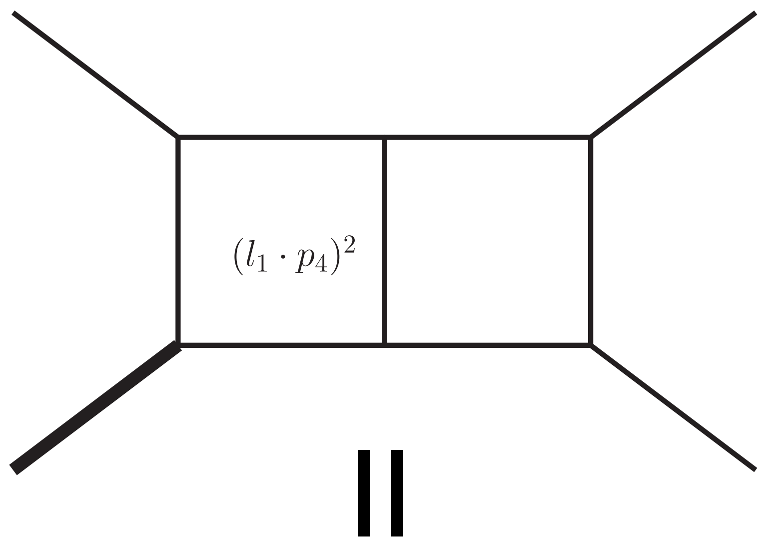
Integration-by-parts reduction from unitarity, an algebraic geometry story

Yang Zhang

Based on work with Alessandro Georgoudis, Kapser Larsen

Amplitudes 2016
Nordita, Stockholm
July. 6, 2016

Integration-by-parts (IBP) reduction



$$\Big\{ \frac{(-10+3d)(-8+3d)\,\text{sunset1}\,(78-24d-10Ms+3d\,Ms+34x-9dx-4Ms\,x+d\,Ms\,x+4x^2-dx^2)}{16(-4+d)^2(-3+d)(-1+Ms)\,x},$$

$$\big((-10+3d)(-8+3d)\,\text{sunset2}\,(-24+6d+28Ms-7d\,Ms-4Ms^2+d\,Ms^2-30x+9dx+34Ms\,x-10d\,Ms\,x-4Ms^2x+d\,Ms^2x+4x^2+dx^2+4Ms\,x^2-d\,Ms\,x^2)\big)\Big/\big(16(-4+d)^2(-3+d)(-1+Ms)^2x\big),\ -\frac{(-10+3d)(-8+3d)\,\text{sunset3}\,(-1+Ms-2x)}{4(-4+d)^2(-3+d)(-1+Ms)^2},$$

$$\frac{(-10+3d)(-8+3d)\,\text{sunset4}\,(4-5Ms+Ms^2+3x-4Ms\,x)}{8(-4+d)^2(-1+Ms)^2Ms},$$

$$\frac{\text{bubtri}\,(-10+3d)(9-11Ms+2Ms^2+2x-5Ms\,x)}{8(-4+d)(-1+Ms)^2},\ -\frac{9(-10+3d)\,\text{tribub}}{8(-4+d)(-1+Ms)},$$

$$-\frac{\text{dbub1}\,(7-2d-7Ms+2d\,Ms+8x-2dx)}{2(-4+d)(-1+Ms)^2},\ -\frac{(-3+d)\,\text{dbub2}\,(-2+Ms-x)}{2(-4+d)(-1+Ms)},$$

$$-\frac{(-4+d)\,\text{tritri}\,(2-5Ms+2Ms^2+Ms^3+x-4Ms\,x-Ms^2x)}{4(-3+d)(-1+Ms)^2},$$

$$-\frac{(-10+3d)\,\text{tribubA}\,(-6+Ms-x)(Ms-x)(-1+x)}{8(-4+d)(-1+Ms)\,x},$$

$$\frac{1}{8(-4+d)(-1+Ms)^2x}(-10+3d)\,\text{tribubB}\,(-6Ms+7Ms^2-Ms^3-6x+10Ms\,x-5Ms^2x+Ms^3x-9x^2+7Ms\,x^2-2Ms^2x^2-x^3+Ms\,x^3),$$

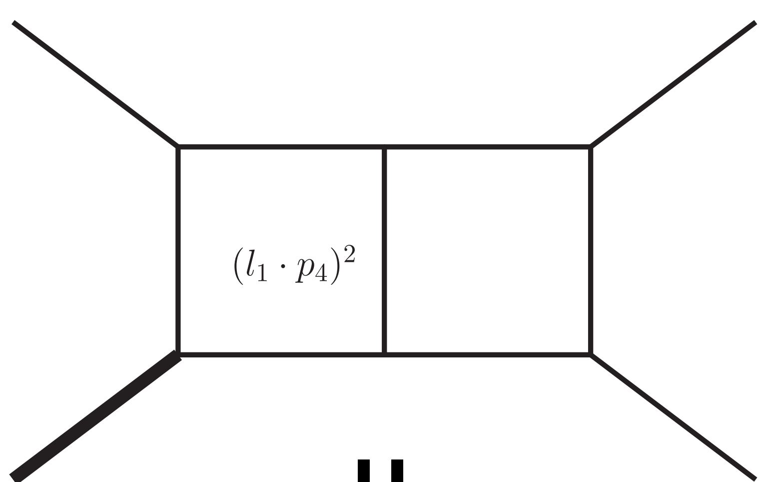
$$\text{boxbub},\ \frac{\text{bubbox}\,(-7+Ms-3x)}{2(-1+Ms)},\ -\frac{(-4+d)\,\text{slashedA}\,(-7+7Ms-9x)}{4(-3+d)(-1+Ms)},$$

$$\frac{1}{16(-3+d)(-1+Ms)}\,\text{slashedB1}\,(-92+26d+178Ms-51d\,Ms-28Ms^2+8d\,Ms^2-134x+37dx+54Ms\,x-15d\,Ms\,x-26x^2+7dx^2),$$

$$\frac{(-10+3d)\,\text{slashedB2}\,(-6+Ms-x)(1+x)^2}{8(-3+d)(-1+Ms)\,x},\ -\frac{(-4+d)\,\text{tribox}\,(-1+Ms-x)(Ms-x)}{8(-3+d)(-1+Ms)},$$

$$-\frac{(-4+d)\,\text{dbox1}\,x}{8(-3+d)(-1+Ms)},\ -\frac{\text{dbox2}\,(12-3d-12Ms+3d\,Ms+2x)}{4(-3+d)(-1+Ms)}\Big\}$$

Integration-by-parts (IBP) reduction



$$=$$

$$\Bigg\{ \frac{(-10+3d)(-8+3d)\,\text{sunset1}\,(78-24d-10\,\text{Ms}+3d\,\text{Ms}+34x-9dx-4\,\text{Ms}\,x+d\,\text{Ms}\,x+4x^2-dx^2)}{16(-4+d)^2(-3+d)(-1+\text{Ms})\,x},$$

$$\big((-10+3d)(-8+3d)\,\text{sunset2}\,(-24+6d+28\,\text{Ms}-7d\,\text{Ms}-4\,\text{Ms}^2+d\,\text{Ms}^2-30x+9dx+34\,\text{Ms}\,x-10d\,\text{Ms}\,x-4\,\text{Ms}^2x+d\,\text{Ms}^2x+4x^2+dx^2+4\,\text{Ms}\,x^2-d\,\text{Ms}\,x^2)\big)\Big/\big(16(-4+d)^2(-3+d)(-1+\text{Ms})^2x\big),\ -\frac{(-10+3d)(-8+3d)\,\text{sunset3}\,(-1+\text{Ms}-2x)}{4(-4+d)^2(-3+d)(-1+\text{Ms})^2},$$

$$\frac{(-10+3d)(-8+3d)\,\text{sunset4}\,(4-5\,\text{Ms}+\text{Ms}^2+3x-4\,\text{Ms}\,x)}{8(-4+d)^2(-1+\text{Ms})^2\,\text{Ms}},$$

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$$\frac{1}{8(-4+d)(-1+\text{Ms})^2x}(-10+3d)\,\text{tribubB}\,(-6\,\text{Ms}+7\,\text{Ms}^2-\text{Ms}^3-6x+10\,\text{Ms}\,x-5\,\text{Ms}^2x+\text{Ms}^3x-9x^2+7\,\text{Ms}\,x^2-2\,\text{Ms}^2x^2-x^3+\text{Ms}\,x^3),$$

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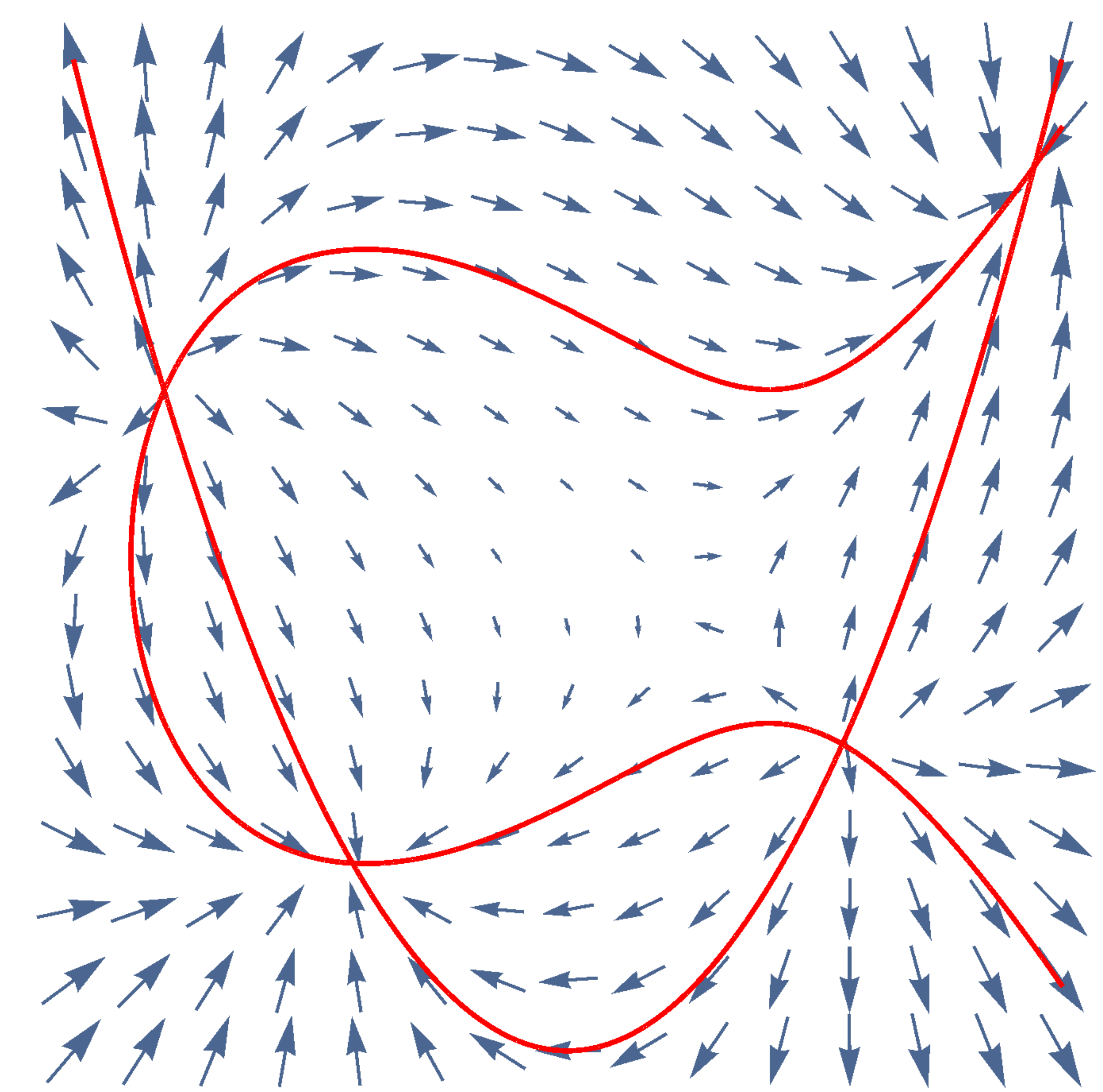
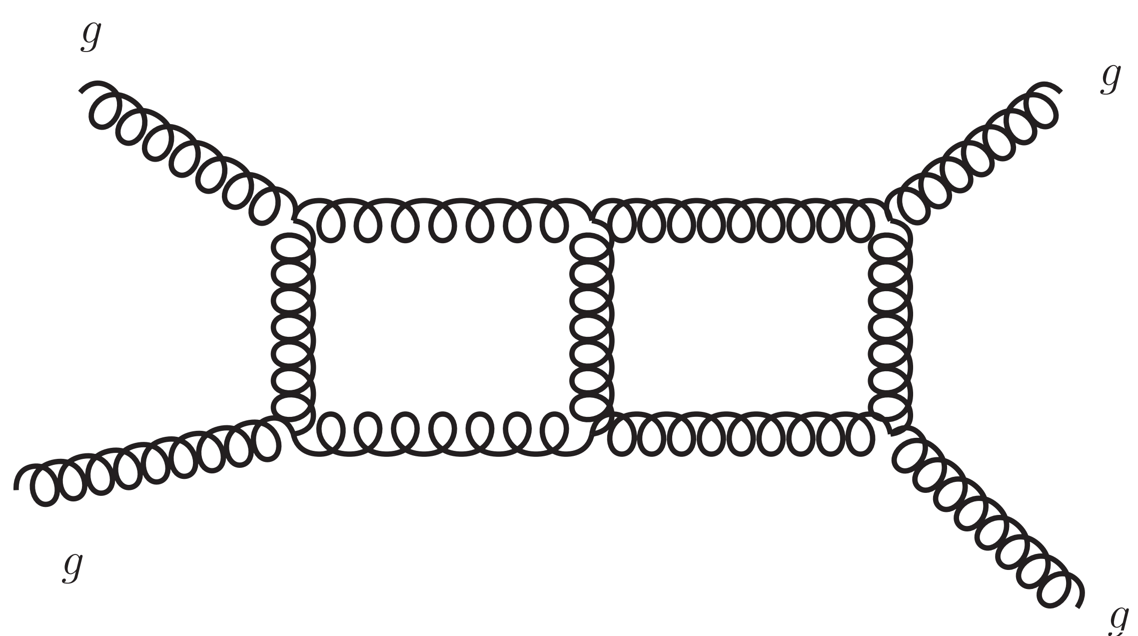


Tangent Algebra (blue arrows)
of
Affine Varieties (red curves)

Multi-loop Integration-by-parts reduction



massless/massive, supersymmetric/non-supersymmetric

crucial for the next-to-next-to-leading (NNLO) order of LHC processes

$$\int \frac{dl_1^D}{i\pi^{D/2}} \cdots \int \frac{dl_L^D}{i\pi^{D/2}} \frac{N(l_i \cdot p_j)}{D_1 \ldots D_k}$$

