Multi-Regge kinematics and configurations of points on the Riemann sphere

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Planar N=4 SYM

- Planar N=4 SYM is our favorite playground to explore the structure of scattering amplitudes in gauge theories.
- Many structures have been uncovered over the last years:
  - ➡ Singularities described by cluster algebra. [Golden, Goncharov, Spradlin, Vergu, Volovich]
  - ➡ Perturbative amplitudes are iterated integrals.
  - ➡ Dynamics encoded into collinear OPE. [Basso, Sever, Vieira]
  - ➡ Building blocks of OPE can be obtained from integrability.
- Very successful for six-point amplitude!
  - ➡ Hexagon bootstrap program. [Dixon, CD, Drummond, Henn, van Hippel, McLeod, Pennington]
  - ➡ Heptagon amplitude and cluster polylogarithms.

[Golden, Spradlin; Drummond, Papathanasiou, Spradlin]

Planar N=4 SYM

- Despite this progress:
  - ➡ Only 6 & 7 point MHV amplitudes known.
  - ➡ Only 6 point NMHV amplitude known.
- Reasons (among others):
  - ➡ Cluster algebra infinite starting from 8 points.
  - ➡ Non-polylogarithmic functions expected to appear for non-MHV amplitudes.
- Aim of this talk: Present a limit where all the mathematical structures act in harmony.
  - ➡ Results for many loops and legs for complicated helicity configurations.

Outline

- Multi-Regge Kinematics
- The moduli space $\mathfrak{M}_{0,n}$
  - ➡ The geometry of multi-Regge kinematics
- MHV amplitudes in multi-Regge kinematics
  - ➡ Convolutions & Factorisation
- Non-MHV amplitudes in multi-Regge kinematics
  - ➡ Helicity flips & leading singularities

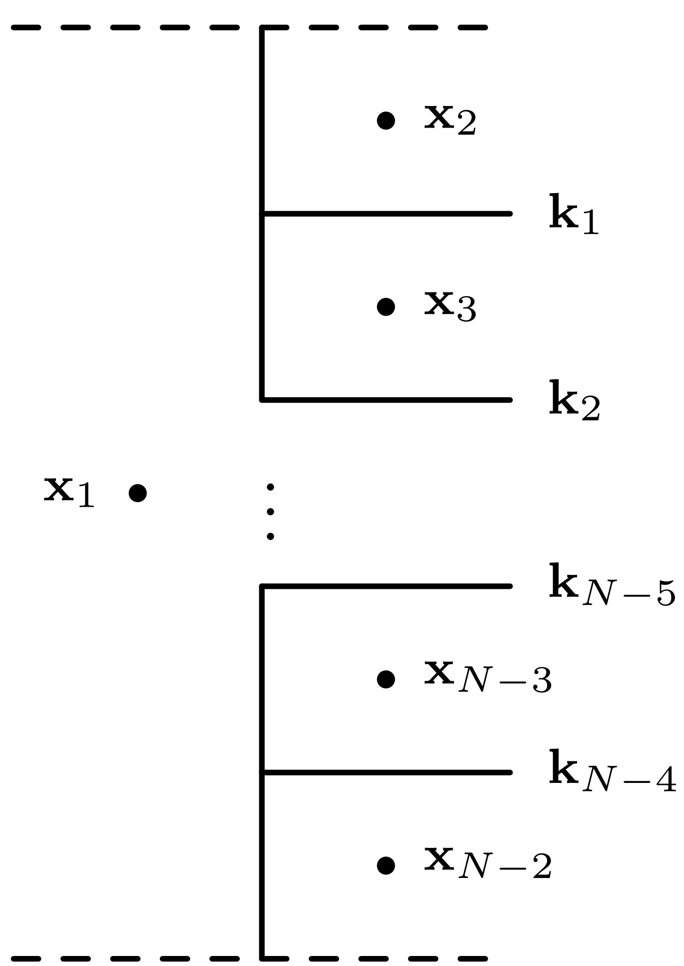
Multi-Regge kinematics

Multi-Regge kinematics

- Definition of MRK:

$$p_3^+ \gg p_4^+ \gg \dots p_{N-1}^+ \gg p_N^+ \,, \qquad |\mathbf{p}_3| \simeq \dots \simeq |\mathbf{p}_N| \qquad \mathbf{p}_k = p_k^x + i p_k^y$$

- Non-trivial kinematical dependence in transverse momenta.



