot A(\lambda\beta)\right\}$$

IR singularities from Wilson lines

<u>Factorization at fixed angles:</u>

all kinematic invariants are simultaneously taken large $p_i \cdot p_j = Q^2 \beta_i \cdot \beta_j \gg \Lambda^2$

Soft singularities factorise to all orders & computed from a product of Wilson lines:



$$\mathcal{M}_J(p_i, \epsilon_{\rm IR}) = \sum_K \mathcal{S}_{JK}\left(\gamma_{ij}, \epsilon_{\rm IR}\right) H_K(p_i)$$

$\mathcal{S}$ is a product of Wilson lines: $\mathcal{S} = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$ **— Process Independent!**

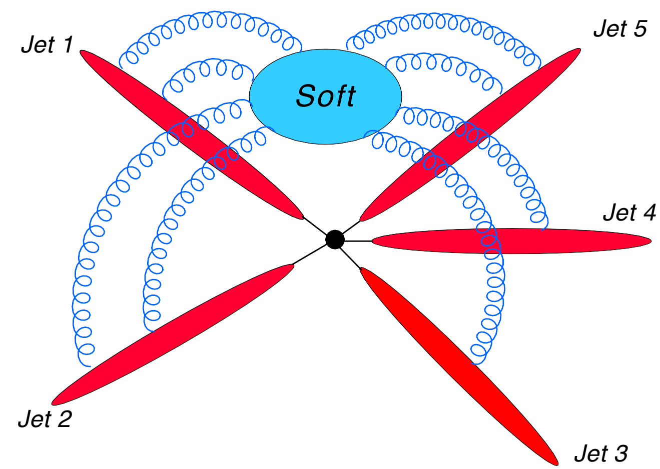
The *soft anomalous dimension* Γ is the logarithmic derivative of $\mathcal{S}$

Due to rescaling symmetry it only depends on angles: $\gamma_{ij} = \frac{2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \, \beta_j^2}}$

Factorization of amplitudes with massless legs

Fixed angle scattering $p_i \cdot p_j = Q^2 \beta_i \cdot \beta_j \gg \Lambda^2$
with **lightlike partons** $p_i^2 = 0$

IR singularities can be factorised
- all originate in soft and collinear regions



Lightlike Wilson lines

Jets (colour singlet)

$$\mathcal{M}_N\left(p_i/\mu,\epsilon\right) = \sum_L \mathcal{S}_{NL}\left(\beta_i\cdot\beta_j,\epsilon\right)\, H_L\left(\frac{2p_i\cdot p_j}{\mu^2},\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}\right)\prod_{i=1}^{n}\frac{J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

The soft function now depends on $\beta_i \cdot \beta_j$, violating rescaling symmetry.
This collinear anomaly is restored by the eikonal jets.
This implies an **all-order constraint on the soft function,**
leading to the **Dipole Formula**.

Becher & Neubert,
EG & Magnea (2009)

IR singularities for amplitudes with massless legs

Solving a renormaliaztion-group equation ⇒ **Exponentiation:**

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s,\epsilon\right) = \exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma\left(\lambda^2/s_{ij},\alpha_s(\lambda^2,\epsilon)\right)\right\}\mathcal{H}\left(\frac{p_i}{\mu},\alpha_s\right)$$

The Dipole Formula:

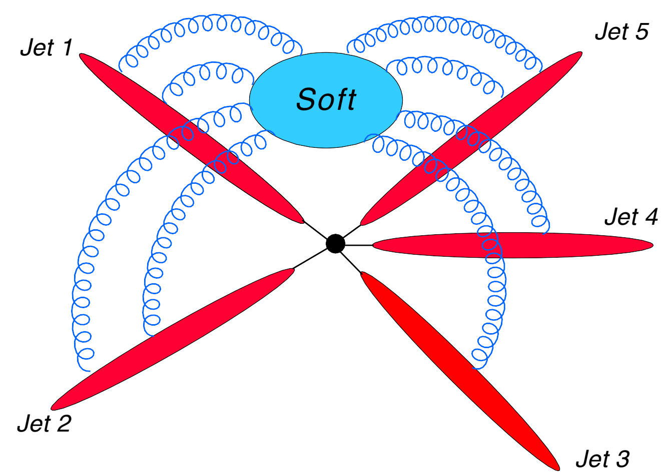
simple ansatz for the singularity structure of multi-leg massless amplitudes

$$\Gamma_{\rm Dip.}(\lambda,\alpha_s) = \frac{1}{4}\,\widehat{\gamma}_K\left(\alpha_s\right)\sum_{(i,j)}\ln\left(\frac{\lambda^2}{-s_{ij}}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s\right)$$

Lightlike Cusp anomalous dimension

Complete two-loop calculation by Dixon, Mert-Aybat and Sterman in 2006 (confirming Catani's predictions from 1998).

Generalization to all orders motivated by constraints based on **soft/jet factorisation** and **rescaling symmetry.** Becher & Neubert, EG & Magnea (2009)



CORRECTIONS TO THE DIPOLE FORMULA

There are two types of possible **corrections to the dipole formula**:

1. Corrections induced by higher Casimir contributions to the cusp anom. dim — starting at 4 loops.
2. Functions of **conformally-invariant cross ratios** — starting at 3-loops:

$$\Gamma = \Gamma_{\text{Dip.}} + \Delta(\rho_{ijkl}) \qquad\qquad \rho_{ijkl} = \frac{(p_i \cdot p_j)(p_k \cdot p_l)}{(p_i \cdot p_k)(p_j \cdot p_l)}$$

Constraints on $\Delta(\rho_{ijkl})$:

Non-Abelian exponentiation theorem [EG, Smillie, White (2013)] **implies that $\Delta(\rho_{ijkl})$ has *fully connected* colour factors, such as** $f^{abe} f^{cde} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d$

Bose symmetry
Transcendental weight
Collinear limits
Regge limit

EG & Magnea, Becher & Neubert (2009)
Dixon, EG & Magnea (2010)
Del Duca, Duhr, EG, Magnea & White (2011)
Ahrens & Neubert &Vernazza (2012)
Caron-Huot (2013)

The complete 3-loop correction to the dipole formula

Ø. Almelid, C. Duhr, EG 1507.00047 (v2)

$$\Delta(z,\bar z) = 16\left(\frac{\alpha_s}{4\pi}\right)^3 f_{abe}f_{cde}\Bigg\{\sum_{1\le i<j<k<l\le n}\Big[\ \mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\left(F\left(1-1/z\right)-F\left(1/z\right)\right)$$

$$+\mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\left(F(1-z)-F(z)\right)$$

$$+\mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\left(F\left(1/(1-z)\right)-F\left(1-1/(1-z)\right)\right)\Big]$$

$$-\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n\\ j,k\ne i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\left(\zeta_5+2\zeta_2\zeta_3\right)\Bigg\}$$

$$F(z)=\mathcal{L}_{10101}(z)+2\zeta_2\left(\mathcal{L}_{100}(z)+\mathcal{L}_{001}(z)\right)$$

$$\rho_{1234}=z\bar z$$

$$\rho_{1432}=(1-z)(1-\bar z)$$

$\mathcal{L}_{10\ldots}(z)$ are the single-valued harmonic polylogarithms introduced by Francis Brown in 2009. They are single-valued in the region where $\bar z = z^*$