→ Large number of terms

Integration-by-parts (IBP)

$$\int \frac{dl_1^D}{i\pi^{D/2}} \cdots \int \frac{dl_L^D}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \left(\frac{v_i^\mu}{D_1 \ldots D_k} \right) = 0$$

Integrand reduction ⇓

Integrand Basis

IBP ⇓

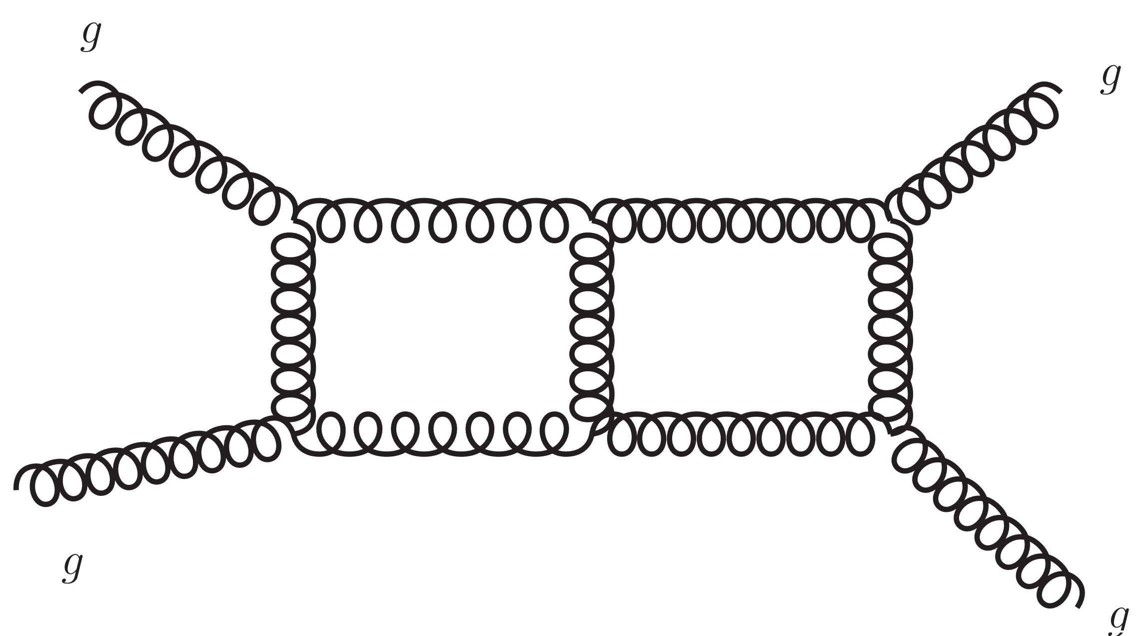
IBP ⇓

master integrals

Differential equations (IBP) ⇒

MPLs, ...

Multi-loop Integration-by-parts reduction



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IBP: Chetyrkin, Tkachov 1981, Laporta 2001, …

IBP codes: FIRE (Smirnov), Reduze (von Manteuffel, Studerus), LiteRed (Lee)

Syzygy approach: Gluza, Kjada, Kosower 2010, Schabinger 2011

Chen, Liu, Xie, Zhang, Zhou 2015

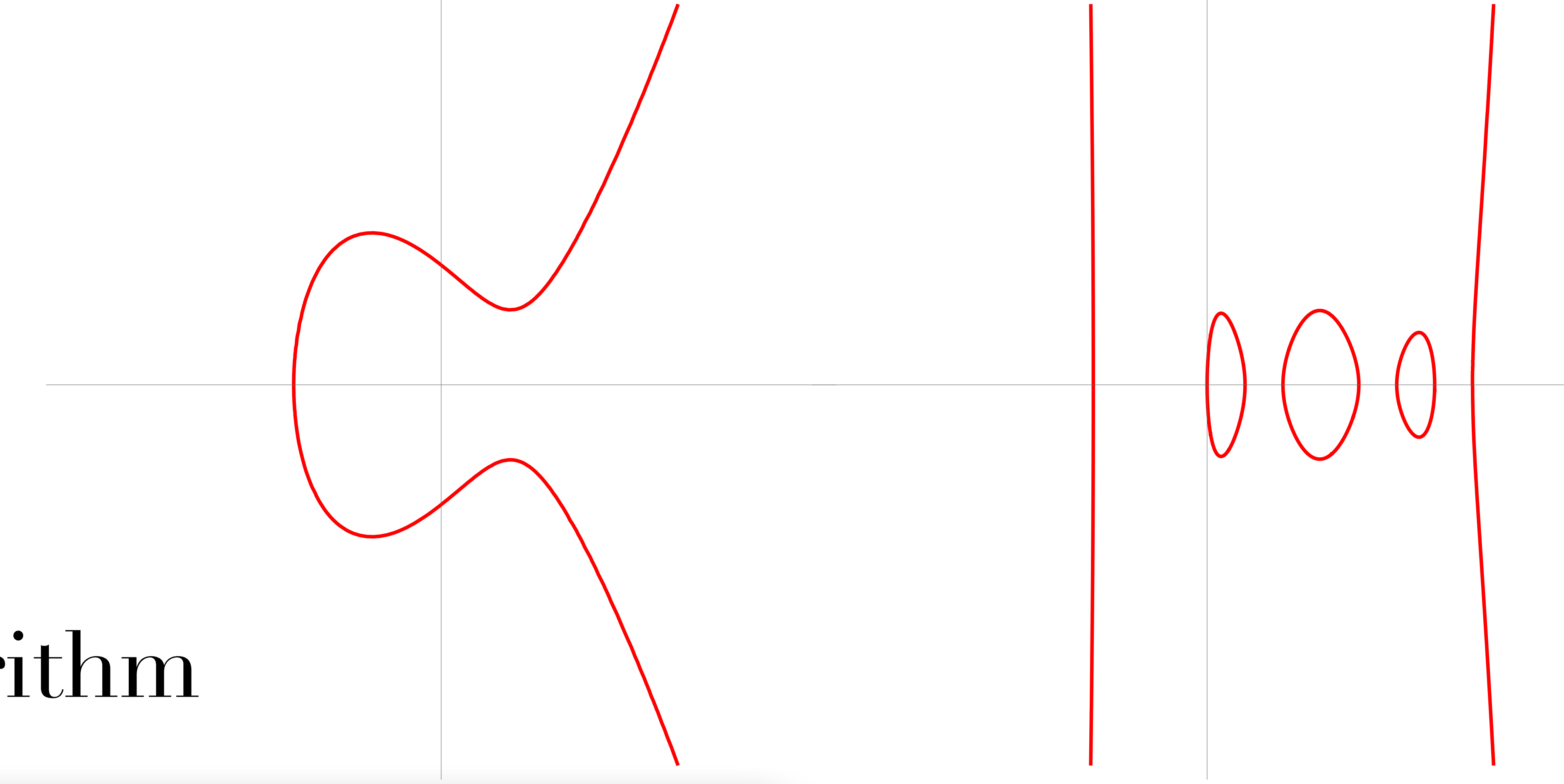
Integrand reduction ⇓ Integrand Basis ⇓ IBP ⇓ master integrals

IBP ⇓ master integrals

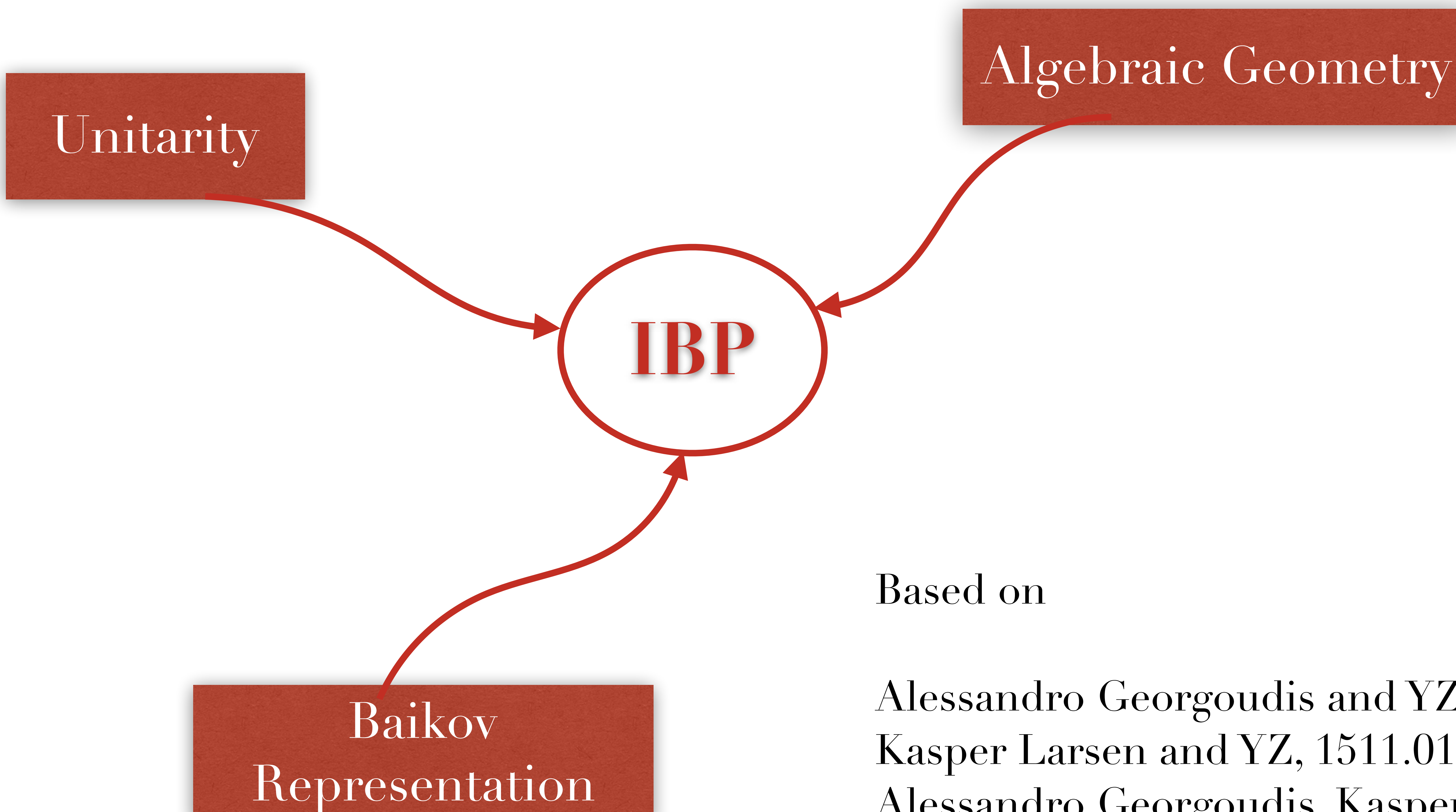
master integrals ⇒ Differential equations (IBP) ⇒ MPLs, …

also Schabinger and Smirnov's talk

Outline



IBP: Unitarity + Tangent algebra (syzygy) algorithm



Based on

Alessandro Georgoudis and YZ, 1507.06310
Kasper Larsen and YZ, 1511.01071.
Alessandro Georgoudis, Kasper Larsen and YZ, to appear

Set up

K. Larsen and YZ, 1511.01071

See also: Ita 1510.05626

Dimensional Regularization $D = 4 - 2\epsilon$

External momenta are in 4D

2-loop $l_1 = l_1^{[4]} + l_1^{\perp}, \quad l_2 = l_2^{[4]} + l_2^{\perp}$

$\mu_{11} = -(l_1^{\perp})^2, \quad \mu_{22} = -(l_2^{\perp})^2, \quad \mu_{12} = -l_1^{\perp} \cdot l_2^{\perp}$

Dimensional decomposition

$$\int \frac{d^D l_1}{i\pi^{D/2}} \int \frac{d^D l_2}{i\pi^{D/2}} \frac{N}{D_1 \ldots D_k} \propto \int_0^\infty d\mu_{11} \int_0^\infty d\mu_{22} \int_{-\sqrt{\mu_{11}\mu_{22}}}^{\sqrt{\mu_{11}\mu_{22}}} d\mu_{12} \left(\mu_{11}\mu_{22} - \mu_{12}^2\right)^{\frac{D-7}{2}} \int d^4 l_1 d^4 l_2 \frac{N}{D_1 \ldots D_k}$$

L-loop

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{N}{D_1 \ldots D_k} \propto \int \prod_{1 \le i \le j \le L} d\mu_{ij} \det(\mu_{ij})^{\frac{D-5-L}{2}} \int d^4 l_1 \ldots d^4 l_L \frac{N}{D_1 \ldots D_k}$$

Set up

K. Larsen and YZ, 1511.01071

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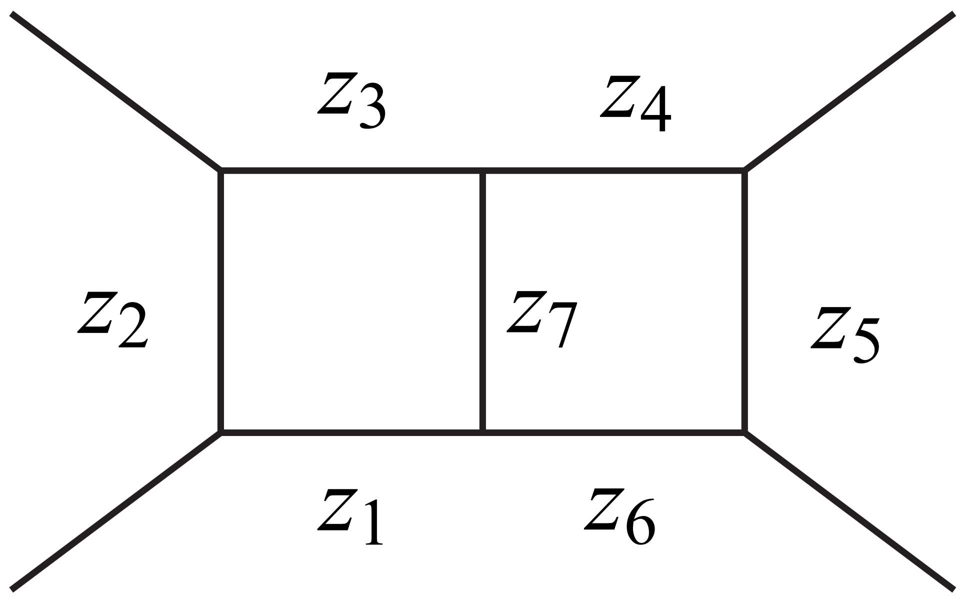
n external legs. If $n \le 4$, $5 - n$ orthogonal directions can be integrated out,

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{N}{D_1 \ldots D_k} \propto \int \prod_{1 \le i \le j \le L} d\mu_{ij} \det(\mu_{ij})^{\frac{D-n-L}{2}} \int d^{n-1} l_1 \ldots d^{n-1} l_L \frac{N}{D_1 \ldots D_k}$$

Baikov parametrization

Baikov 1996



$$z_i \equiv D_i, \quad i = 1, \ldots 7$$
$$z_8 \equiv (l_1 + p_4)^2/2, \quad z_9 \equiv (l_2 + p_1)^2/2$$

4D mu's orthogonal

4-point double box, 8+3-2=9 variables

$(l_1^\mu, l_2^\mu, \mu_{ij}) \mapsto (z_1, \ldots z_9)$ is nonlinear, but the Jacobian is a constant.