➡ Dual conformal invariance in transverse space implies dependence on $N-5$ cross ratios:

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}$$

➡ Strong ordering in rapidities implies no collinear singularities, only soft:

$$\mathbf{k}_i \to 0 \quad \Leftrightarrow \quad \mathbf{x}_{i+1} \to \mathbf{x}_{i+2}$$

Multi-Regge kinematics

- In the Euclidean region, the amplitude vanishes in MRK.
- After analytic continuation to Mandelstam region, the amplitude does no longer vanish in the limit.
- Here: Leading logarithmic accuracy (LLA):

$$\mathcal{R}^{[p,q]}_{h_1\dots h_{N-4}} = 1 + a\,i\pi\,r^{[p,q],(1)}_{h_1\dots h_{N-4}} + a\,i\pi\,(-1)^{q-p}\left[\prod_{k=p}^{q-1}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar{z}_k}\right)^{n_k/2}\int_{-\infty}^{+\infty}\frac{d\nu_k}{2\pi}|z_k|^{2i\nu_k}\right]$$

$$\times\left[-1+\prod_{k=p}^{q-1}\tau_k^{aE_{\nu_k n_k}}\right]\chi^{h_p}(\nu_p,n_p)\left[\prod_{k=p}^{q-2}C^{h_{k+1}}(\nu_k,n_k,\nu_{k+1},n_{k+1})\right]\chi^{-h_q}(\nu_{q-1},n_{q-1})$$

$E_{\nu n}$: BFKL eigenvalue
large $\log\tau_k$ at every order.

Impact factors &
central emission blocks

$$= 1 + a\,i\pi\,r^{[p,q],(1)}_{h_1\dots h_{N-4}} + 2\pi i\sum_{i=2}^{\infty}\sum_{i_1+\ldots+i_{N-5}=i-1} a^i\left(\prod_{k=1}^{N-5}\frac{1}{i_k!}\log^{i_k}\tau_k\right) g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}$$

Perturbative coefficient (labelled by powers of logs and helicities)

Known results at LLA

- Six-point MHV and NMHV amplitudes.
  - ➡ Known to arbitrary loop order at LLA.
  - ➡ Can be expressed via single-valued harmonic polylogarithms.

[Lipatov, Prygarin; Dixon, Drummond, Henn; Dixon, CD, Pennington; Pennington; Brödel, Sprenger; Bartels, Kormilitzin, Lipatov, Prygarin]

- Two-loop MHV amplitudes at LLA.
  - ➡ Factorise into six-point amplitudes.

[Prygarin, Spradlin, Vergu, Volovich; Bartels, Prygarin, Lipatov]

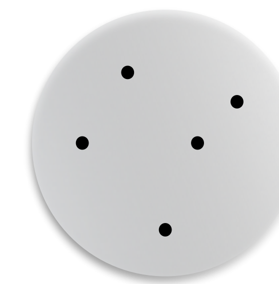
- Aim of this talk:. Generalise the LLA results to higher loop orders and to arbitrary helicity configurations.

The moduli space $\mathfrak{M}_{0,n}$

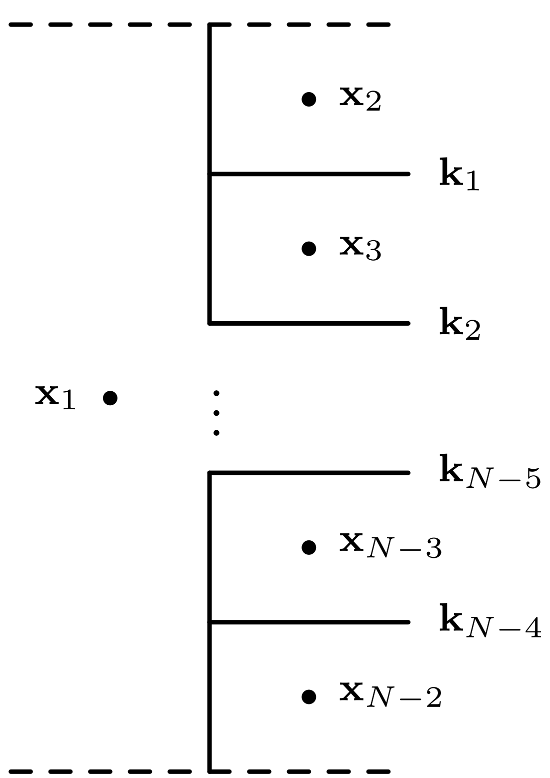
The geometry of multi-Regge kinematics

The moduli space $\mathfrak{M}_{0,n}$

- $\mathfrak{M}_{0,n}$ = moduli space space of Riemann spheres with n marked points.
  = space of configurations of n points on the Riemann sphere.



