A Worst Case Survival Guide For The Four-Loop QCD Form Factors

Robert M. Schabinger

with Andreas von Manteuffel and Erik Panzer (for this talk: Phys.Lett. B744 (2015) 101)

Trinity College Dublin

Outline

What We're Doing

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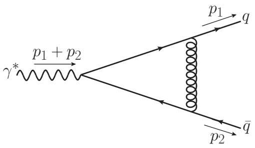

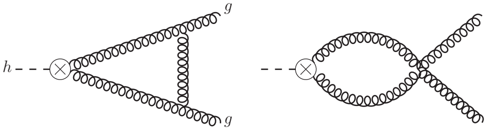

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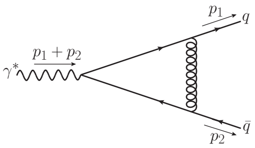

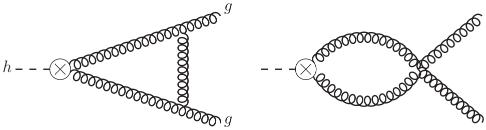

(Yes, we do use Feynman diagrams)

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L. Magnea and G. Sterman, Phys.Rev. D42 (1990) 4222

$$q^2 \frac{\partial}{\partial q^2} \ln\left(\mathcal{F}\left(q^2/\mu^2, \alpha_s, \epsilon\right)\right) = 1/2\mathcal{K}(\alpha_s) + 1/2\mathcal{G}\left(q^2/\mu^2, \alpha_s, \epsilon\right)$$

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At L loops, Γ_L characterizes the leading IR divergences which cannot be understood as exponentiated lower-loop contributions.

$\Longrightarrow$ Γ_L is a very important and universal quantity.

A Dipole Formula For Gauge Theory IR Divergences?

S. Catani, Phys. Lett. **B427** (1998) 161; S. Mert Aybat *et. al.*, Phys. Rev. **D74** (2006) 074004

T. Becher and M. Neubert, **JHEP** 0906 (2009) 081; E. Gardi and L. Magnea, **JHEP** 0903 (2009) 079

The IR divergences of the simplest non-Abelian gauge theory, planar $SU(N_c)$ $\mathcal{N} = 4$ super Yang-Mills, are believed to be of the form:

$$\mathcal{A}_1^{\mathcal{N}=4}(p_1, \ldots, p_n) = \exp\left\{-\frac{1}{2}\sum_{L=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^L \mu_\epsilon^{2L\epsilon}\int_0^{\mu_\epsilon^2} \mathrm{d}\mu^2 \left(\mu^2\right)^{-1-L\epsilon} \sum_{\substack{\mathrm{i,j}=1\\ \mathrm{i}<\mathrm{j}}}^{n}\left(\Gamma_{1;L}^{\mathcal{N}=4}\ln\left(\frac{\mu^2}{-s_{ij}}\right) + \mathcal{G}_{1;L}^{\mathcal{N}=4}\right)\frac{\mathbf{T}_\mathrm{i}\cdot\mathbf{T}_\mathrm{j}}{N_c}\right\}\sum_{L=0}^{\infty}\mathbf{H}_{1;L}^{\mathcal{N}=4}(\epsilon; p_1, \ldots, p_n)$$

At four points, this structure has been realized explicitly at strong coupling (L. F. Alday and J. Maldacena, **JHEP** 0706 (2007) 064). In a nutshell, the dipole conjecture is the suggestion that, with minor modifications, the above structure could hold for more general gauge theories like QCD.

When Something Sounds Too Good To Be True...

Although some three-loop evidence was collected by Dixon (Phys. Rev. **D79** (2009) 091501) for the n_f terms, it is now clear that the dipole conjecture fails for QCD due to a Regge limit four-loop calculation and an eikonal three-loop calculation which probe the structure of the soft anomalous dimension matrix.

S. Caron-Huot, **JHEP** 1505 (2015) 093; Ø. Almelid *et. al.*, arXiv:1507.00047

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In fact, Casimir scaling for the light-like cusp anomalous dimension

$$\Gamma_L^g \stackrel{?}{=} C_A/C_F \Gamma_L^q$$

is still very much an open problem at four loops.