Inverse map exists, and has the form of polynomials! (degree-2 case of Jacobian conjecture, proven)

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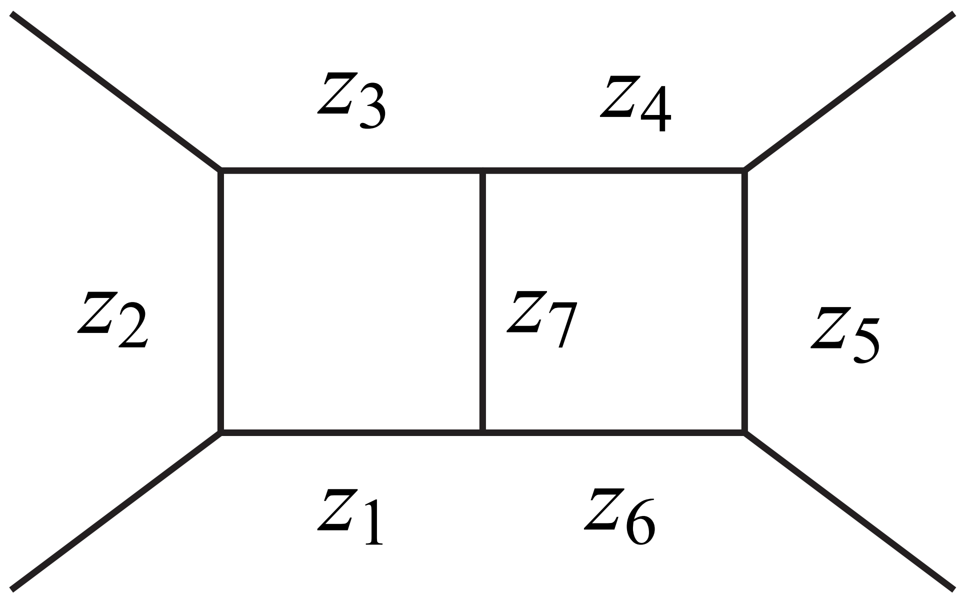
Baikov 1996



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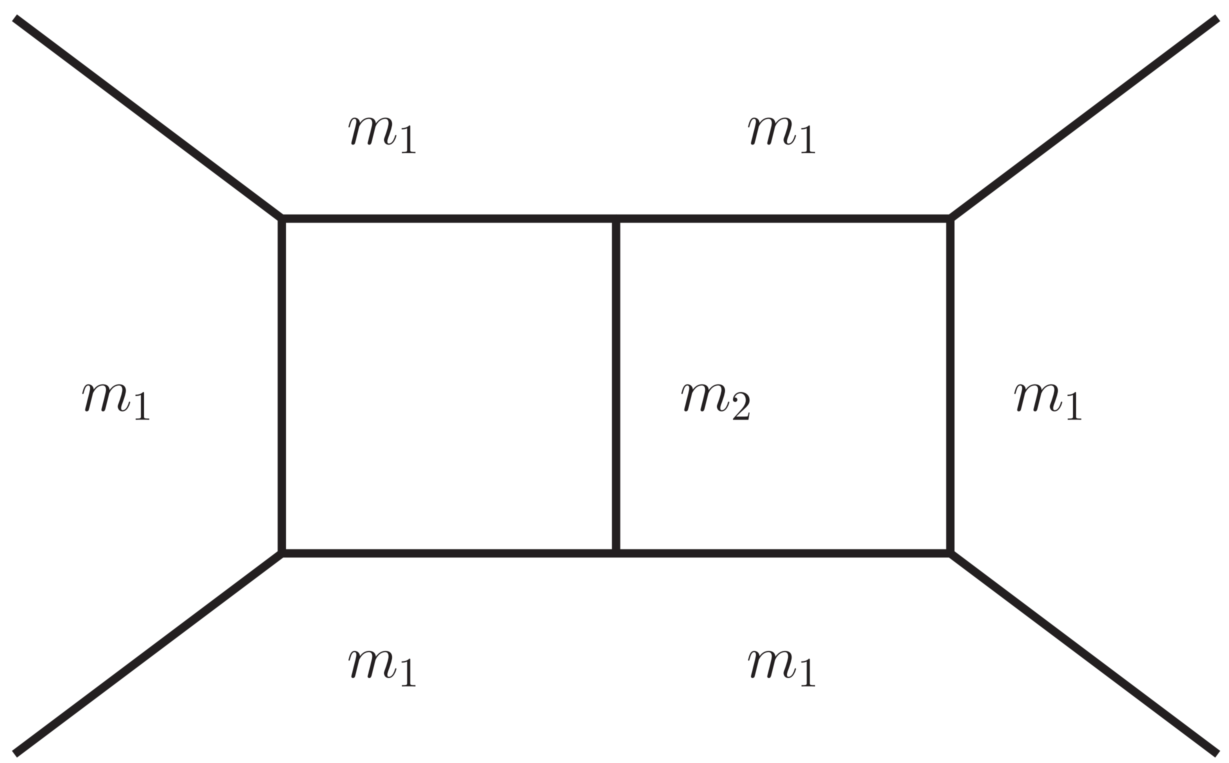
$$\int \frac{d^D l_1}{\pi^{D/2}} \frac{d^D l_2}{\pi^{D/2}} \frac{N}{D_1 D_2 \dots D_7} \propto \int \left(\prod_{i=1}^{9} dz_i \right) F(z)^{\frac{D-6}{2}} \frac{N(z)}{z_1 \dots z_7}$$

Linear

polynomial

- Easy to apply unitarity cut
- Adaptive integrand reduction (Mastrolia, Peraro, Primo 2016)
- Works for any loop order

Maximal cut



Unitarity cut $\quad \left.\frac{1}{D_1 \ldots D_k}\right|_{\text{cut}} \propto \delta(D_1) \ldots \delta(D_k)$

$$I^D_{\text{dbox}}|_{\text{cut}} \propto \int\int dz_8 dz_9 F(z_8, z_9)^{\frac{D-6}{2}} N(z_8, z_9)$$

measure on the cut

		$F(x, y) = 0$
Case I	$m_1 = m_2 = 0$	reducible curve: two lines plus one conic
Case II	$m_1 \neq 0, m_2 = 0$	deformed elliptic curve
Case III	$m_1 \neq 0, m_2 \neq 0$	elliptic curve

Integral reduction

$$0 = \int d[(-\alpha_9 dz_8 + \alpha_8 dz_9) F^{\frac{D-6}{2}}]$$
$$= \int \left[\left(\frac{\partial \alpha_8}{\partial z_8} + \frac{\partial \alpha_9}{\partial z_9}\right) F^{\frac{D-6}{2}} + (\alpha_8 F_{z_8} + \alpha_9 F_{z_9}) \left(\frac{D-6}{2}\right) F^{\frac{D-8}{2}}\right] dx \wedge dy$$

dimension shifted

Require

$$\boxed{\alpha_8 F_{z_8} + \alpha_9 F_{z_9} + \alpha F = 0}$$

Syzygy (συζυγία) equation

Gluza, Kjada, Kosower 2010

Tangent algebra

"Affine varieties and Lie algebras of vector fields"
Hauser, Müller 1993

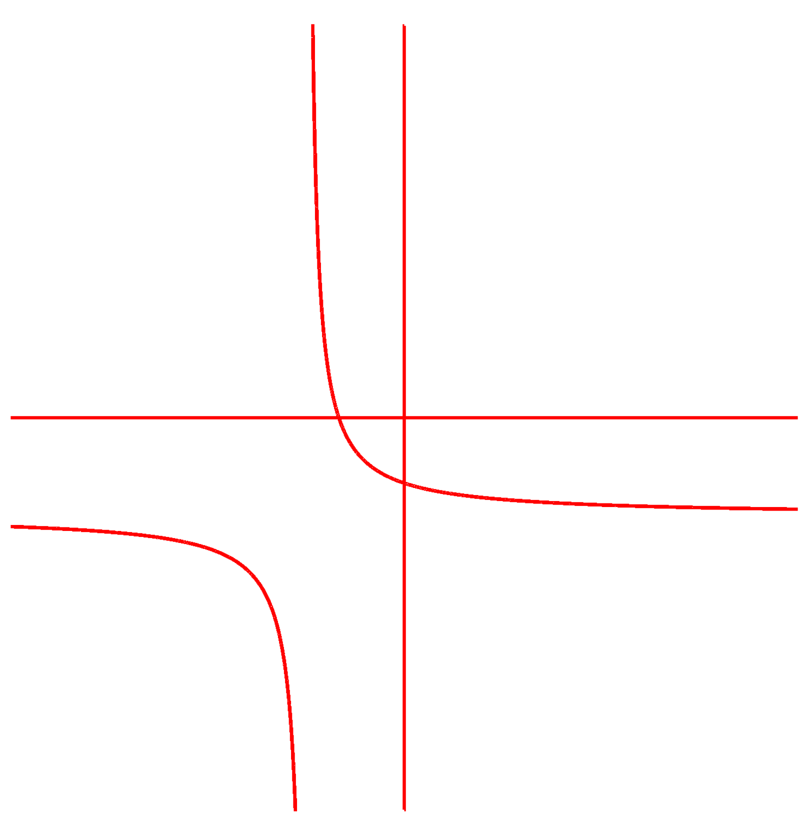
$F = 0$ defines an affine variety V. The solution set of $\alpha_8 F_{z_8} + \alpha_9 F_{z_9} + \alpha F = 0$ is the tangent algebra of V, i.e., polynomial vector fields such that

$$\left(\alpha_8 \frac{\partial}{\partial z_8} + \alpha_9 \frac{\partial}{\partial z_9}\right) F \in \langle F \rangle.$$

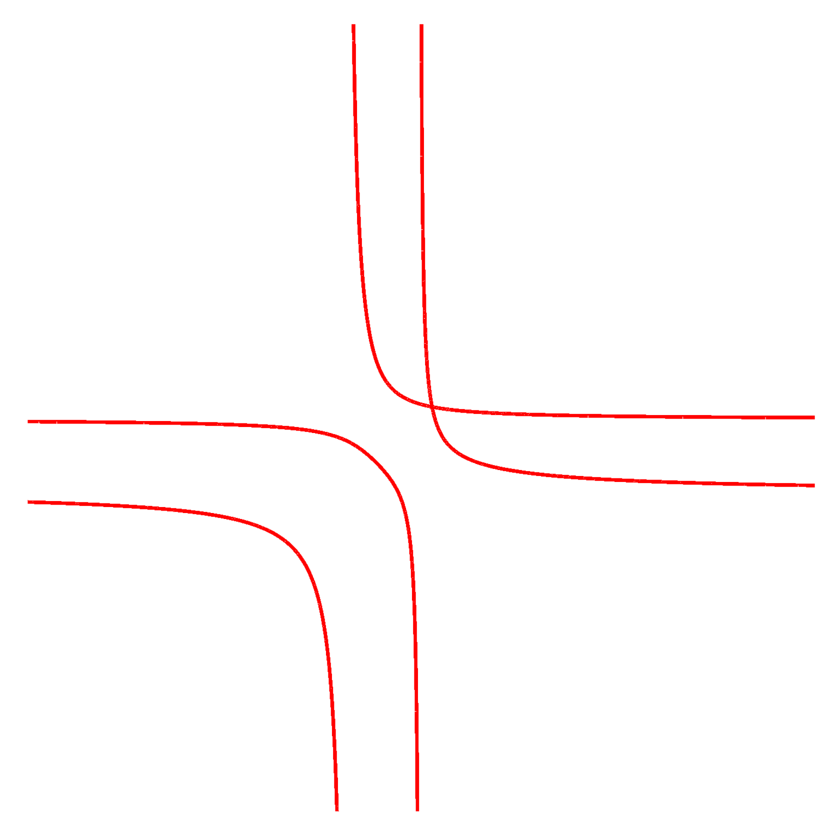
- (infinite-dimensional) Lie algebra
- Module over polynomial ring

$\alpha_8 F_{z_8} + \alpha_9 F_{z_9} + \alpha F = 0$ defines syzygy for the singular ideal $J = \langle F_{z_8}, F_{z_9}, F \rangle$.