- MRK is defined by a configuration of $n = N-2$ points in the transverse place, which we identify with the Riemann sphere.



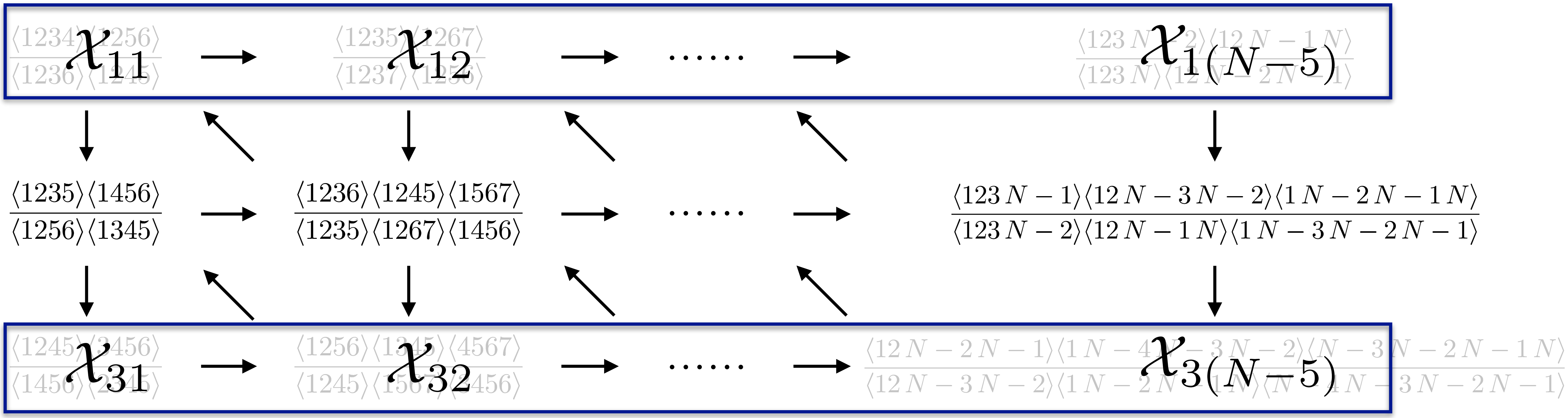
➡ $\dim_{\mathbb{C}} \mathfrak{M}_{0,n} = n-3$

➡ Coordinates are collection of $n-3 = N-5$ cross ratios

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}$$

➡ Singularities when points become equal (cf. soft): $\mathbf{x}_i = \mathbf{x}_j$

Cluster algebra picture



$$\mathcal{X}_{1j} = \frac{(\overline{\mathbf{x}}_2 - \overline{\mathbf{x}}_{j+2})(\overline{\mathbf{x}}_{j+3} - \overline{\mathbf{x}}_{j+4})}{(\overline{\mathbf{x}}_2 - \overline{\mathbf{x}}_{j+4})(\overline{\mathbf{x}}_{j+2} - \overline{\mathbf{x}}_{j+3})} \qquad \mathcal{X}_{3j} = \frac{(\mathbf{x}_1 - \mathbf{x}_{j+1})(\mathbf{x}_{j+2} - \mathbf{x}_{j+3})}{(\mathbf{x}_1 - \mathbf{x}_{j+3})(\mathbf{x}_{j+1} - \mathbf{x}_{j+2})}$$

[See Spradlin's talk]

- Two complex conjugate copies of the cluster algebra A_{N-5} .
- A_{N-5} is the cluster algebra associated to

$$\mathrm{Conf}_{N-2}(\mathbb{CP}^1) \simeq \mathfrak{M}_{0,N-2}$$

MRK and $\mathfrak{M}_{0,n}$

- Iterated integrals on $\mathfrak{M}_{0,n}$ have alphabet $d\log(\mathbf{x}_i - \mathbf{x}_j)$.

- Gauge fixing (Simplicial coordinates): [Brown]

$$(\mathbf{x}_1, \ldots, \mathbf{x}_n) = (0, 1, \infty, t_1, \ldots, t_{n-3})$$

➡ Alphabet reduces to $d\log t_i$, $d\log(1 - t_i)$, $d\log(t_i - t_j)$.

- Alphabet is linear in all simplicial coordinates.