R. Boels *et. al.*, **JHEP** 1302 (2013) 063; Nucl. Phys. **B902** (2016) 387;

A. Grozin *et. al.*, **JHEP** 1601 (2016) 140; J. Henn *et. al.*, JHEP 1605 (2016) 066

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$\Longrightarrow$ new approaches to multi-loop calculations are required!

see Smirnov's talk based on J. Henn *et. al.*, JHEP 1403 (2014) 088 and paper above

How To Survive The Calculation

- Use a decent-sized cluster to do numerator algebra ($\sim$ 50,000 diagrams)
- Crunch lots of integral reductions for up to twelve line integrals allowing for up to 6 inverse propagators (this talk)
- Use the reductions to write the raw integrand as a linear combination of scalar master integrals
- Construct an alternative basis of finite integrals and rewrite everything in terms of it using the reductions (talk last year)
- Evaluate all finite master integrals using either `HyperInt` (ask Erik Panzer about his cool program) or `FIESTA 4`

Integration By Parts Reduction

F. Tkachov, Phys. Lett. **B100** (1981) 65; K. Chetyrkin and F. Tkachov, Nucl. Phys. **B192** (1981) 159

It is well-known that one can generate recurrence relations by considering families of Feynman integrals and then integrating by parts in d spacetime dimensions, *e.g.*

$$\begin{aligned} 0 &= \int \frac{d^d q}{(2\pi)^d} \frac{\partial}{\partial q_\mu} \left(\frac{q_\mu}{(q^2 - m^2)^a} \right) \\ &= \int \frac{d^d q}{(2\pi)^d} \left(\frac{d}{(q^2 - m^2)^a} - \frac{2 a q^2}{(q^2 - m^2)^{a+1}} \right) \\ &= (d - 2a) I(a) - 2 a m^2 I(a+1) \end{aligned}$$

In this case, the recurrence relation can be directly solved. Usually, one employs Laporta's algorithm (S. Laporta, Int. J. Mod. Phys. **A15** (2000) 5087) to reduce some particular integrals to masters using linear algebra.

The Complexity of Gaussian Elimination

- It is widely believed that the computational complexity of Gaussian elimination is $\mathcal{O}\left(n^3\right)$ for $n \times n$ rational matrices.
- This is far too simplistic and is true only if each arithmetic operation takes essentially the same amount of time.
- In a finite field, the "grade school" algorithm does have $\mathcal{O}\left(n^3\right)$ complexity but, even over the rational numbers, the situation is much worse because the numerators and denominators of the rational numbers typically increase in size after every operation.
- Intermediate expression swell is a severe problem for the grade school Gaussian elimination algorithm and can lead to run-times and run-time storage requirements which are exponential in n!

X. G. Fang and G. Havas, ISSAC '97, 28, (1997)

And It Gets Worse...

- The linear systems that one obtains from Laporta's algorithm will always have *polynomial* entries at the outset and this introduces additional complications.
- Avoiding unrecognized zeros during the course of the elimination procedure requires a very large number of polynomial greatest common divisor (GCD) computations.
- These operations actually account for a substantial fraction of the total run-time of most currently available integration by parts reduction codes.
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Luckily for us, an enormous amount of mathematical research has been devoted to ameliorating these problems!

The General Idea

To save time, we restrict ourselves in this talk to linear systems with coefficients in $\mathbb{Q}$. Let us stress, however, that a univariate rational function version is implemented.

- As we will see, the core of our improved linear system solver can be straightforwardly worked out for any Euclidean domain.
- Map system to “machine-sized” prime fields and row reduce over them to avoid intermediate expression swell.
- Sew the solutions together by “Chinese remaindering.”
- Reconstruct the rational null space vectors of interest using the rational reconstruction algorithm.

The Image Of A Rational Number Under The Canonical Homomorphism of $\mathbb{Z}$ onto $\mathbb{Z}_m$

First, one needs to figure out how to map rational numbers down to elements of $\mathbb{Z}_m$ in a faithful manner:

$$\phi_m(a/b) = \phi_m(a)\phi_m(1/b)$$

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$\Longrightarrow$ the extended Euclidean algorithm solves the problem

The Extended Euclidean Algorithm

Begin with $(g_0, s_0, t_0) = (a, 1, 0)$ and $(g_1, s_1, t_1) = (b, 0, 1)$

$$\begin{aligned} q_i &= g_{i-1} \text{ quotient } g_i \\ g_{i+1} &= g_{i-1} - q_i g_i \\ s_{i+1} &= s_{i-1} - q_i s_i \\ t_{i+1} &= t_{i-1} - q_i t_i \end{aligned}$$

The algorithm terminates when $g_{k+1} = 0$ for some k. At that point

$$s_k\, a + t_k\, b = g_k = \mathrm{GCD}(a, b)$$

What About Going The Other Way?