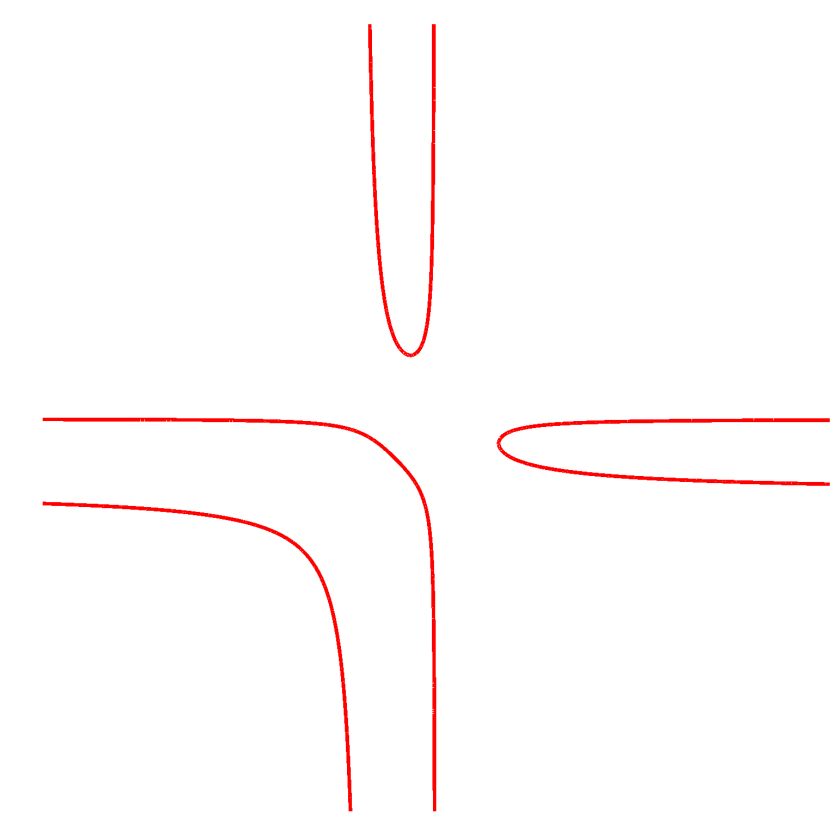
characterizes singular points of V



Case I, 3 singular points



Case II, 1 singular point



Case III, no singular point

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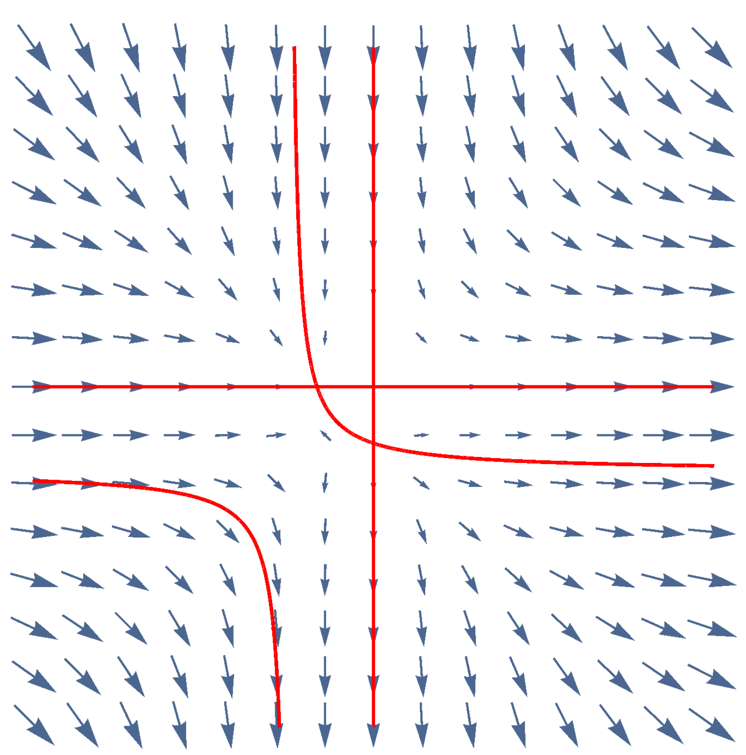
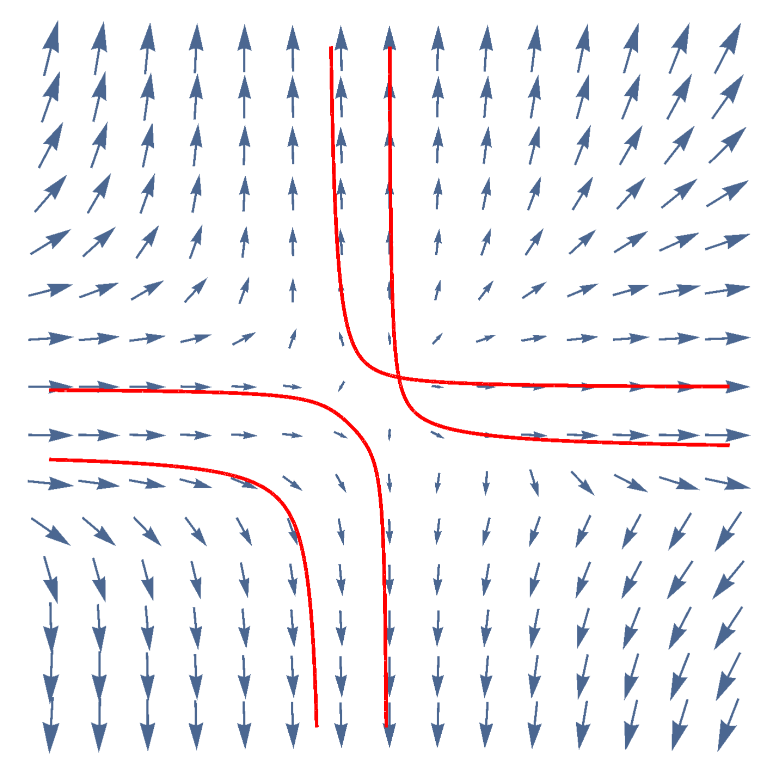
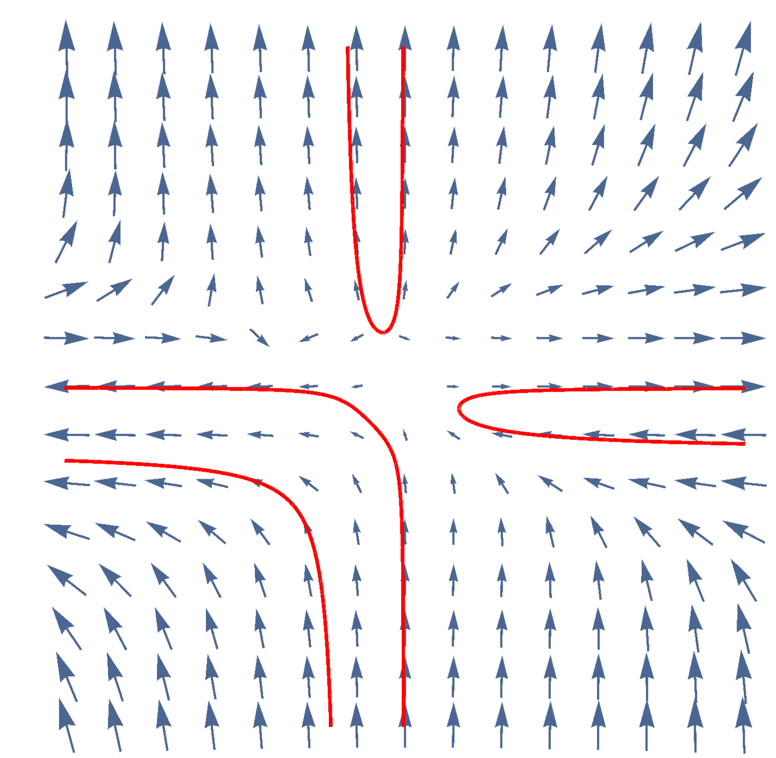


Case I, 3 singular points



Case II, 1 singular point



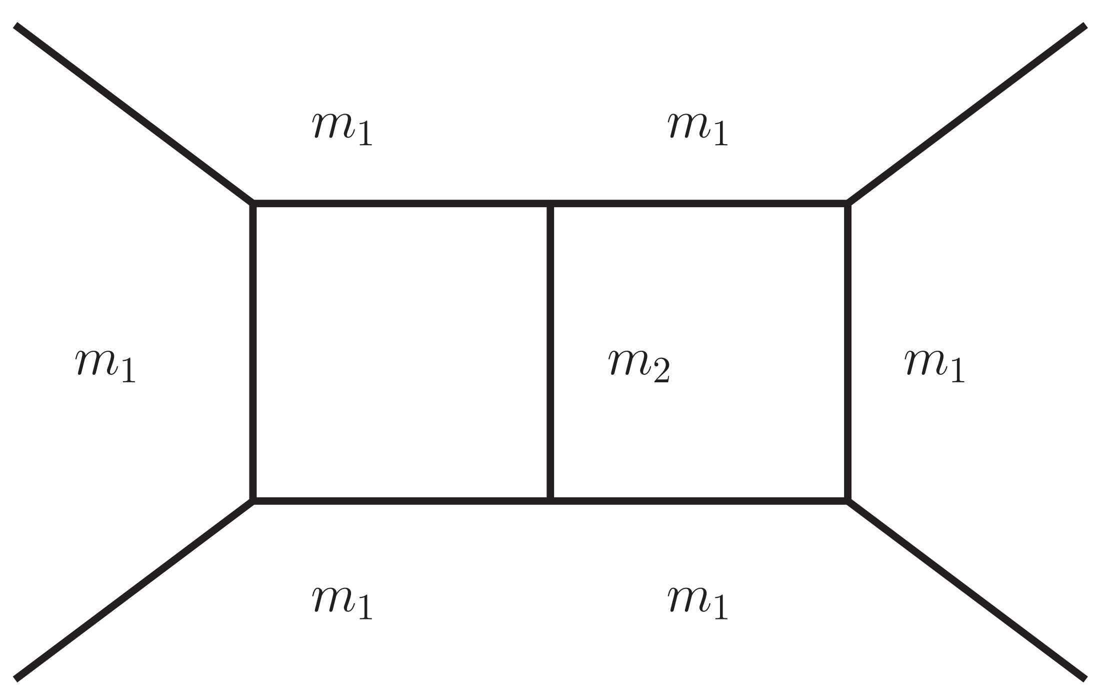
Case III, no singular point

We can find tangent algebras in all these case, but before the calculation…

Tangent algebra and singular points

Quillen–Suslin theorem: Syzygy for polynomials without common root is a free module.

$F=0$ is smooth → $F_{z_8}=F_{z_9}=F=0$ has no solution → Tangent algebra is a free module, generated by principle syzygies.



Case III, $m_1\neq 0, m_2\neq 0$ has the simplest tangent algebra (generated by principle syzygies). For case I, II, the tangent algebras are generated by principle syzygy + weighted Euler vectors around the singular points.

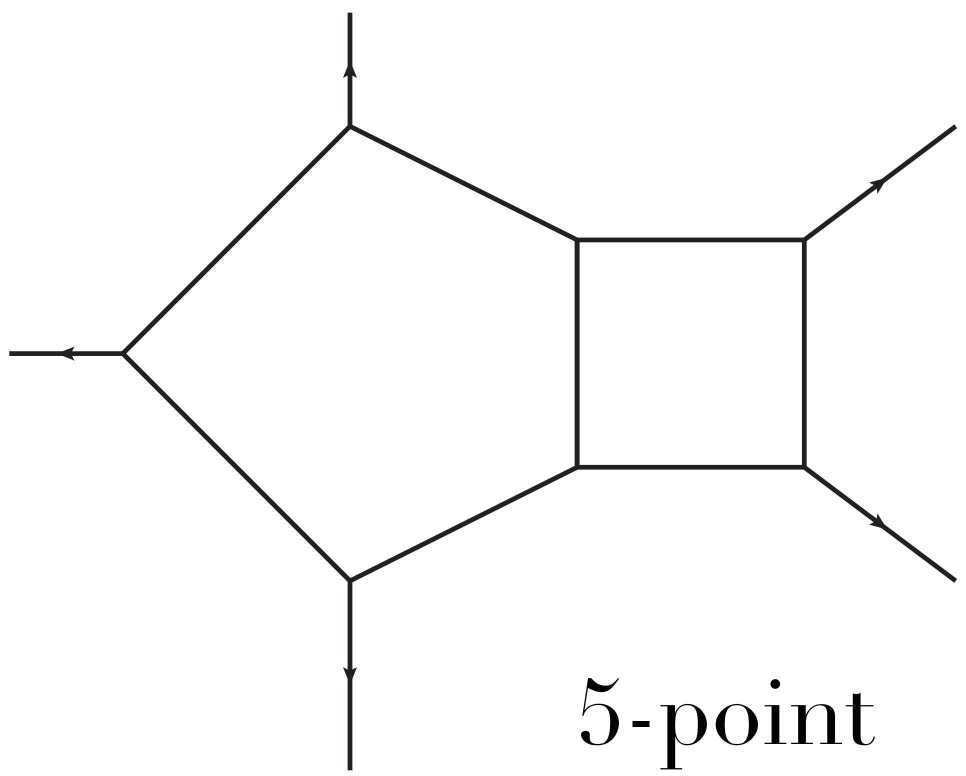
All cases' algebra can be automatically found by algebraic geometry softwares Macaulay2/Singular

$$\int \frac{dl_1^D}{i\pi^{D/2}} \int \frac{dl_2^D}{i\pi^{D/2}} \frac{-\alpha(D-6)/2 + \partial\alpha_8/\partial z_8 + \partial\alpha_9/\partial z_9}{D_1 \ldots D_7} = 0 + \ldots$$

get all on-shell part of D-dim IBPs

Maximal cut

$F(x, y_1, y_2) = 0$ surface



5-point

$$I^D_{\text{pentabox}}\big|_{\text{cut}} \equiv \int\int\int dx dy_1 dy_2 N(x, y_1, y_2) F(x, y_1, y_2)^{\frac{D-7}{2}}$$

$$\begin{aligned} 0 &= \int d[(\alpha dy_1 \wedge dy_1 + \beta dy_2 \wedge dx + \gamma dx \wedge dy_1) F^{\frac{D-7}{2}}] \\ &= \int \left[\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y_1} + \frac{\partial \gamma}{\partial y_2}\right) - \delta\left(\frac{D-7}{2}\right)\right] F^{\frac{D-7}{2}} dx \wedge dy \end{aligned}$$

$$\boxed{\alpha F_x + \beta F_{y_1} + \gamma F_{y_2} + \delta F = 0}$$

Syzygy equation

Public Package “Azurite”

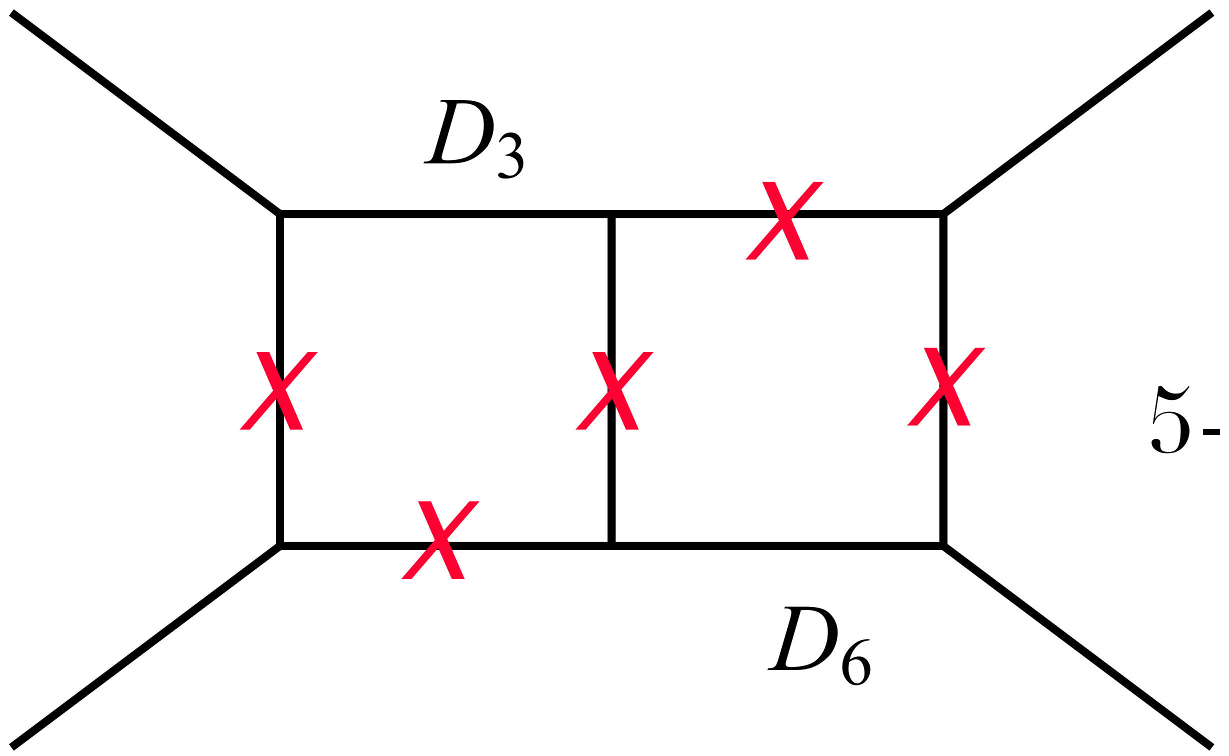
- Find all master integrals quickly (see also MINT package, Lee, Pomeransky 2013)
- Obtain IBPs at the maximal cut level