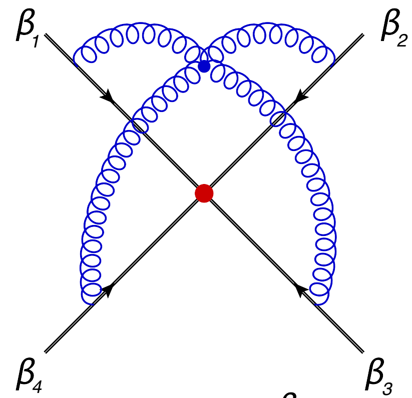
Symbol alphabet: $\{z,\bar z,1-z,1-\bar z\}$ — crossing particles between initial and final state involves taking monodromies around $z=\{0,1,\infty\}$

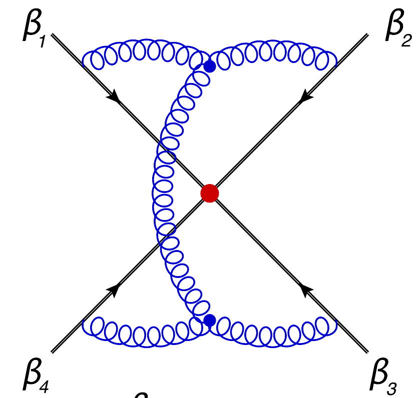
Computing IR singularities at 3-loops

Classes of three-loop webs connecting four Wilson lines

Single connected subgraph

Each web depends on all six angles - **can form conformally-invariant cross ratios** (cicrs)

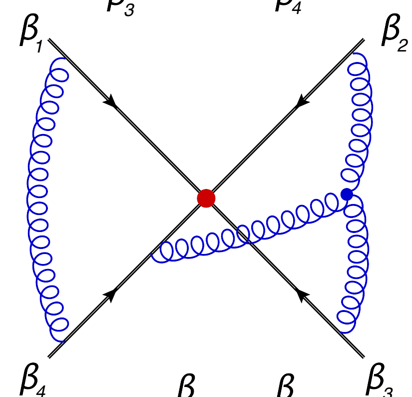




Two connected subgraphs

Depends on $\gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34}$ only.

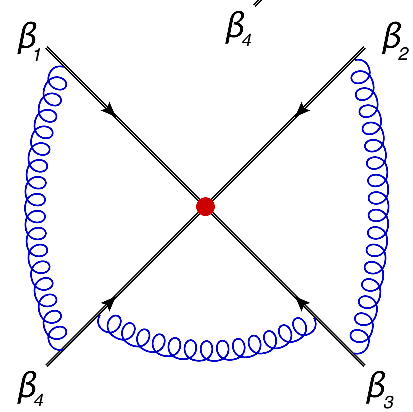
Cannot form cicrs - yields products of logs for near lightlike kinematics

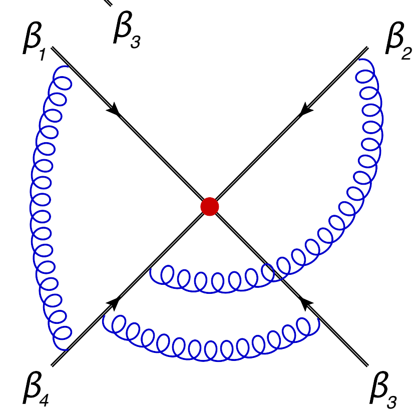


Three connected subgraphs (multiple gluon exchanges)

Depends on 3 angles only!

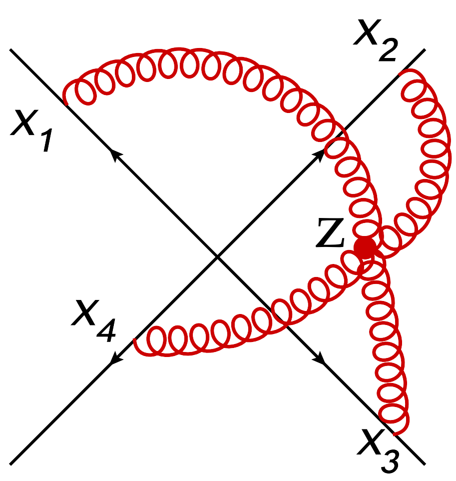
Cannot form cicrs - yields products of logs for near lightlike kinematics

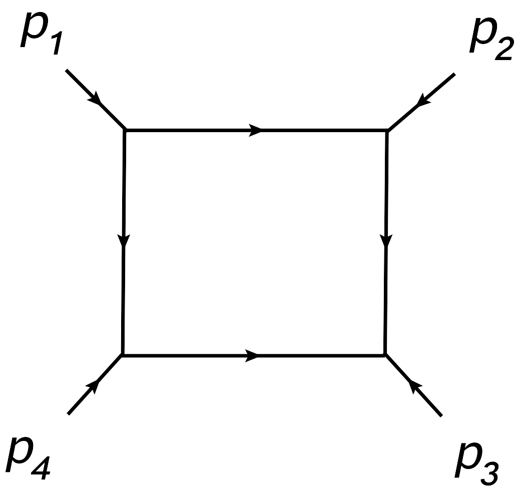




DUAL MOMENTUM BOX INTEGRAL

$W_{4g} =$





Parametrise the positions along the Wilson lines by $x_i^\mu = \beta_i^\mu s_i$

Define auxiliary momenta $p_i = x_i - x_{i-1}$

The z integral is a 4-mass $\text{Box}(p_1, p_2, p_3, p_4)$

$$C_{4g} = T_1^a T_2^b T_3^c T_4^d \Big[f^{abe} f^{cde} (\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23}) + f^{ade} f^{bce} (\gamma_{12}\gamma_{34} - \gamma_{13}\gamma_{24}) + f^{ace} f^{bde} (\gamma_{12}\gamma_{34} - \gamma_{14}\gamma_{23}) \Big]$$