Mapping our rational linear systems to prime fields is not going to help unless we have some way to invert the map $\phi_m(z)$.

$$g_i = s_i m + t_i u \text{ for all } i$$

when one applies the extended Euclidean algorithm to u and $m > 0$.

If, as will usually be the case, $(m, t_i) = 1$ for all i,

$$g_i/t_i \equiv u \bmod m \text{ for all } i$$

which implies that, typically, $\phi_m^{-1}(z)$ cannot be defined.

Rational Reconstruction

P. S. Wang, SYMSAC '81, ACM Press, 212 (1981);

P. S. Wang *et. al.* SIGSAM Bulletin **16**, No. 2, 2 (1982)

- Remarkably, under appropriate conditions, the map $\phi_m(z)$ *does* have an inverse.
- For a given rational number, a/b, one can invert $\phi_m(z)$ if $m > 2\ \max\{a^2, b^2\}$.
- In this situation, the unique solution to the rational reconstruction problem is given by:

$$\frac{a}{b} = \frac{g_j}{t_j}$$

where g_j is the first g_i in the extended Euclidean algorithm to violate

$$|g_i| > \lfloor \sqrt{m/2} \rfloor$$

Linear System Solving Over $\mathbb{Q}$

We now have all the ingredients we need to describe a fast and memory-efficient algorithm for the solution of linear systems over $\mathbb{Q}$.

- On a computer cluster, select some number of cores, n_c, likely to be larger than the length of the nastiest integer expected in the result (measured in machine words).
- On each core, choose a largish machine-sized prime ($e.g.$ $2^{64} - 59$), p_i, take the image of the linear system modulo p_i, and then solve the system n_c times in parallel.
- In this fashion, a solution, $\mathbf{k}_i$, is generated on each core modulo the corresponding p_i and these solutions can be sewn together using the Chinese remainder algorithm to produce a lifted solution, $\mathbf{K}$, modulo $p_1 \cdots p_{n_c}$.
- Finally, we can attempt a rational reconstruction on the coefficients. If the procedure succeeds we are done. Otherwise, we have to compute additional samples and then try once again.

see *e.g.* M. Kauers, Nuclear Physics B (Proc. Suppl.), **183**, 245 (2008)

How Well Does The Algorithm Actually Perform?

- Despite the maturity of the subject, it is surprisingly difficult to find a working public implementation. Manuel Kauers's `Mathematica` package `LinearSystemSolver.m` is the only example known to us.
- A new code, provisionally called FinRed, was painstakingly developed by my collaborator Andreas von Manteuffel to achieve near-optimal scaling behavior and memory usage.
- Unlike `Reduze 2`, the parallelization is rather trivial since all finite field samples are completely independent of one another.
- On an old desktop, all of the rank five three-loop form factor reductions run in the time it takes to go get a hot dog.
- We could compute all rank four massless two-loop $2 \to 2$ integral reductions at a particular phase space point in about 20 seconds.
- Four-loop form factor reductions are under way, first results coming soon to a conference near you.

Outlook

Overall, our IBP algorithm seems strong enough to significantly ameliorate one of the biggest performance bottlenecks to calculating the four-loop cusp anomalous dimensions in QCD. Several obvious improvements and further applications come to mind:

- The syzgzy-based approach to integration by parts, J. Gluza *et. al.*, Phys.Rev. D83 (2011) 045012, has recently been developed to the point where applications look like they are just around the corner. The work discussed in Zhang's talk (based on a paper with K. Larsen, Phys.Rev. D93 (2016) no.4, 041701) appears to be of immense practical value and must be investigated further.
- In fact the "numerical unitarity" idea of H. Ita, arXiv:1510.05626, may be realizable by combining their method with ours.
- A tool like the one described here can be used to solve problems where a big ansatz must be constrained and then fit.