Georgoudis, Larsen, YZ, to appear July, 2016

Non-maximal cut

$$2\times 4+3-2-5=4 \text{ variables left, } z_3, z_6, z_8, z_9$$

4D   mu's   spurious   5-cut



5-cut

$$I^D_{\text{dbox}}|_{5\text{-cut}} \propto \int dz_3 dz_6 dz_8 dz_9 \; N\, F(z)^{\frac{D-6}{2}} z_3^{-1} z_6^{-1}$$

$$0=\int d\Big((\alpha_3 dz_6\wedge dz_8\wedge dz_9-\alpha_6 dz_8\wedge dz_9\wedge dz_3$$
$$+\alpha_8 dz_9\wedge dz_3\wedge dz_6-\alpha_9 dz_3\wedge dz_6\wedge dz_8)NF(z)^{\frac{D-6}{2}} z_3^{-1} z_6^{-1}\Big)$$

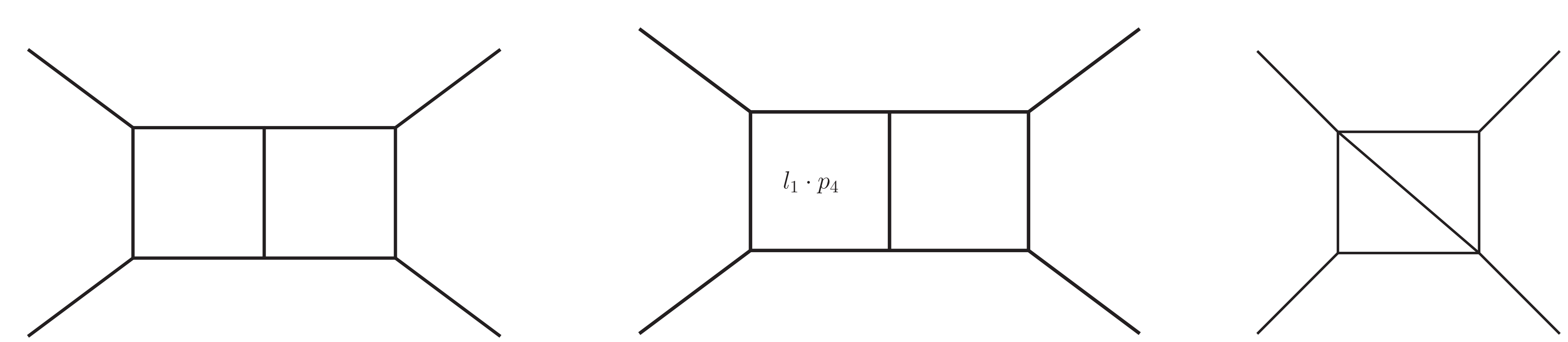
$$\alpha_i \frac{\partial F}{\partial z_i}+\beta F=0$$
$$\alpha_3+\beta_3 z_3=0$$
$$\alpha_6+\beta_6 z_6=0$$

Syzygy for polynomials

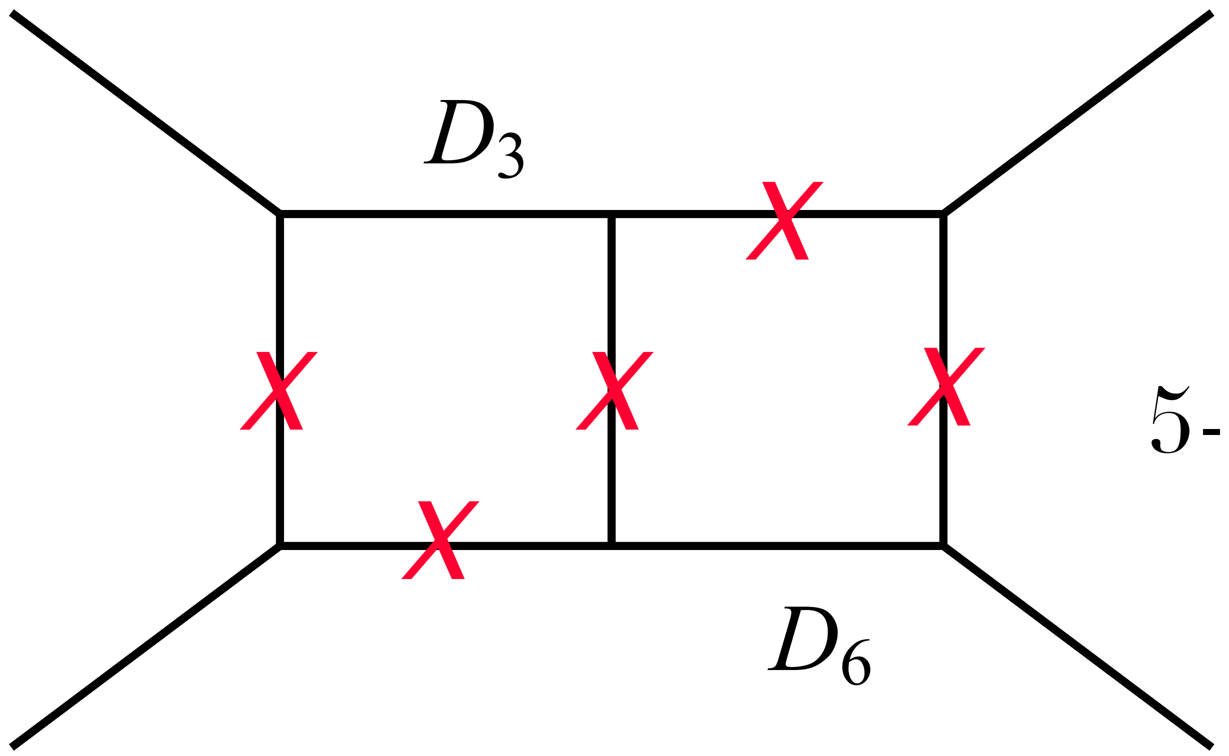
$\{z_3 F_{z_3}, z_6 F_{z_6}, F_{z_8}, F_{z_9}, F\}$

} Tangent algebra of $z_3 z_6 F=0$

Reduce to 3 MIs



Non-maximal cut



5-cut

$$2 \times 4 + 3 - 2 - 5 = 4 \text{ variables left, } z_3, z_6, z_8, z_9$$

4D mu's spurious 5-cut

$$I^D_{\text{dbox}}|_{\text{5-cut}} \propto \int dz_3 dz_6 dz_8 dz_9 \; N \, F(z)^{\frac{D-6}{2}} z_3^{-1} z_6^{-1}$$

$$0 = \int d\Big((\alpha_3 dz_6 \wedge dz_8 \wedge dz_9 - \alpha_6 dz_8 \wedge dz_9 \wedge dz_3 + \alpha_8 dz_9 \wedge dz_3 \wedge dz_6 - \alpha_9 dz_3 \wedge dz_6 \wedge dz_8) N F(z)^{\frac{D-6}{2}} z_3^{-1} z_6^{-1} \Big)$$

$$\alpha_i \frac{\partial F}{\partial z_i} + \beta F = 0$$

$$\alpha_3 + \beta_3 z_3 = 0$$

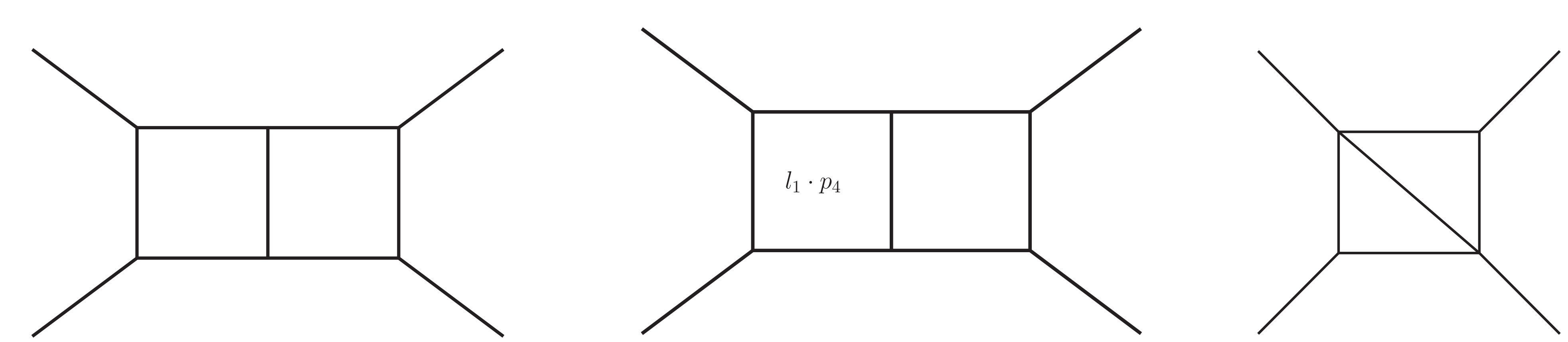
$$\alpha_6 + \beta_6 z_6 = 0$$

Syzygy for polynomials

$\{z_3 F_{z_3}, z_6 F_{z_6}, F_{z_8}, F_{z_9}, F\}$

} Tangent algebra of $z_3 z_6 F = 0$

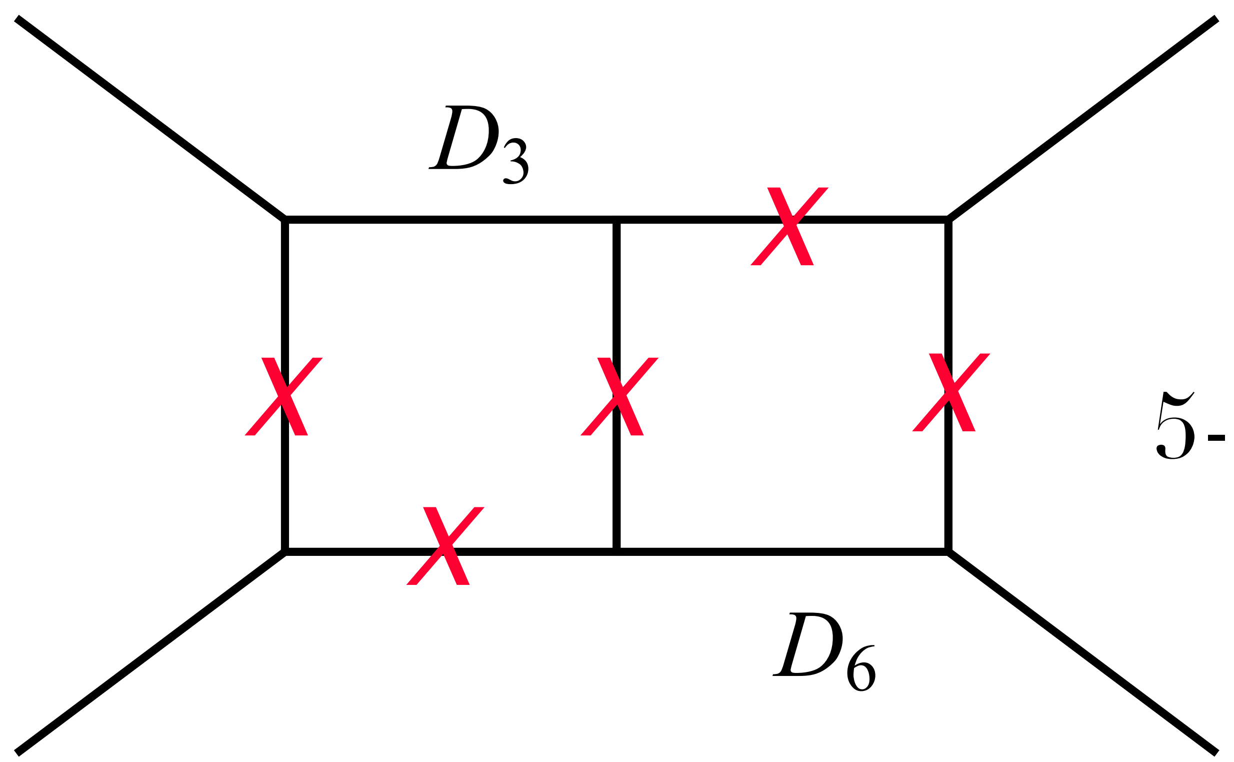
Reduce to 3 MIs



Non-maximal cut

$2 \times 4 + 3 - 2 - 5 = 4$ variables left, z_3, z_6, z_8, z_9

4D → 2×4; mu's → 3; spurious → 2; 5-cut → 5



5-cut

$$I^D_{\text{dbox}}|_{\text{5-cut}} \propto \int dz_3 dz_6 dz_8 dz_9 \; N \, F(z)^{\frac{D-6}{2}} z_3^{-1} z_6^{-1}$$

$$0 = \int d\Big((\alpha_3 dz_6 \wedge dz_8 \wedge dz_9 - \alpha_6 dz_8 \wedge dz_9 \wedge dz_3$$
$$+ \alpha_8 dz_9 \wedge dz_3 \wedge dz_6 - \alpha_9 dz_3 \wedge dz_6 \wedge dz_8) N F(z)^{\frac{D-6}{2}} z_3^{-1} z_6^{-1} \Big)$$