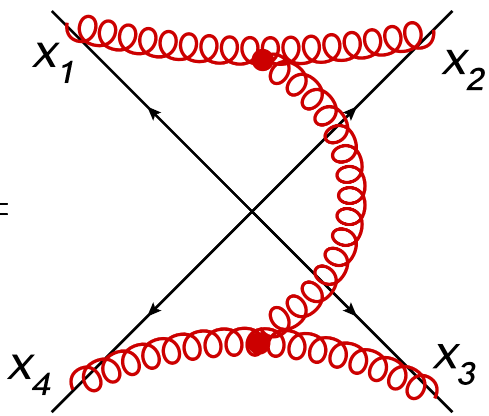
$$W_{4g} = g_s^6 \mathcal{N}^4 C_{4g} \int_0^{\text{``}\infty\text{''}} ds_1 ds_2 ds_3 ds_4 \ \text{Box}(x_1 - x_4, x_2 - x_1, x_3 - x_2, x_4 - x_3)$$

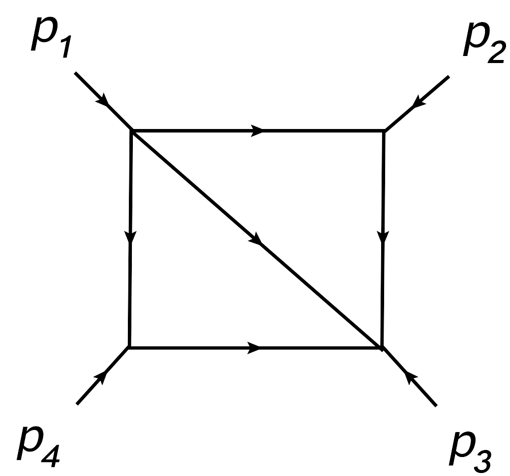
$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \lambda \begin{pmatrix} ca \\ c(1-a) \\ (1-c)b \\ (1-c)(1-b) \end{pmatrix}$$

Integration over λ yields an overall $1/\epsilon$ UV pole.
Remaining integrations can be done in 4 dimensions.

Connected three-loop webs with two 3-gluon vertices

A similar mapping - but with a diagonal box

$W_{(3g)^2} =$ 



We extract the asymptotic near-lightlike behaviour using Mellin-Barnes techniques. The remaining MB integral is three-fold, and can be converted into an iterated parameter integral and expressed in terms of multiple polylogarithms:

W_{4g} and $W_{(3g)^2}$ have non-trivial kinematic dependence in the limit $\beta_i^2 \to 0$

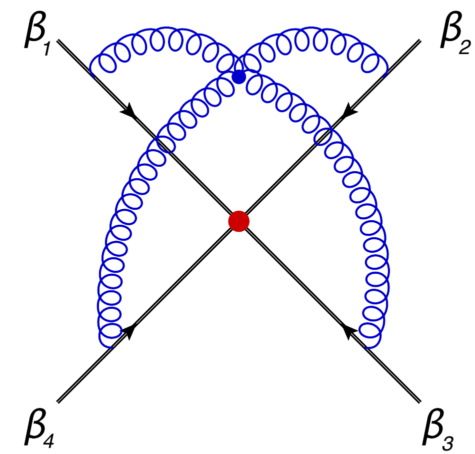
$$\rho_{ijkl} = \frac{\gamma_{ij}\,\gamma_{kl}}{\gamma_{ik}\,\gamma_{jl}} = \frac{(\beta_i \cdot \beta_j)\,(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$$

$$\rho_{1234} = z\bar{z}$$

$$\rho_{1432} = (1-z)(1-\bar{z})$$

Connected webs: results and Bose symmetry

$$w_{4g}^{(3,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \Big[f^{abe} f^{cde} \left(z\bar{z} - z - \bar{z}\right) + f^{ade} f^{bce} \left(1 - z\bar{z}\right) + f^{ace} f^{bde} \left(1 - z - \bar{z}\right) \Big] \frac{1}{z - \bar{z}}\, g_1(z, \bar{z}, \{\gamma_{ij}\})$$



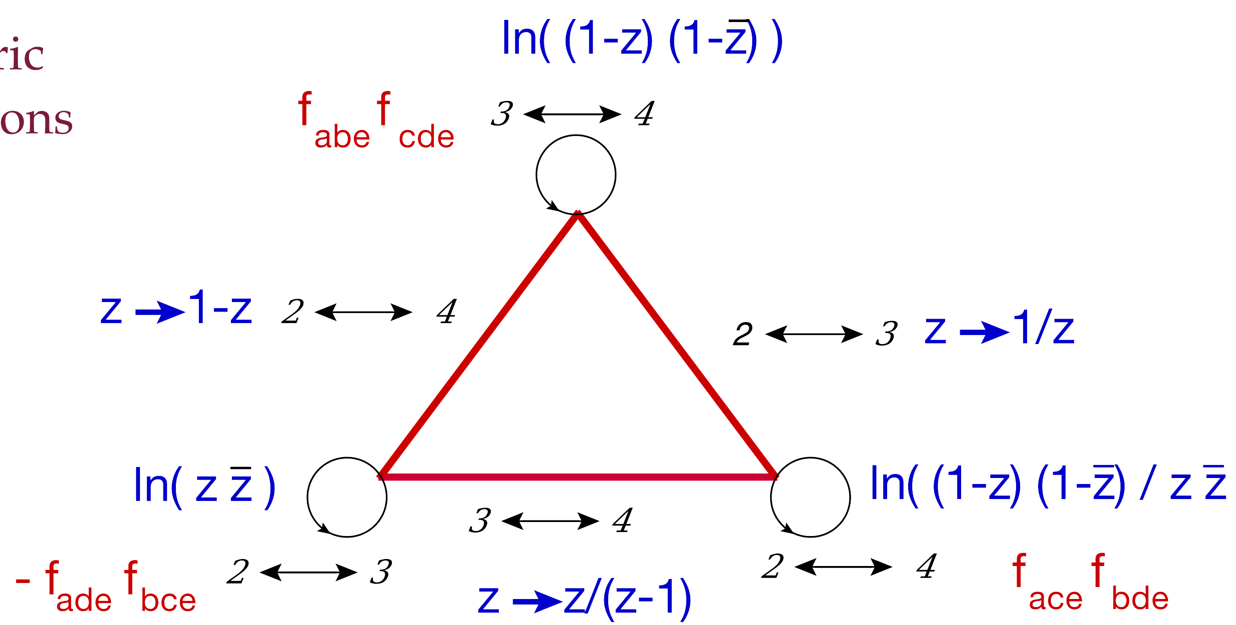
$$\rho_{ijkl} = \frac{\gamma_{ij}\,\gamma_{kl}}{\gamma_{ik}\,\gamma_{jl}} = \frac{(\beta_i \cdot \beta_j)\,(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$$

$$\rho_{1234} = z\bar{z}$$

$$\rho_{1432} = (1 - z)(1 - \bar{z})$$

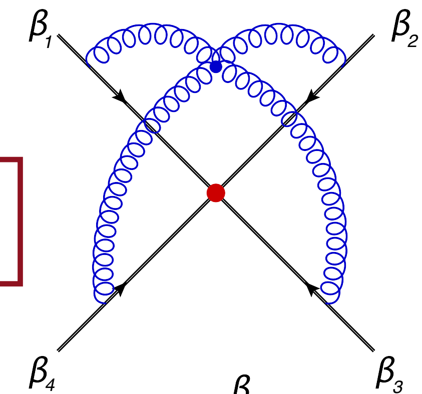
The permutation symmetry of the colour factors is mapped onto the kinematics