➡ Can always be evaluated in terms of multiple polylogarithms.

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; z)$$

$$G(a; z) = \log\left(1 - \frac{z}{a}\right) \qquad G(0; z) = \log z \qquad G(0, 1; z) = -\mathrm{Li}_2(z)$$

MRK and $\mathfrak{M}_{0,n}$

- Issues:
  - ➡ We have two complex conjugate copies of this alphabet.
  - ➡ Branch cuts are constraint by unitarity.
- Unitarity forces the iterated integrals in MRK to be single-valued functions on $\mathfrak{M}_{0,N-2}$.
- Conclusion:

> N-point scattering amplitudes in planar N=4 SYM in MRK are single-valued iterated integrals on $\mathfrak{M}_{0,N-2}$.

- Generalises the 6-point SVHPL story to arbitrary numbers of points, loops and helicity configurations.
- Only polylogarithms appear in MRK!

Single-valued functions

- Single-valued polylogarithms = combinations of poylogarithms and their complex conjugates such that all branch cuts cancel.
- One way to construct them: A map $\mathbf{s}$ that assigns to each polylogarithm its single-valued version:

$$\mathbf{s} = \mu(\tilde{S} \otimes \mathrm{id})\Delta$$

[cf. Brown for MZV case]

$\mu =$ multiplication

$\Delta =$ coproduct

$\tilde{S} =$ Complex conjugate of the antipode (up to a sign)

- Examples: $\mathcal{G}(\vec{a}; z) = \mathbf{s}(G(\vec{a}; z))$

$$\mathcal{G}_a(z) = G_a(z) + G_{\bar{a}}(\bar{z}) = \log\left|1 - \frac{z}{a}\right|^2$$

$$\mathcal{G}_{a,b}(z) = G_{a,b}(z) + G_{\bar{b},\bar{a}}(\bar{z}) + G_b(a)G_{\bar{a}}(\bar{z}) + G_{\bar{b}}(\bar{a})\,G_{\bar{a}}(\bar{z}) - G_a(b)G_{\bar{b}}(\bar{z}) + G_a(z)G_{\bar{b}}(\bar{z}) - G_{\bar{a}}\left(\bar{b}\right)G_{\bar{b}}(\bar{z}) \,.$$

Single-valued functions

- Preserves multiplication: $\mathbf{s}(a\cdot b)=\mathbf{s}(a)\cdot\mathbf{s}(b)$
- Preserves functional equations.
- Commutes with holomorphic differentiation: $\partial_z\,\mathbf{s}=\mathbf{s}\,\partial_z$
- Antipode corresponds to complex conjugation: $\bar{\mathbf{s}}=\mathbf{s}\,\tilde{S}$
  - ➡ Example:

$$\mathcal{G}(\bar a,\bar b;\bar z)=\bar{\mathbf{s}}(G(a,b;z))$$
$$=\mathcal{G}(b,a;z)+\mathcal{G}(b;a)\,\mathcal{G}(a;z)-\mathcal{G}(a;b)\,\mathcal{G}(b;z)$$

- Does not commute with anti-holomorphic differentiation:
  - ➡ Example:

$$\bar\partial_z\mathcal{G}(a,b;z)=\frac{1}{\bar z-\bar a}\mathcal{G}(b;a)+\frac{1}{\bar z-\bar b}(\mathcal{G}(a;z)-\mathcal{G}(a;b))$$

MHV amplitudes in MRK

Convolutions & Factorisation

Convolutions

- Master formula:

$$g^{(i_1,\dots,i_{N-5})}_{h_1,\dots,h_{N-4}}(z_1,\dots,z_{N-5}) = \frac{(-1)^{N+1}}{2}\left[\prod_{k=1}^{N-5}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar{z}_k}\right)^{n_k/2}\int_{-\infty}^{+\infty}\frac{d\nu_k}{2\pi}|z_k|^{2i\nu_k}E^{i_k}_{\nu_k n_k}\right]$$

$$\times\,\chi^{h_1}(\nu_1,n_1)\left[\prod_{j=1}^{N-6}C^{h_j}(\nu_j,n_j,\nu_{j+1},n_{j+1})\right]\chi^{-h_{N-5}}(\nu_{N-5},n_{N-5})\,.$$

➡ Fourier-Mellin transform: $\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{+\infty}\left(\frac{z}{\bar{z}}\right)^{n/2}\int_{-\infty}^{+\infty}\frac{d\nu}{2\pi}\,|z|^{2i\nu}\,F(\nu,n)$