$$\alpha_i \frac{\partial F}{\partial z_i} + \beta F = 0$$

$$\alpha_3 + \beta_3 z_3 = 0$$

$$\alpha_6 + \beta_6 z_6 = 0$$

Remove doubled propagator, reduce # IBPs

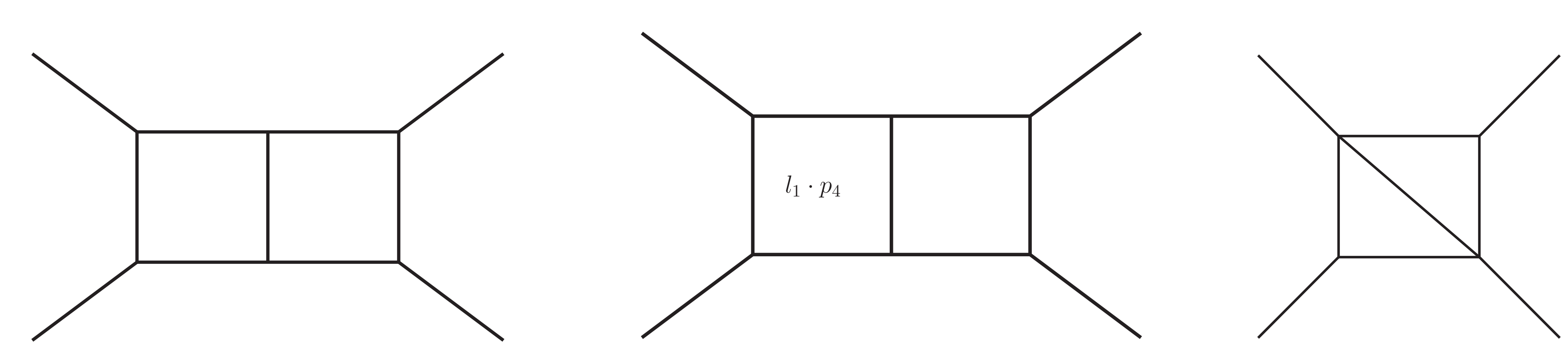
→

Syzygy for polynomials

$\{z_3 F_{z_3}, z_6 F_{z_6}, F_{z_8}, F_{z_9}, F\}$

} Tangent algebra of $z_3 z_6 F = 0$

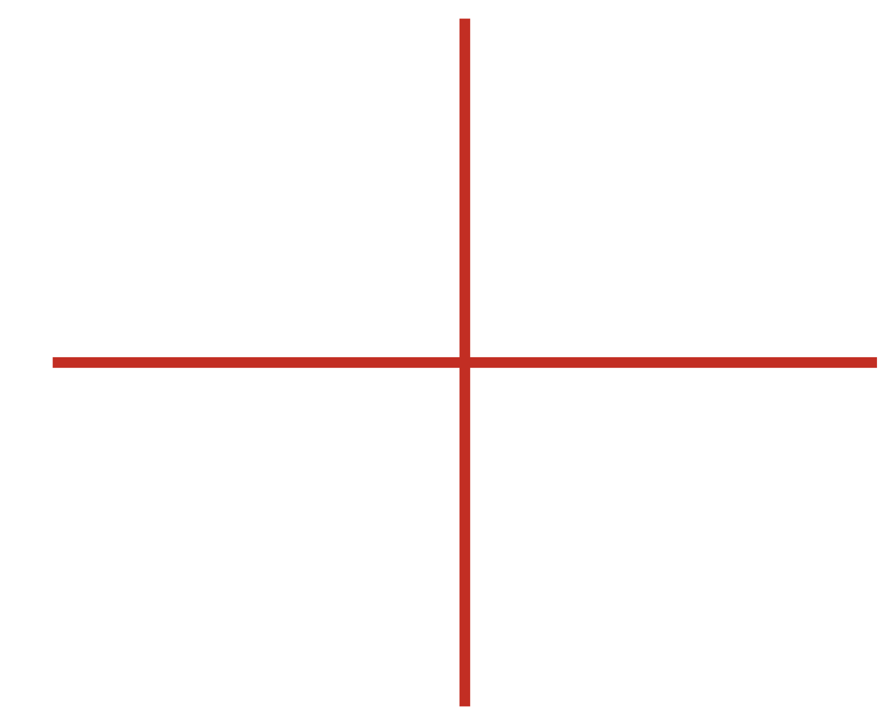
Reduce to 3 MIs



more about tangent algebra

Let X be an affine variety, $X = X_1 \cup X_2 \ldots \cup X_k$ (irreducible components).
The tangent algebra of X, $\mathbb{D}_X$ is,

$$\mathbb{D}_X = \mathbb{D}_{X_1} \cap \mathbb{D}_{X_2} \ldots \cap \mathbb{D}_{X_k}.$$



$X_1 = V(x), \quad \mathbb{D}_{X_1} = \langle x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle$

$X_2 = V(y), \quad \mathbb{D}_{X_2} = \langle \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\rangle$

more about tangent algebra

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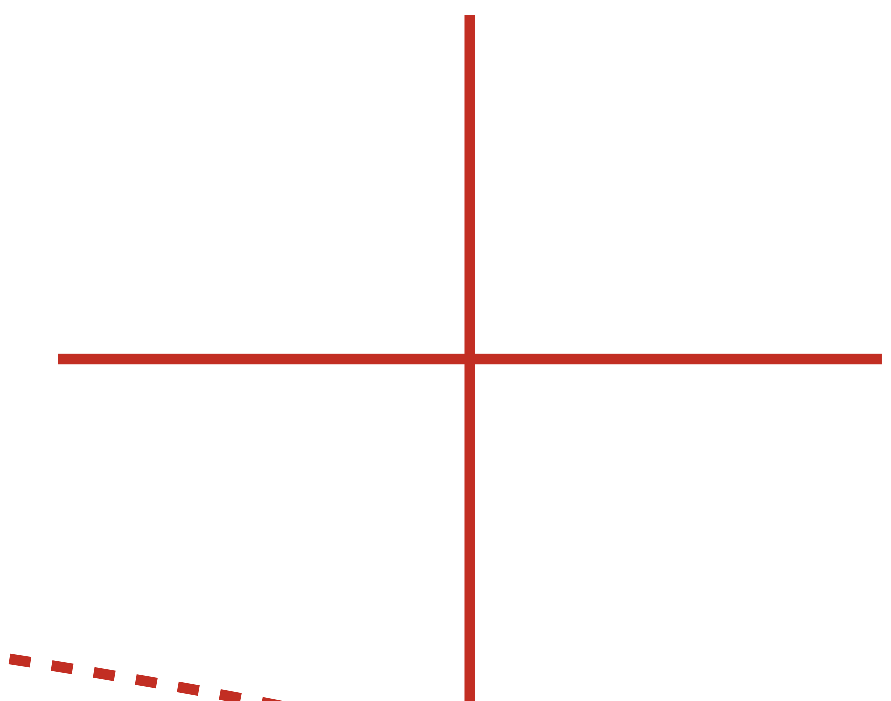

$$X_1 = V(x), \quad \mathbb{D}_{X_1} = \langle x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle$$

$$X_2 = V(y), \quad \mathbb{D}_{X_2} = \langle \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\rangle$$

$$X = X_1 \cup X_2, \quad \mathbb{D}_X = \langle x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\rangle$$

more about tangent algebra

Let X be an affine variety, $X = X_1 \cup X_2 \ldots \cup X_k$ (irreducible components).
The tangent algebra of X, $\mathbb{D}_X$ is,

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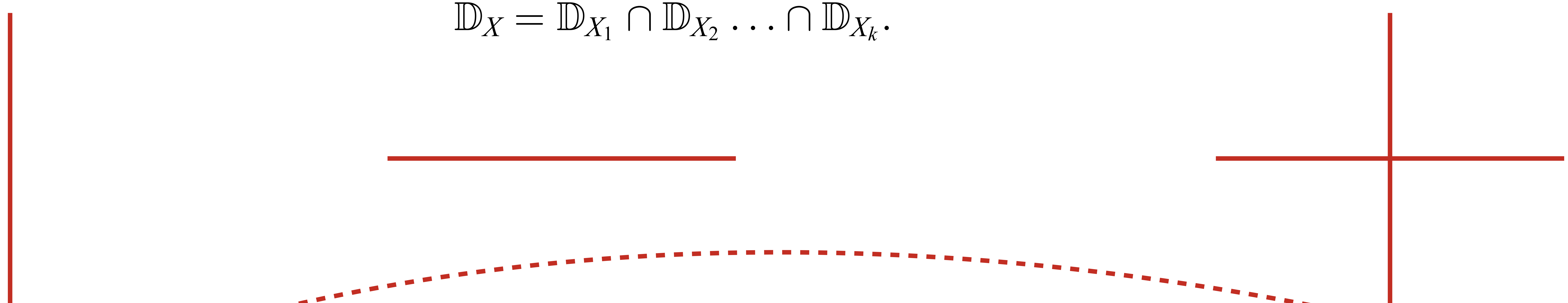


$X_1 = V(x), \quad \mathbb{D}_{X_1} = \langle x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle$

$X_2 = V(y), \quad \mathbb{D}_{X_2} = \langle \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\rangle$

$X = X_1 \cup X_2, \quad \mathbb{D}_X = \langle x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\rangle$

$$\alpha_i \frac{\partial F}{\partial z_i} + \beta F = 0 \qquad \mathbb{D}\text{: Tangent algebra of } V(F)$$

$$\alpha_3 + \beta_3 z_3 = 0 \qquad \mathbb{D}_3\text{: Tangent algebra of } V(z_3)$$

$$\alpha_6 + \beta_6 z_6 = 0 \qquad \mathbb{D}_6\text{: Tangent algebra of } V(z_6)$$

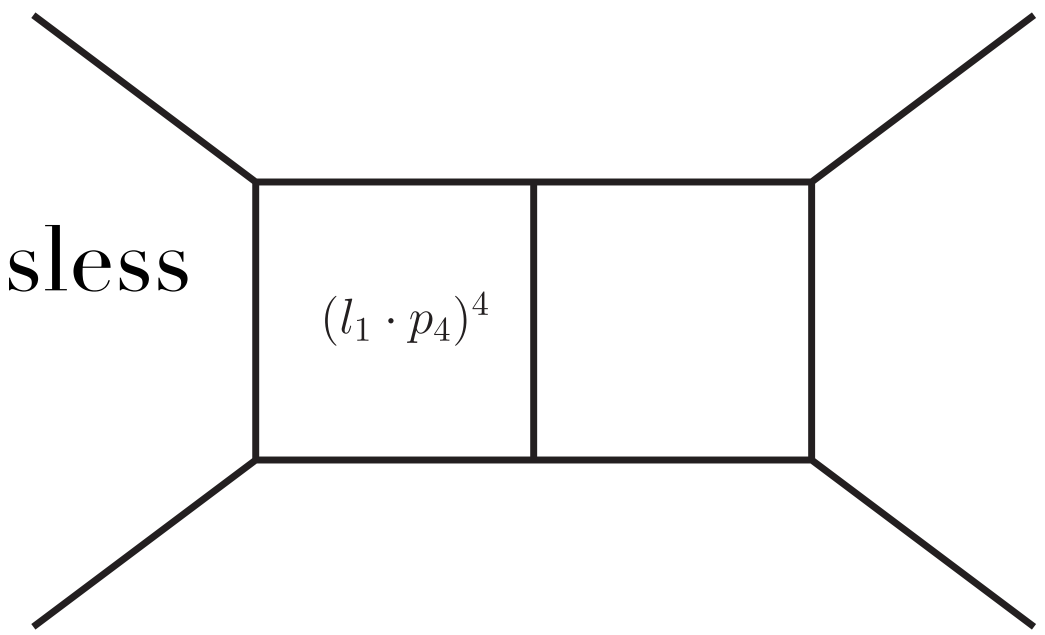
$$\boxed{\mathbb{T} = \mathbb{D} \cap (\mathbb{D}_3 \cap \mathbb{D}_6)}$$

- Algorithm: intersection of modules
- Much more efficient than direct solving

Complete reduction

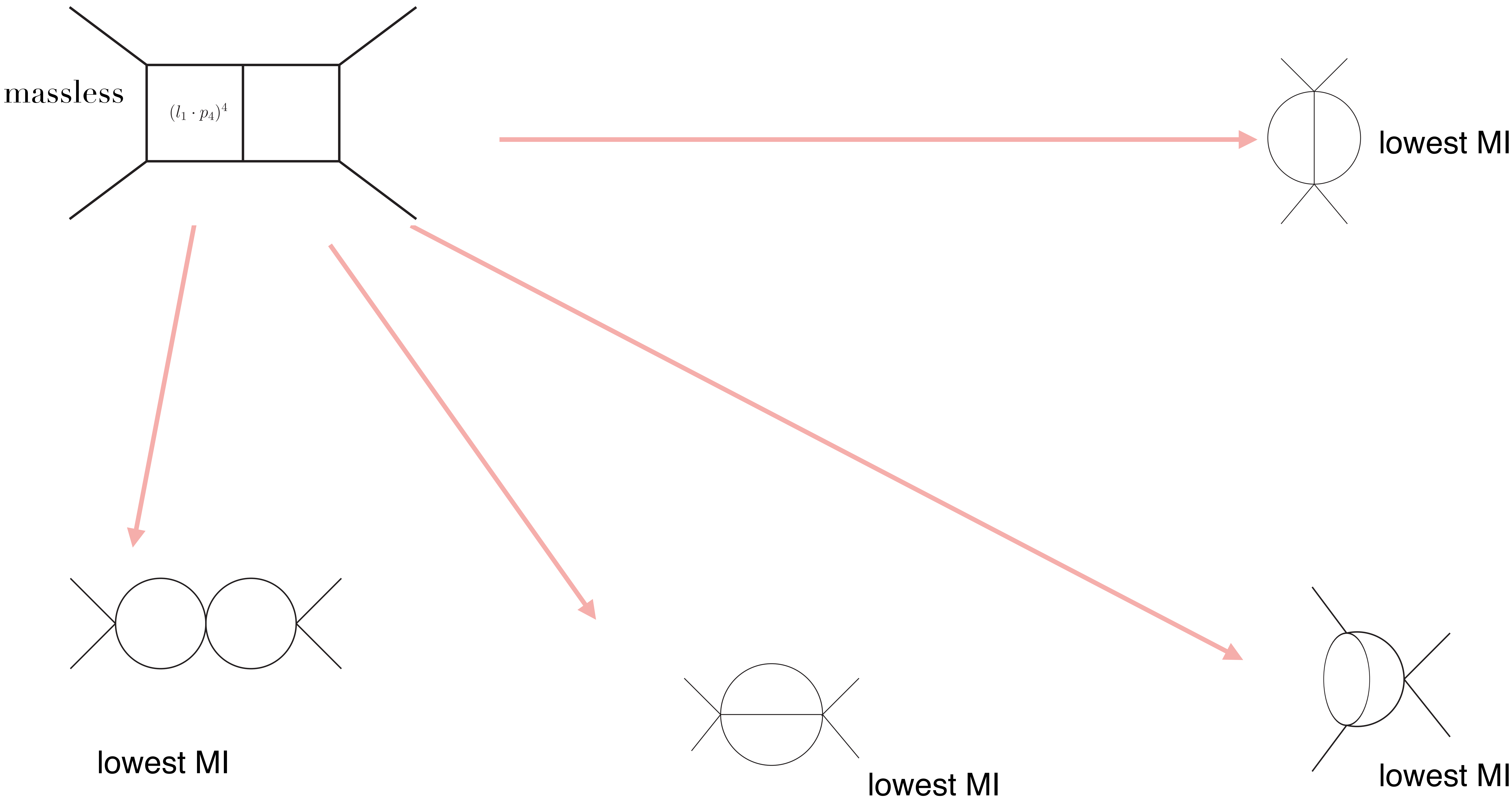
code powered by
Mathematica/Macaulay2/Singular