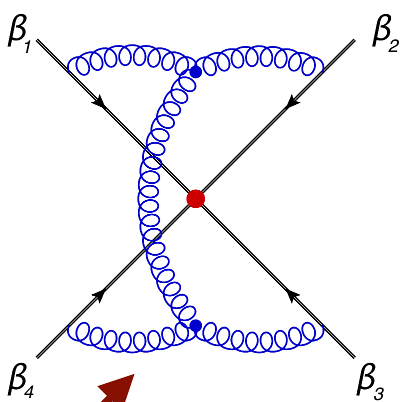
$g_1(z, \bar{z}, \{\gamma_{ij}\})$ is symmetric under these transformations

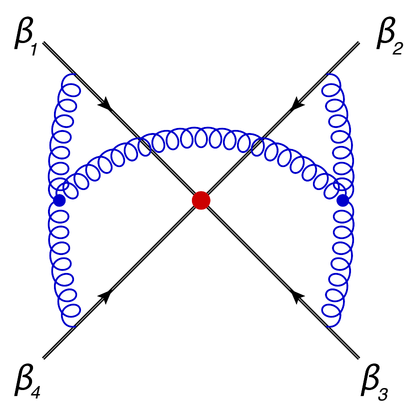


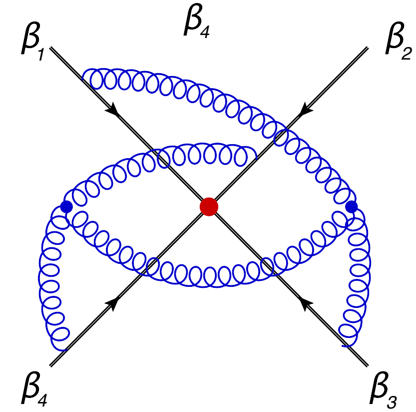
SUMMING THE CONNECTED WEBS RESULTS

$$w_{4g}^{(3,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\Big[f^{abe}f^{cde}\,(z\bar{z}-z-\bar{z})$$
$$+ f^{ade}f^{bce}\,(1-z\bar{z}) + f^{ace}f^{bde}\,(1-z-\bar{z})\Big]\boxed{\frac{1}{z-\bar{z}}\,g_1(z,\bar{z},\{\gamma_{ij}\})}$$









$$w_{(12)(34)}^{(3,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\; f^{abe}f^{cde}\Big[g_0(z,\bar{z},\{\gamma_{ij}\}) - \boxed{\frac{z\bar{z}-z-\bar{z}}{z-\bar{z}}g_1(z,\bar{z},\{\gamma_{ij}\})}\Big]$$

cancels in the sum!

We obtain a pure function of uniform weight 5 ($\mathcal{N}$=4 SYM property).

How is rescaling symmetry realised in the lightlike limit?

After applying Jacobi Identity one finds

$$w_{\text{con.}}^{(3,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \Big[f^{ade}f^{bce}\;\mathcal{F}_1{}^{\text{con.}}(z,\bar{z},\{\gamma_{ij}\}) + f^{abe}f^{cde}\;\mathcal{F}_2{}^{\text{con.}}(z,\bar{z},\{\gamma_{ij}\})\Big]$$

and the functions **separate:**

$$\mathcal{F}_n{}^{\text{con.}}(z,\bar{z},\{\gamma_{ij}\}) = \mathcal{F}_n{}^{\text{con.}}(z,\bar{z}) + Q_n^{\text{con.}}(\{\log(\gamma_{ij})\})$$

a **polylogarithmic** function depending on conformally invariant cross ratios.

a function involving purely logarithmic dependence on individual cusp angles.

Rescaling symmetry implies that the quadrupole contribution to the light-like soft anomalous dimension would depend **exclusively** on $\{z,\bar{z}\}$!

So far put aside non-connected webs, and webs connecting fewer than 4 lines. All these, in the light-like asymptotics, **involve only logarithms**, $\ln(\gamma_{ij})$.

Any kinematic dependence which isn't rescaling invariant must cancel out!

Colour conservation

Colour conservation for n Wilson lines: $(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_3+\ldots\mathbf{T}_n)\;|\mathcal{H}\rangle=0$

Considering the diagrams that connect 4 lines

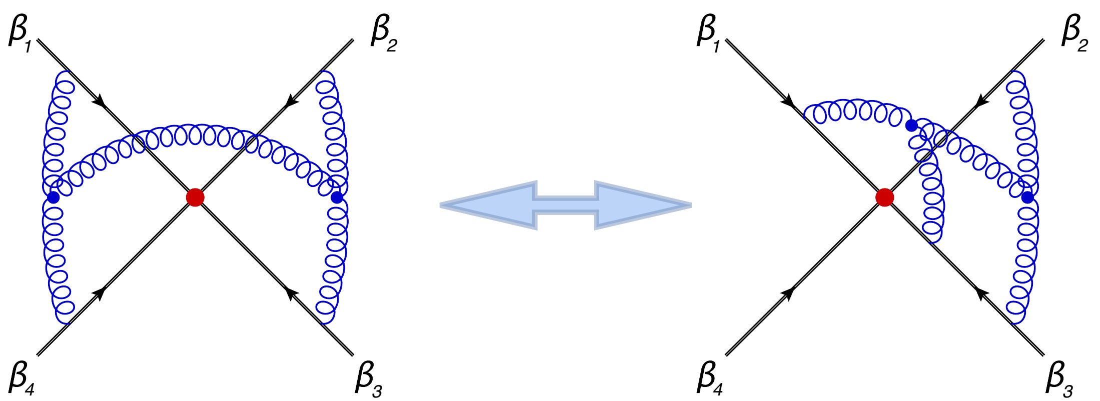
$$G_4(1,2,3,4)=\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\left(f^{abe}f^{cde}H_4[(1,2),(3,4)]+f^{ace}f^{bde}H_4[(1,3),(2,4)]+f^{ade}f^{bce}H_4[(1,4),(2,3)]\right)$$

with permutation symmetry $\quad H_4[(i,j),(k,l)]=-H_4[(j,i),(k,l)]=H_4[(k,l),(i,j)]$