- FM transform maps products to convolutions:

$$\mathcal{F}[F\cdot G] = \mathcal{F}[F]*\mathcal{F}[G] = f*g\ = \frac{1}{\pi}\int\frac{d^2w}{|w|^2}\,f(w)\,g\left(\frac{z}{w}\right)$$

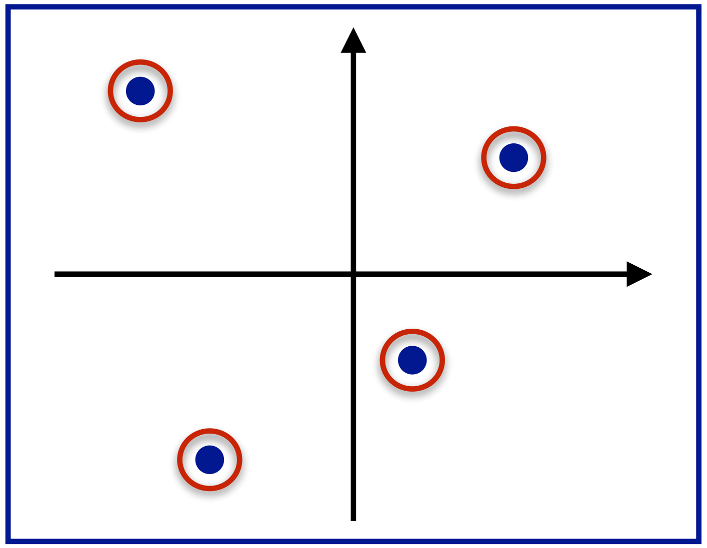
- Translates into a recursion in the number of loops:

$$g^{(i_1,\dots,i_k+1,\dots,i_{N-5})}_{+\dots+}(z_1,\dots,z_{N-5})\ = \mathcal{E}(z_k)*g^{(i_1,\dots,i_{N-5})}_{+\dots+}(z_1,\dots,z_{N-5})$$

$$\mathcal{E}(z) = \mathcal{F}\left[E_{\nu n}\right] = -\frac{z+\bar{z}}{2|1-z|^2}$$

Stokes' theorem & residues

- Single-valuedness implies that the integral can easily be performed in terms of Stokes' theorem.
  - ➡ All singularities are isolated.
  - ➡ Can integrate over the boundary of the punctured complex plane.



$$\int \frac{d^2z}{\pi} f(z) = \text{Res}_{z=\infty} F(z) - \sum_i \text{Res}_{z=a_i} F(z) \qquad \bar{\partial}_z F = f$$

[Schnetz]

- Convolution appearing in the loop recursion reduces to a simple residue computation.
- Start of the recursion: Two-loop MHV amplitude:

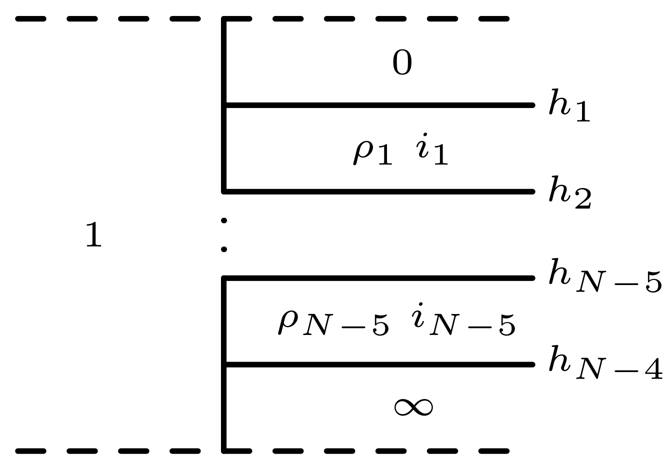
$$\mathcal{R}^{(2)}_{+\ldots+} = \sum_{1\le i\le N-5} \log \tau_i \, g^{(1)}_{++}(\rho_i)$$

[Prygarin, Spradlin, Vergu, Volovich; Bartels, Prygarin, Lipatov; Bargheer, Schomerus, Papathanasiou]