massless



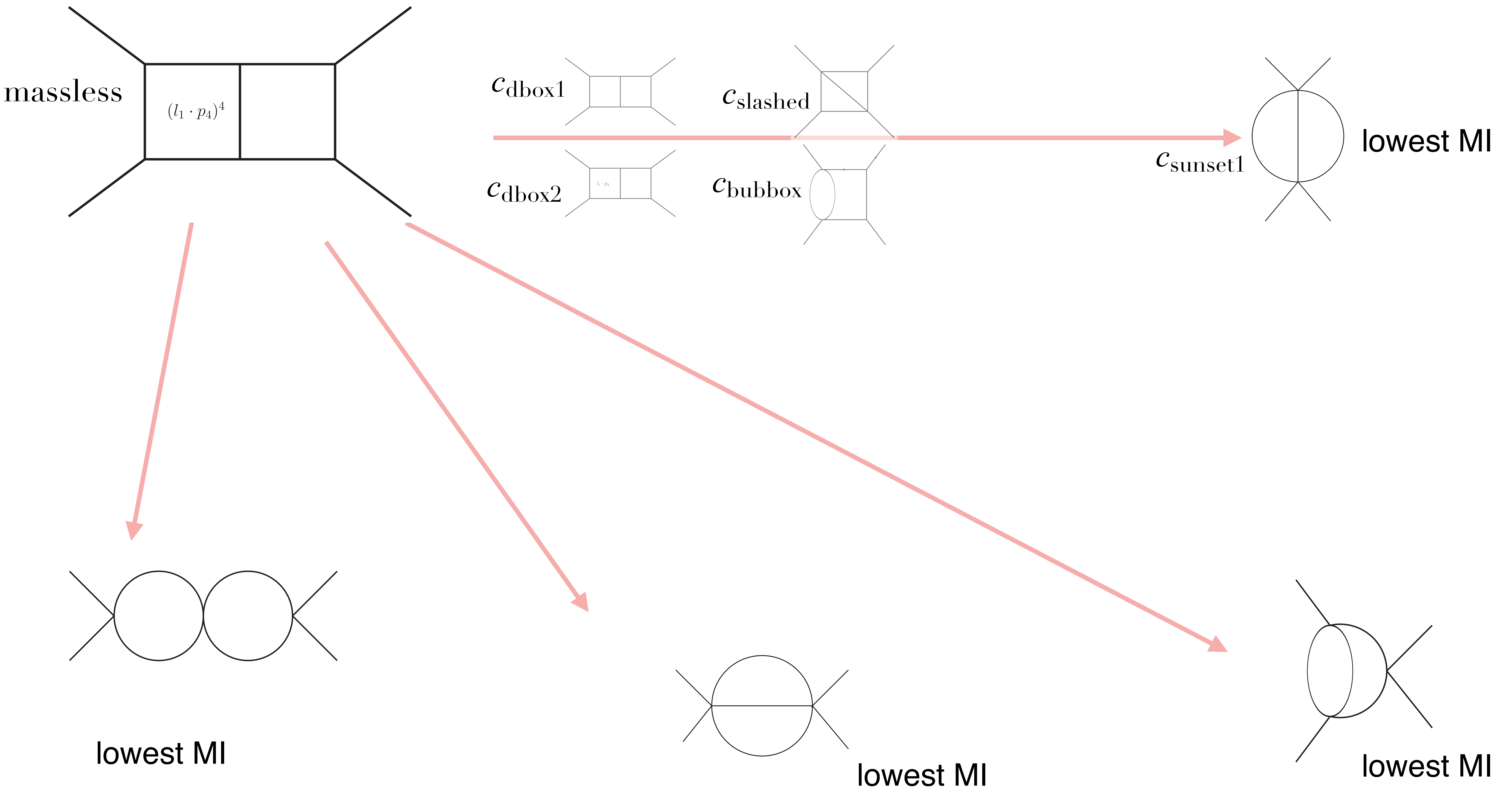
Complete reduction

code powered by
Mathematica/Macaulay2/Singular



Complete reduction

code powered by
Mathematica/Macaulay2/Singular



Complete reduction

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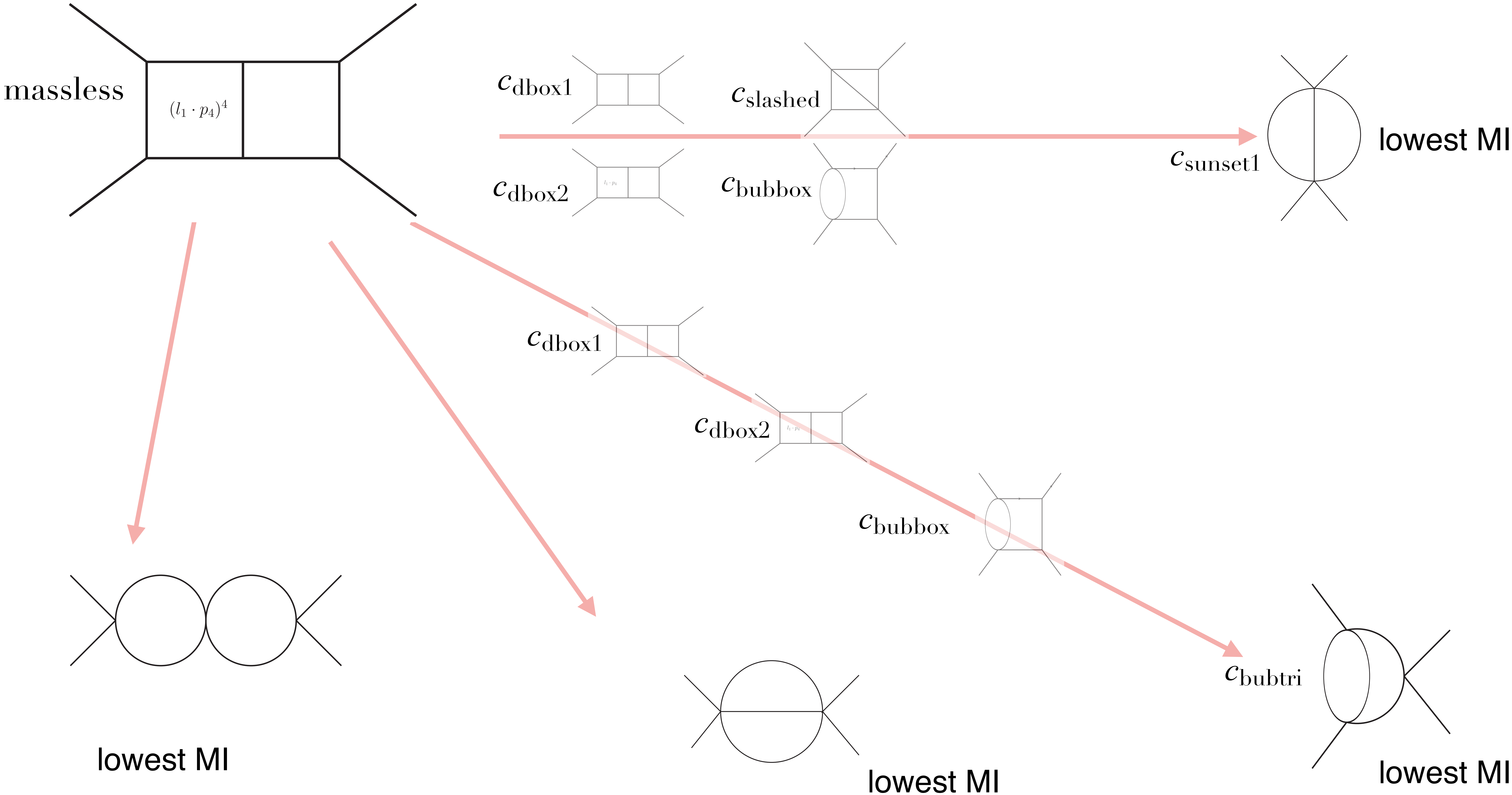
Mathematica/Macaulay2/Singular



Complete reduction

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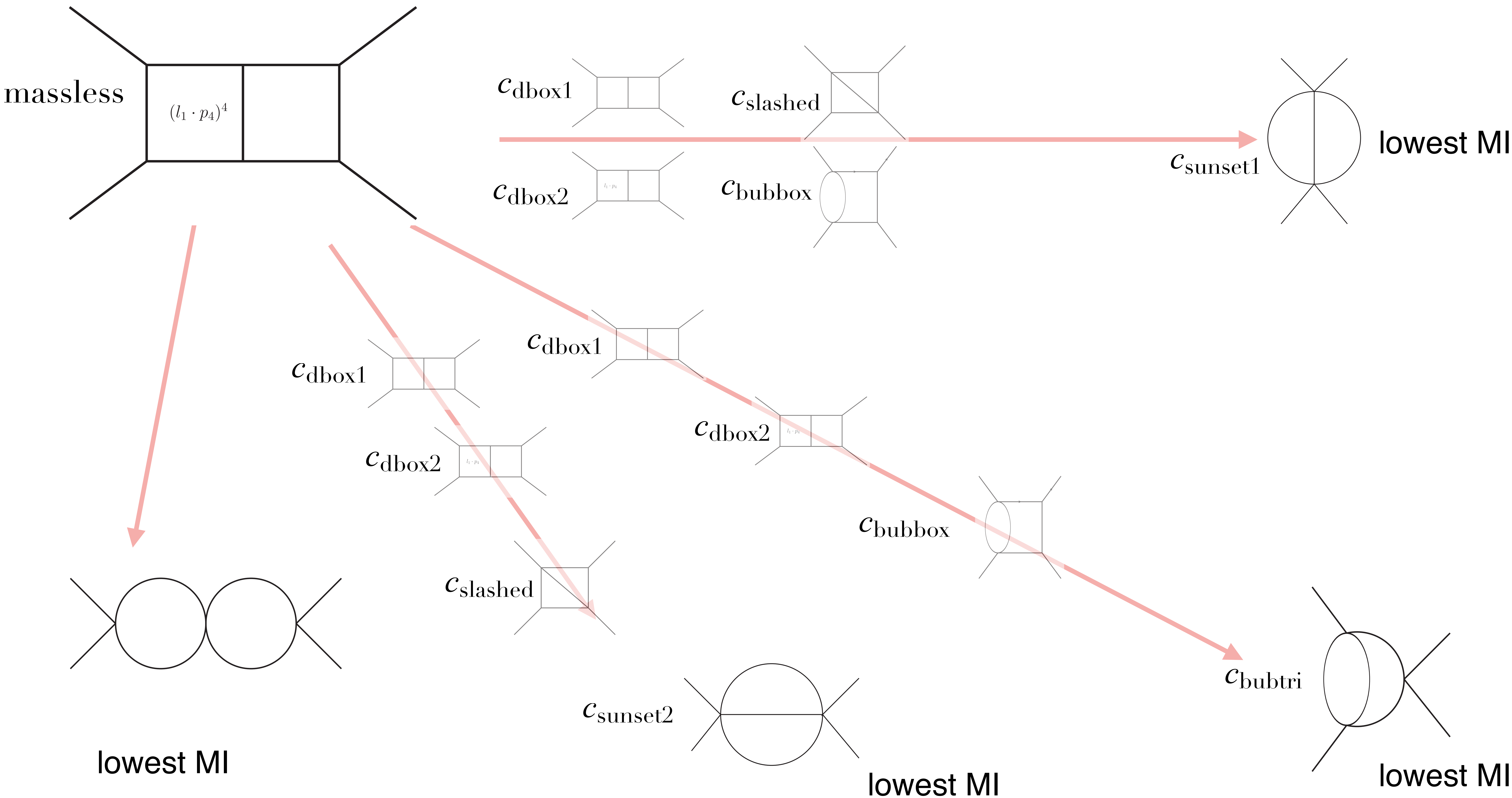

Complete reduction

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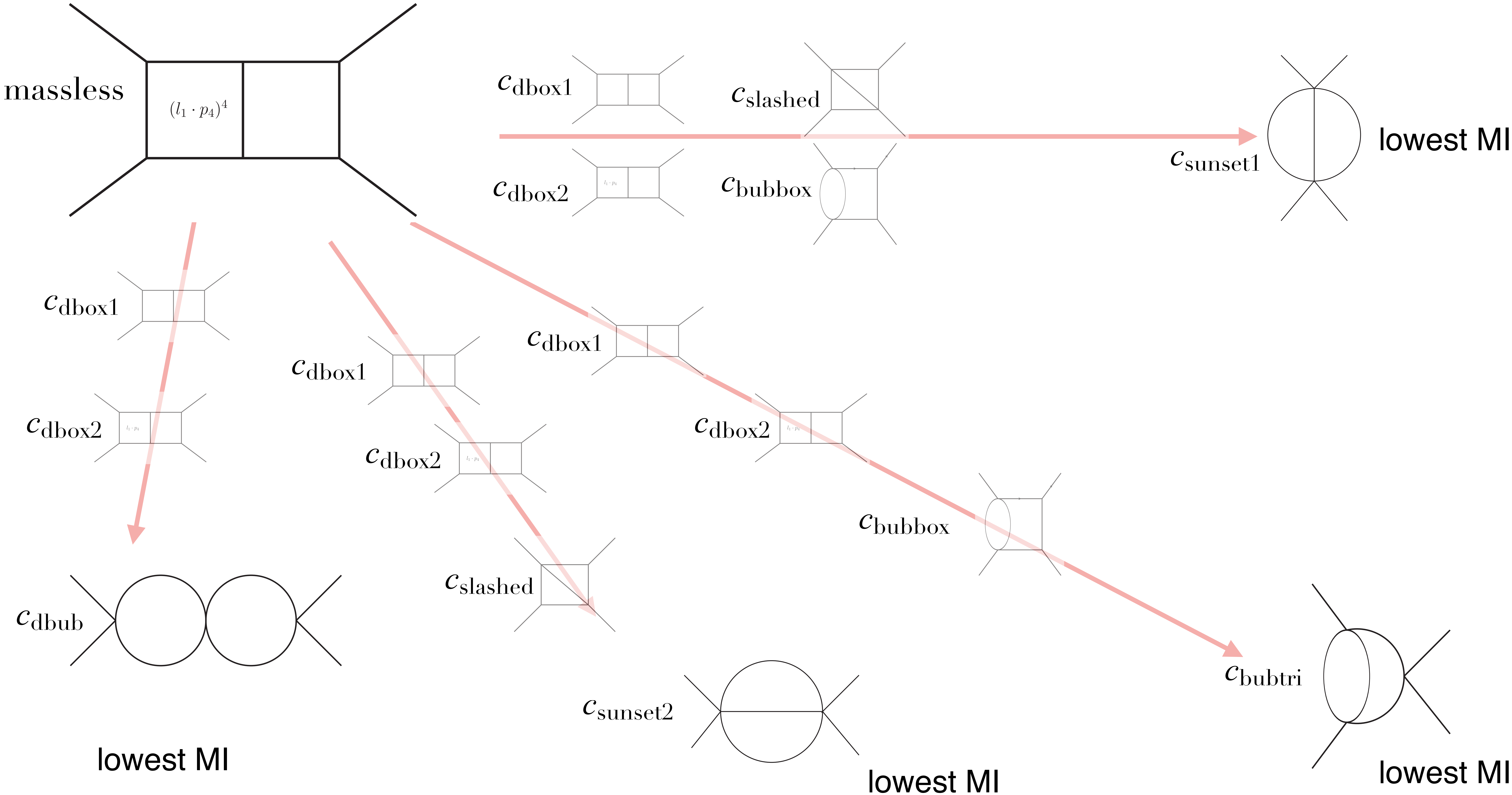