Applying colour conservation to eliminate $\mathbf{T}_4$ — the 4-line result may be expressed as

$$G_4(1,2,3,4)=-\frac{1}{2}f^{abe}f^{cde}\sum_{\substack{(i,j,k)\in(1,2,3)\\ j<k}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\left(H_4[(i,j),(k,4)]+H_4[(i,k),(j,4)]\right)$$

Colour conservation relates 4- and 3-line colour factors:

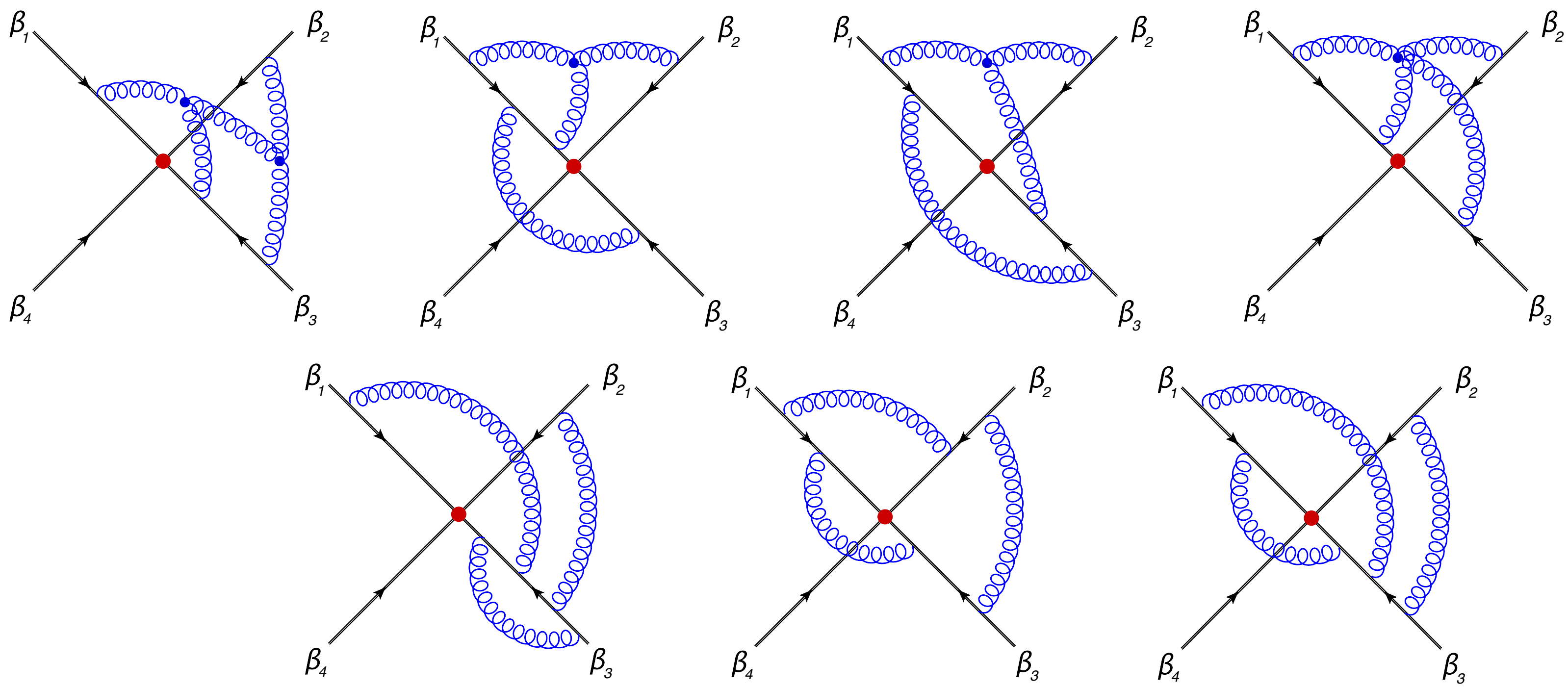


Diagrams connecting fewer Wilson lines are also relevant for Δ_n !

[thanks to Simon Caron-Huot]

Webs with three lines

So we also computed all three-line diagrams:



Colour basis: $f^{abe} f^{cde} \left\{ \mathbf{T}_i^a, \mathbf{T}_i^d \right\} \mathbf{T}_j^b \mathbf{T}_k^c$ → contributes to Δ_n

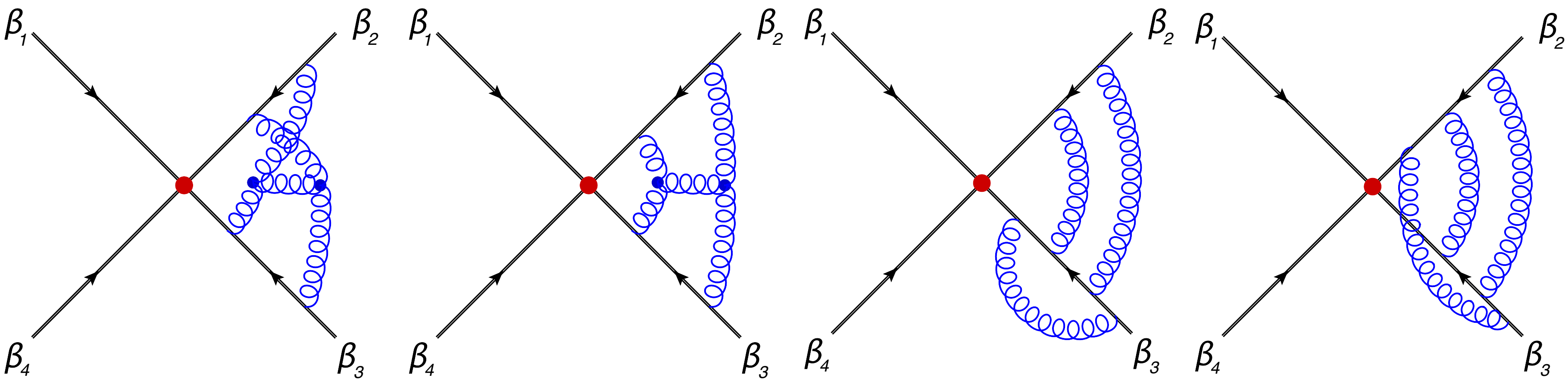
$N_c\, f^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c$ → tripoles do not contribute in the lightlike limit

Sum of all webs connecting lines (1,2,3):

$$G_3(1,2,3) = \textbf{tripole} + f^{abe} f^{cde} \sum_{\substack{(i,j,k)\in(1,2,3) \\ j<k}} \left\{ \mathbf{T}_i^a, \mathbf{T}_i^d \right\} \mathbf{T}_j^b \mathbf{T}_k^c \, H_3[i,\{j,k\}]$$

WEBS WITH TWO LINES

Colour conservation on 3 lines relates to 2 lines, so we also computed 2-line diagrams:



Colour basis: $f^{abe} f^{cde} \{\mathbf{T}_i^a, \mathbf{T}_i^d\} \{\mathbf{T}_j^b, \mathbf{T}_j^c\}$ → contributes to Δ_n

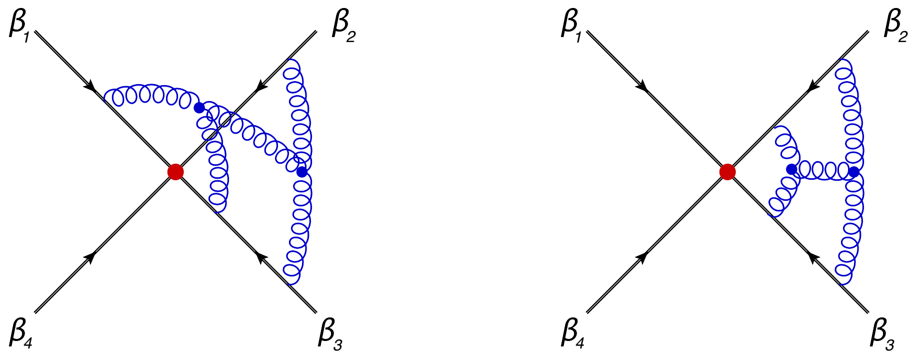
$\mathbf{T}_i \cdot \mathbf{T}_j$ **dipole** → does not contribute

Sum of all webs connecting lines (1,2):

$$G_2(1,2) = \textbf{dipole} - f^{abe} f^{cde} \{\mathbf{T}_1^a, \mathbf{T}_1^d\} \left\{\mathbf{T}_2^b, \mathbf{T}_2^c\right\} H_2(1,2)$$

Colour conservation: contributions from webs connecting 2 or 3 lines

Considering the diagrams that connect **any subset of 2 or 3 lines out of four,**



and eliminating $\mathbf{T}_4$ using $(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4)\,|\mathcal{H}\rangle = 0$ we find

$$G_2(1,2,3,4) + G_3(1,2,3,4) = \textbf{dipoles} + f^{abe} f^{cde} \Bigg[\sum_{\substack{(i,j,k)\in(1,2,3) \\ j<k}} \left\{\mathbf{T}_i^a, \mathbf{T}_i^d\right\} \mathbf{T}_j^b \mathbf{T}_k^c \; \overline{U}(i,\{j,k\},4) - \frac{1}{2} \sum_{1\leq i\leq j\leq 3} \left\{\mathbf{T}_i^a, \mathbf{T}_i^d\right\} \left\{\mathbf{T}_j^b, \mathbf{T}_j^c\right\} \left(\bar{H}_3[i,\{j,4\}] + \bar{H}_3[j,\{4,i\}] + \bar{H}_3[4,\{j,i\}]\right) \Bigg]$$

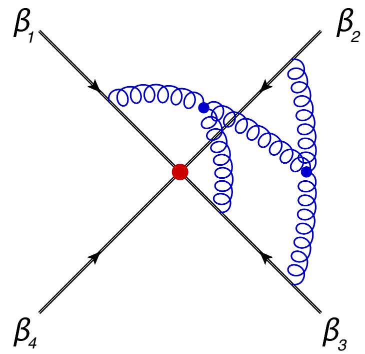
where we defined

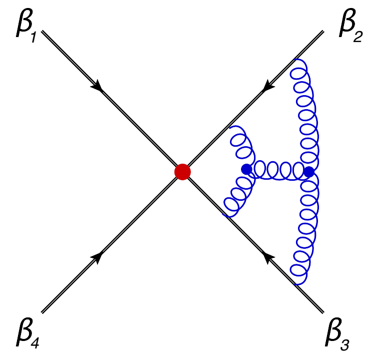
$$\bar{H}_3[i,\{j,k\}] = H_3[i,\{j,k\}] + H_2[\{i,j\}] + H_2[\{i,k\}]$$

$$\overline{U}(i,\{j,k\},4) \equiv \bar{H}_3[i,\{j,k\}] - \bar{H}_3[i,\{j,4\}] - \bar{H}_3[i,\{k,4\}] - \bar{H}_3[4,\{i,j\}] - \bar{H}_3[4,\{i,k\}] + \bar{H}_3[4,\{j,k\}]$$

COLOUR CONSERVATION: CONTRIBUTIONS FROM WEBS CONNECTING 2 OR 3 LINES

Considering the diagrams that connect **any subset of 2 or 3 lines out of four,**





and eliminating $\mathbf{T}_4$ using $(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_3+\mathbf{T}_4)\,|\mathcal{H}\rangle = 0$ we find

$$G_2(1,2,3,4)+G_3(1,2,3,4)=\mathbf{dipoles}\;+\;f^{abe}f^{cde}\Bigg[\sum_{\substack{(i,j,k)\in(1,2,3)\\ j<k}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\;\overline{U}(i,\{j,k\},4)$$
$$-\frac{1}{2}\sum_{1\le i\le j\le 3}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\left\{\mathbf{T}_j^b,\mathbf{T}_j^c\right\}\left(\bar{H}_3[i,\{j,4\}]+\bar{H}_3[j,\{4,i\}]+\bar{H}_3[4,\{j,i\}]\right)\Bigg]$$

$\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\;\overline{U}(i,\{j,k\},4)$ can combine with 4-line diagrams to form CICRs.

$\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\left\{\mathbf{T}_j^b,\mathbf{T}_j^c\right\}\left(\bar{H}_3[i,\{j,4\}]+\bar{H}_3[j,\{4,i\}]+\bar{H}_3[4,\{j,i\}]\right)$ cannot form CICRs…
Can only be consistent with rescaling symmetry if the sum conspires to be constant $\bar{H}_3[i,\{j,k\}]+\bar{H}_3[j,\{k,i\}]+\bar{H}_3[k,\{j,i\}]=3C$ — which indeed holds!

Surprise with three lines

Consider now the soft anomalous dimension for three coloured lines, subject to the colour-conservation constraint: $\left(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3\right) |\mathcal{H}\rangle = 0\,.$

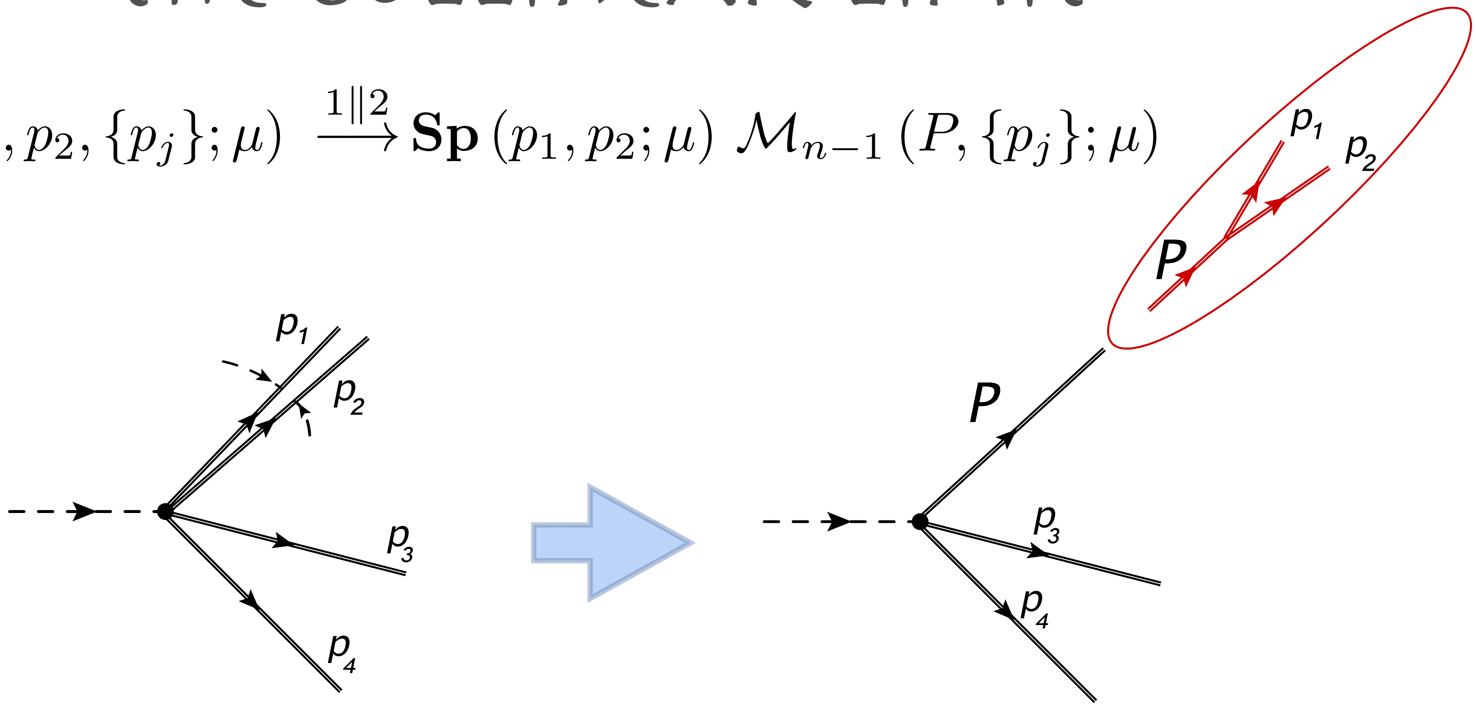
Given that **no conformal cross ratios can be formed**, the expectation was: *no corrections beyond the dipole formula, i.e.* $\Delta_3 = 0$.

Summing all 2- and 3-line webs we get, instead, a non-zero constant:

$$\Delta_3 = -16\left(\frac{\alpha_s}{4\pi}\right)^3 \left(\zeta_5 + 2\zeta_2\zeta_3\right) f^{abe} f^{cde} \sum_{\substack{(i,j,k)\in(1,2,3)\\ j<k}} \left\{\mathbf{T}_i^a, \mathbf{T}_i^d\right\} \mathbf{T}_j^b \mathbf{T}_k^c$$

THE COLLINEAR LIMIT

$$\mathcal{M}_n\left(p_1, p_2, \{p_j\}; \mu\right) \xrightarrow{1\|2} \mathbf{Sp}\left(p_1, p_2; \mu\right) \mathcal{M}_{n-1}\left(P, \{p_j\}; \mu\right)$$



In particular, IR singularities of the splitting amplitude are those present in n-parton scattering (with 1 ∥ 2) and not in (n-1)-parton scattering:

$$\Gamma_{\mathbf{Sp}} = \Gamma_n - \Gamma_{n-1}$$

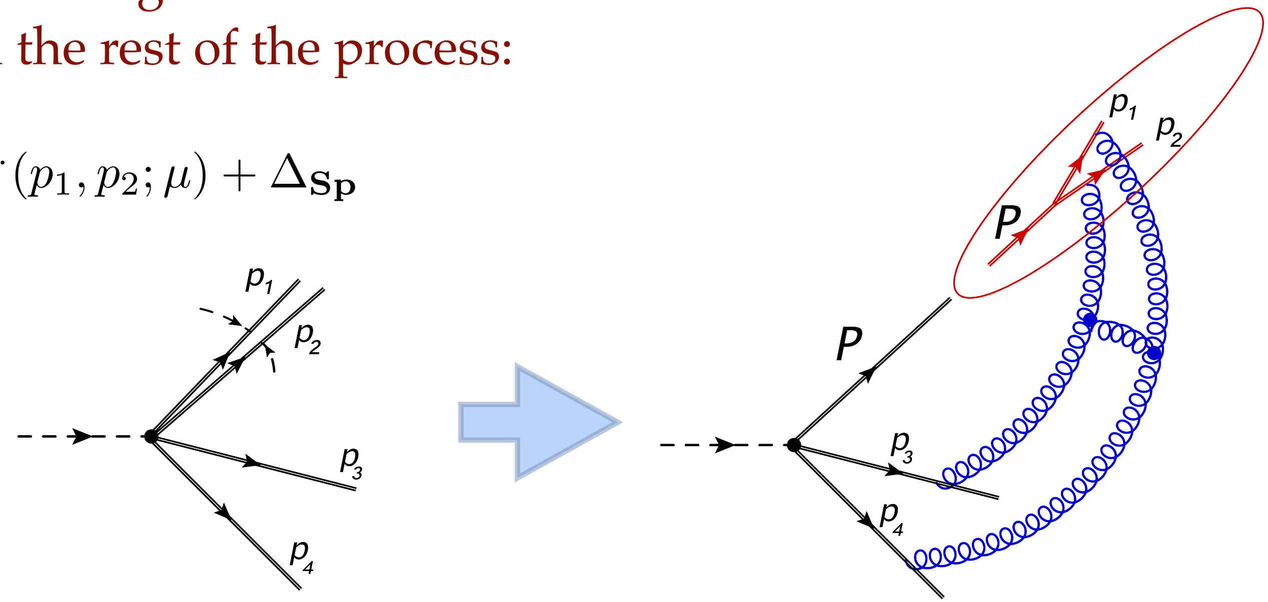
Becher & Neubert (2009)
Dixon, EG & Magnea (2010)

The expectation (see e.g. [Feige & Schwartz 1403.6472]) is that the splitting amplitude depends exclusively on the variables of the collinear pair.
This is *automatically realised* by the dipole formula for the singularities.