Complete reduction

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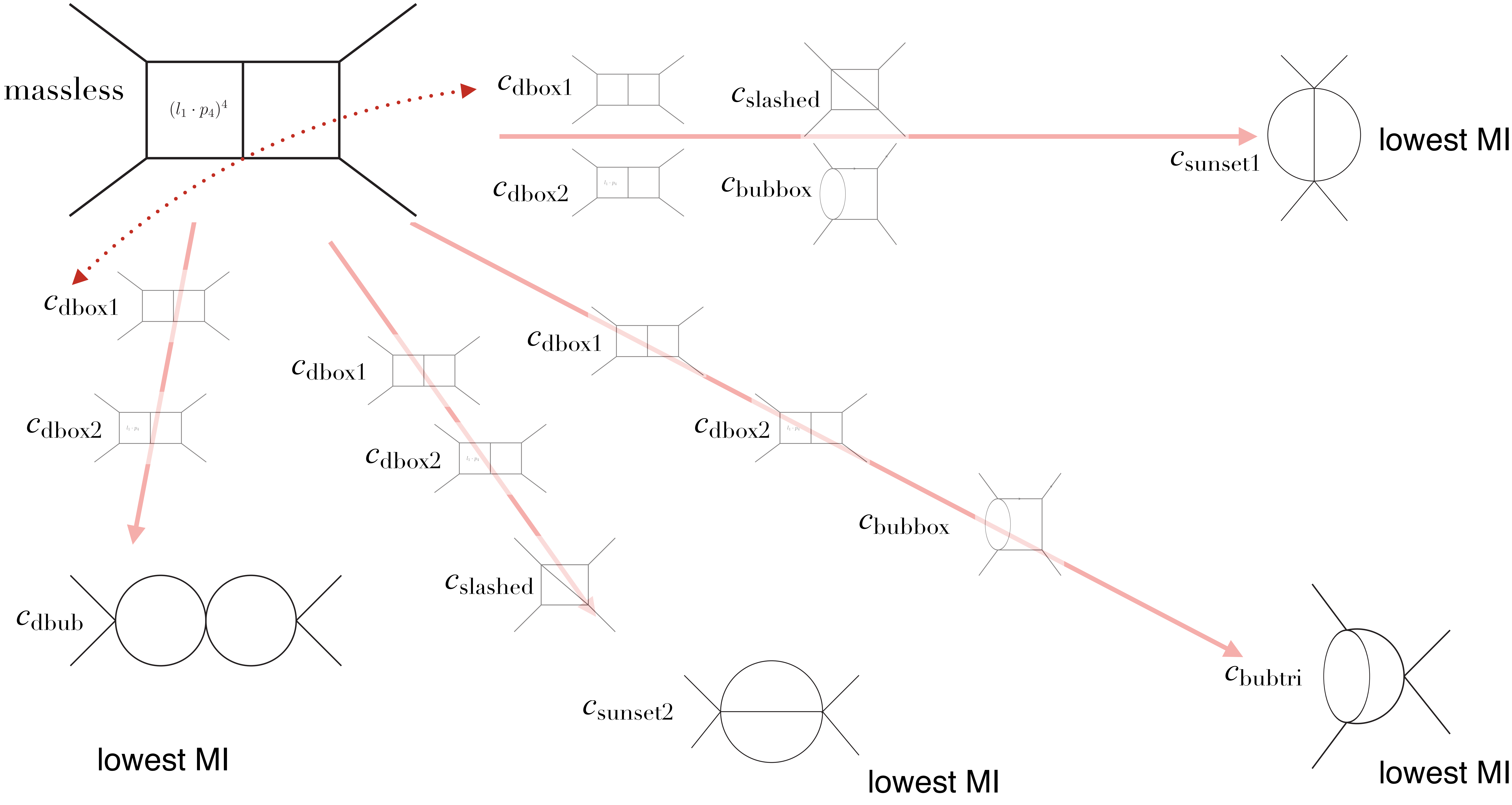
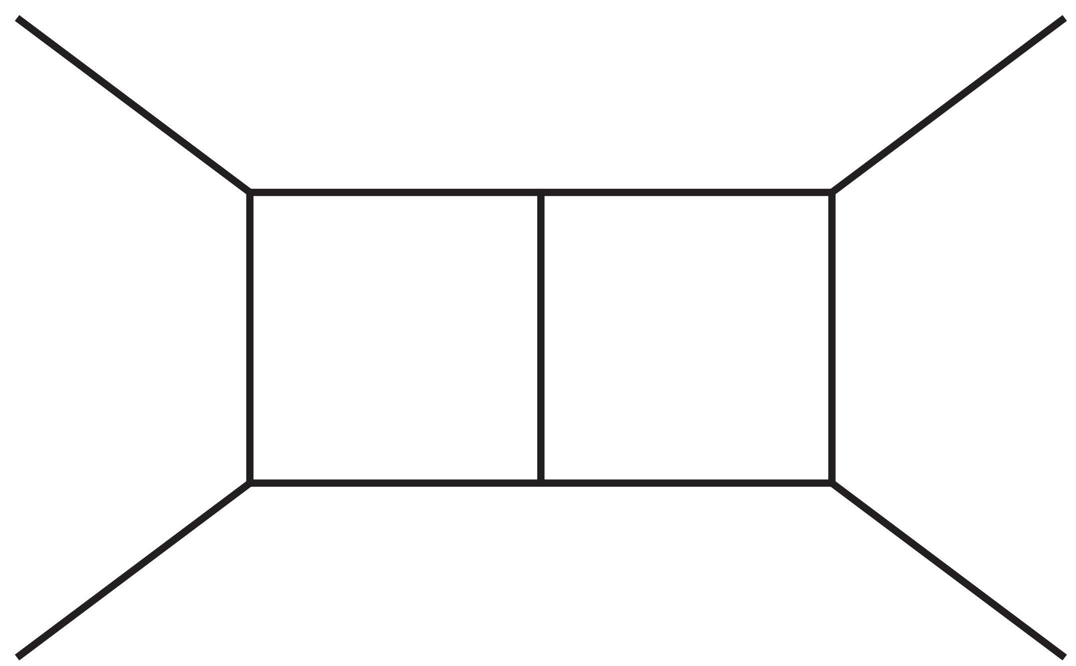
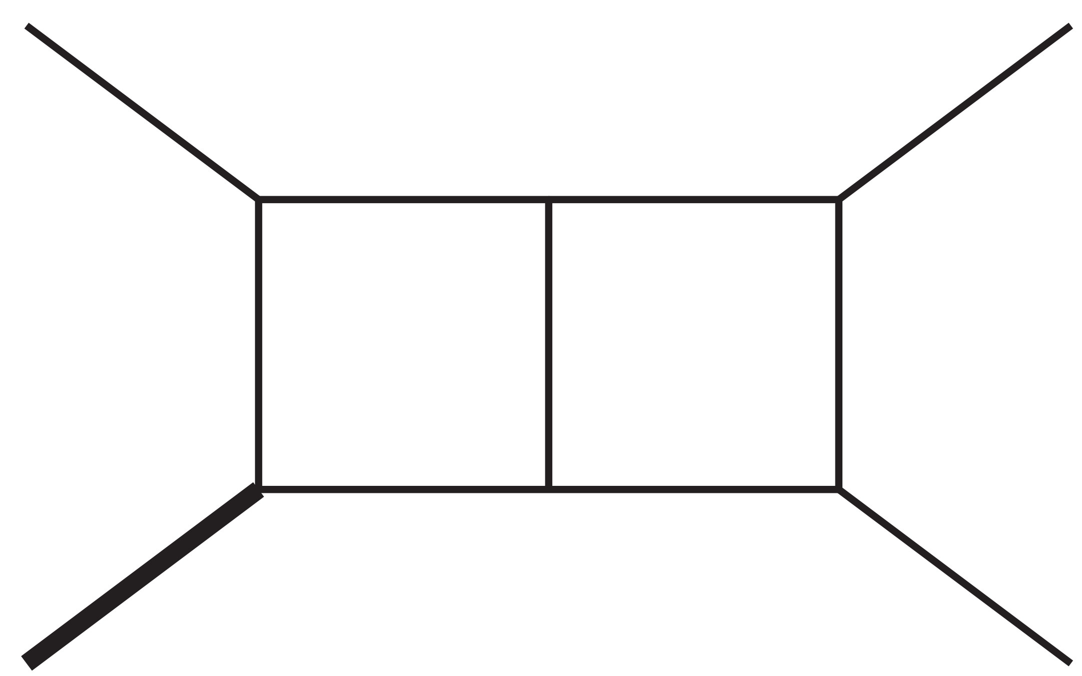


Complete IBP reduction, examples

primitive implementation powered by
Mathematica/Macaulay2/Singular

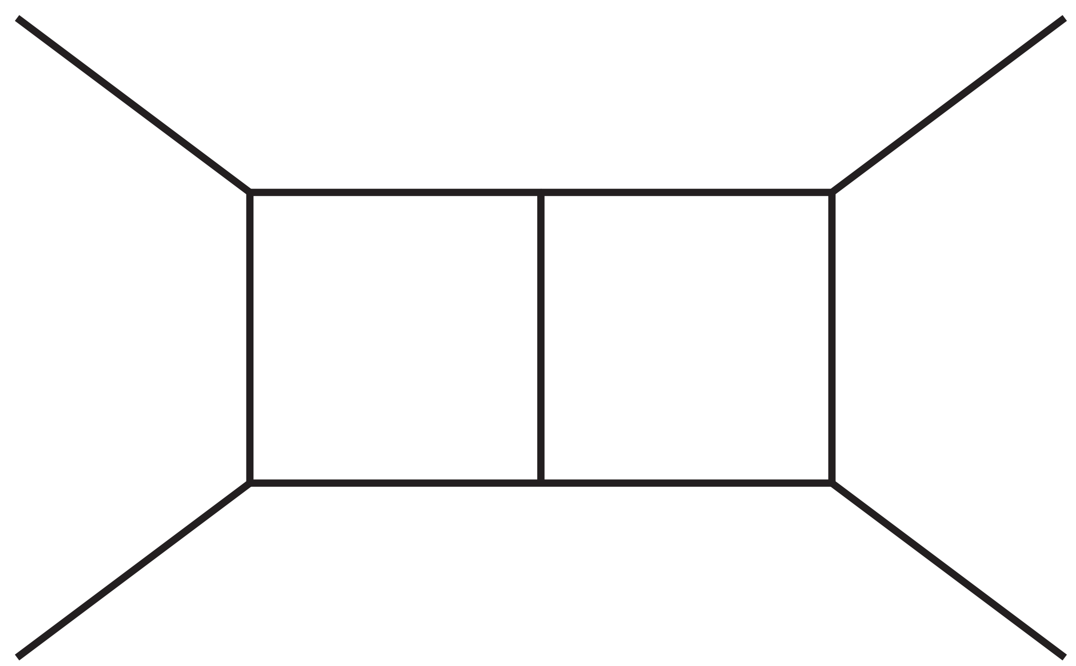


Massless double box
complete reduction
of all integrals with rank<=4
to 8 MIs in 39 seconds



Complete IBP reduction, examples

primitive implementation powered by
Mathematica/Macaulay2/Singular



Massless double box
complete reduction
of all integrals with rank<=4
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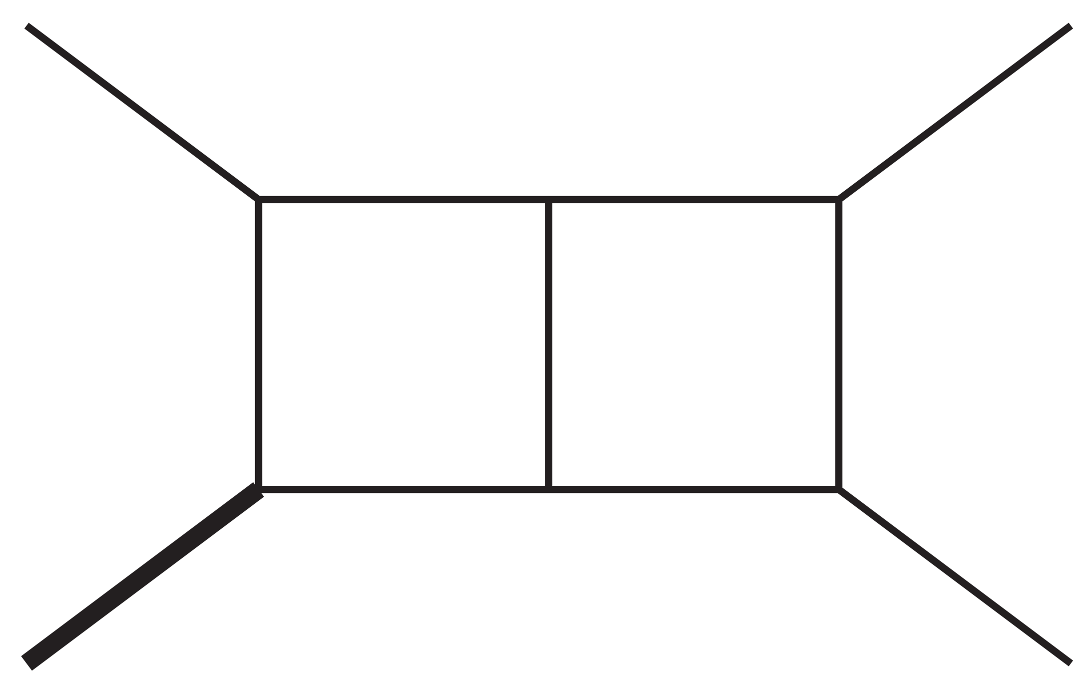

One-Mass double box
complete reduction
of all integrals with rank<=4
to 19 MIs in 162 seconds

Summary

- Algebraic geometry approach for IBP reduction
- highly efficient for examples tested
- arbitrary kind of unitarity cut

Future directions

- Large dimension limit
- Weyl algebra and D-modules approach
- Recursive reduction
- A fully automatic program