The collinear limit at 3 loops

At three loops there are diagrams that could introduce correlation between collinear partons and the rest of the process:

$$\Gamma_{\mathbf{Sp}}(p_1, p_2; \mu) = \Gamma^{\text{dip.}}_{\mathbf{Sp}}(p_1, p_2; \mu) + \Delta_{\mathbf{Sp}}$$



But through intricate cancellations the correction is a *constant* depending **only** on the colour degrees of freedom of the collinear pair:

$$\Delta_{\mathbf{Sp}} = (\Delta_n - \Delta_{n-1})|_{1\|2} = -24\left(\frac{\alpha_s}{4\pi}\right)^3 (\zeta_5 + 2\zeta_2\zeta_3)\left[f^{abe}f^{cde}\left\{\mathbf{T}_1^a, \mathbf{T}_1^c\right\}\left\{\mathbf{T}_2^b, \mathbf{T}_2^d\right\} + \frac{1}{2}C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2\right]$$

Ø. Almelid, C. Duhr, EG 1507.00047 (v2)

Conclusion:

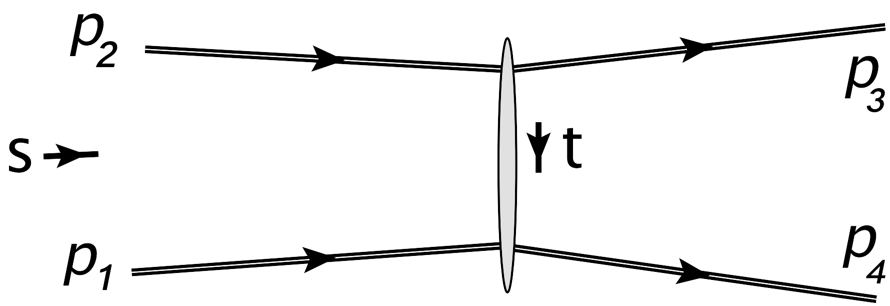
The splitting amplitudes singularities are independent of the rest of the process. Consistent with expectations!

HIGH-ENERGY (REGGE) LIMIT

Expanding Δ_4 at large s/(-t) we get ***no log-enhanced terms,*** just a constant.
This can be contrasted with dedicated calculations of the high-energy limit.

The Regge limit is dominated by t-channel gluon exchange. Leading logs of (-t/s) are summed through Reggeization:

$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$



The gluon Regge pole is

$$\alpha(t) = \frac{1}{4}(\mathbf{T}_2+\mathbf{T}_3)^2\int_0^{-t}\frac{d\lambda^2}{\lambda^2}\widehat{\gamma}_K(\alpha_s(\lambda^2,\epsilon))$$

Korchemskaya and Korchemsky (1996)
Del Duca, Duhr, EG, Magnea & White (2011)

which is **fully consistent with the dipole formula**. This consideration excludes any quadrupole contribution $\alpha_s^3\log^n(-t/s)$ with $n\geq 2$ for the Re part.

$i\alpha_s^3\log^2(-t/s)$ is excluded by an explicit **two Reggeon** calculation

Caron-Huot 1309.6521

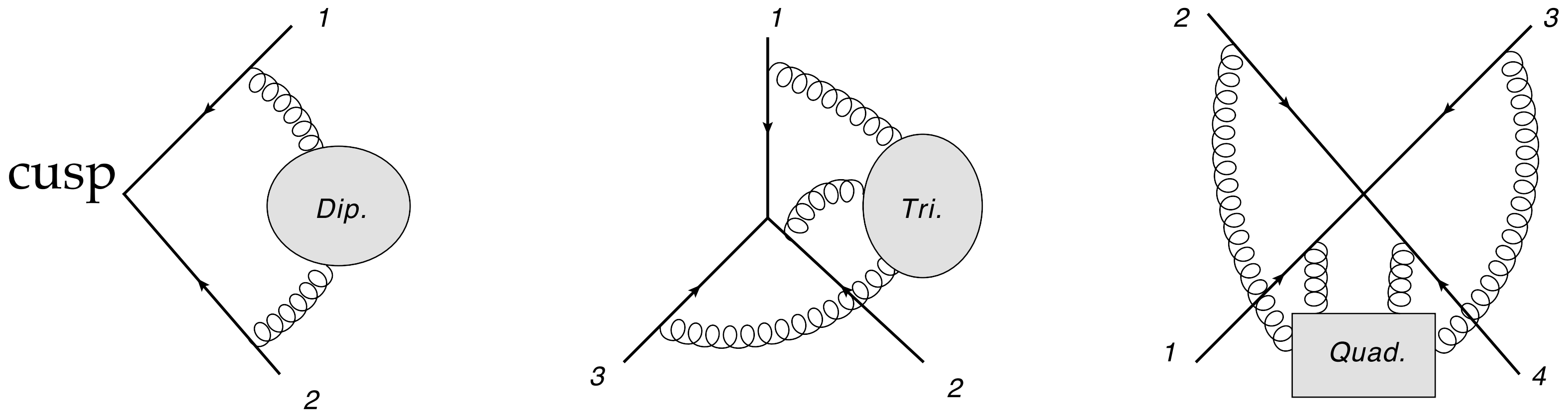
$\alpha_s^3\ln(-t/s)$ is excluded by a dedicated **three Reggeon** calculation.

Caron-Huot, EG, Vernazza - to appear

Conclusions

- IR singularities of massless scattering amplitudes are now known to **3-loops**.
- As expected, the first correction to the dipole formula occurs at three loops. For three partons it is a **constant**, while for four or more, a quadrupole interaction correlating simultaneously colour and kinematics of 4 patrons.
- The quadrupole term is expressed in terms of single-valued harmonic polylogarithms of weight 5, depending on $\{z, \bar{z}\}$. These variables are simple algebraic functions of conformally-invariant cross ratios, and they manifest the symmetries and analytic structure of the quadruple interaction.
- Splitting amplitudes receive a kinematic-independent correction beyond the dipole formula at 3-loop, but remains independent of the rest of the process!
- Regge limit: consistency with known results at LL and NLL and new predictions at NNLL and beyond.

The Structure of the Soft Anomalous Dimension: Massless vs. Massive Partons



$\mathbf{T}_i \cdot \mathbf{T}_j$ / planar

$f^{abc}\, \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c$

$f^{abe} f^{cde}\, \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d$

massless	known @ 3-loop (& Nf planar 4-loop*)	forbidden to all loops	**3-loop done!** ***
massive	known @ 3-loop**	known @ 2-loops progress @ 3-loop	starts @ 3-loop - in progress

* Henn, A. Smirnov, V. Smirnov & Steinhauser (2016)

** Grozin, Henn, Korchemsky & Marquard (2015)

*****Almelid, Duhr, EG - 1507.00047 (v2)**