Factorisation

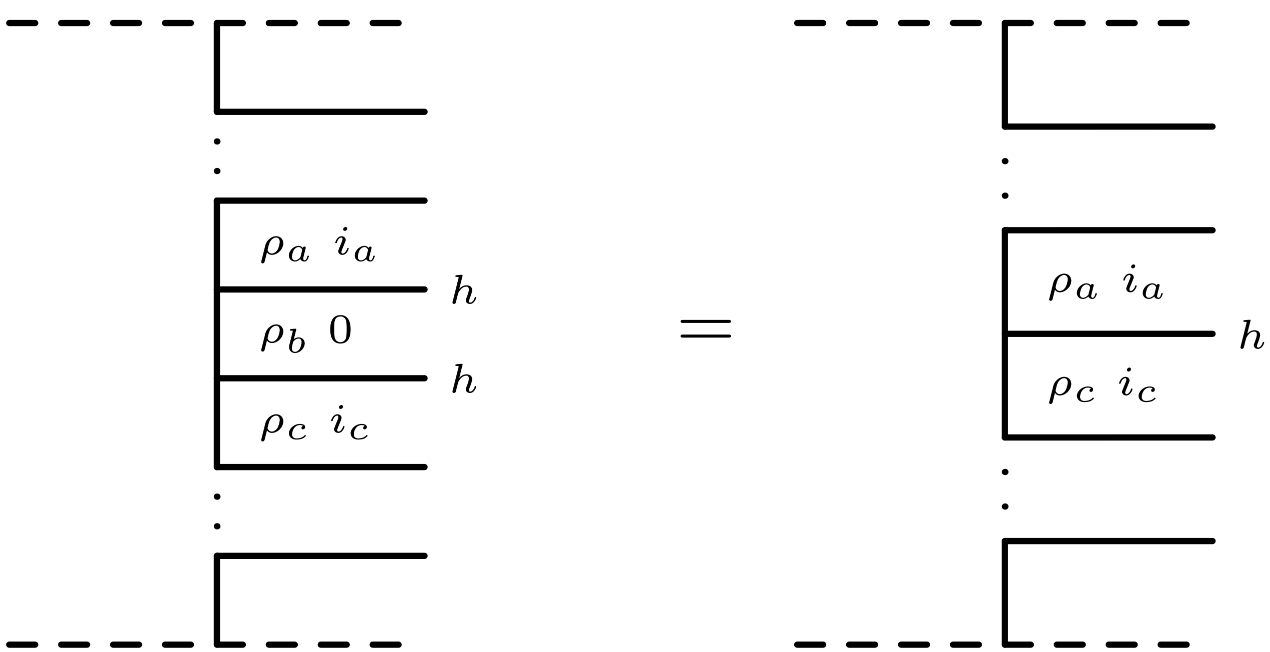
- Convolutions imply a factorisation theorem!

$$g^{(i_1,\ldots,i_{N-5})}_{h_1\ldots h_{N-4}}(\rho_1,\ldots,\rho_{N-5}) = \quad 1$$



$$z_i = \frac{(\rho_i - \rho_{i-1})(\rho_{i+1} - 1)}{(\rho_i - \rho_{i+1})(\rho_{i-1} - 1)}$$

- Factorisation theorem:



[Del Duca, Druc, Drummond, CD ,Dulat, Marzucca, Papathanasiou, Verbeek; see also recent work by Bargheer on 7-point MHV three-loop symbol]

- N.B.: Factorisation is not restricted to MHV!

Factorisation for MHV

- For MHV amplitudes, the factorisation implies that we can drop all 0's:

$$g_{+\dots+}^{(0,\dots,0,i_{a_1},0,\dots,0,i_{a_2},0,\dots,0,i_{a_k},0,\dots,0)}(\rho_1,\dots,\rho_{N-5}) = g_{+\dots+}^{(i_{a_1},i_{a_2},\dots,i_{a_k})}(\rho_{i_{a_1}},\rho_{i_{a_2}},\dots,\rho_{i_{a_k}})$$

- Consequence: At L loops an MHV amplitudes in MRK at LLA is determined by amplitudes with at most (L+4) external legs.

- Two loops: Reduces to known factorisation:

$$\mathcal{R}_{+\dots+}^{(2)} = \sum_{1\le i\le N-5} \log\tau_i\, g_{++}^{(1)}(\rho_i)$$

- Three loops:

$$\mathcal{R}_{+\dots+}^{(3)} = \frac{1}{2}\sum_{1\le i\le N-5} \log^2\tau_i\, g_{++}^{(2)}(\rho_i) + \sum_{1\le i<j\le N-5} \log\tau_i\,\log\tau_j\, g_{+++}^{(1,1)}(\rho_i,\rho_j)\,.$$

Factorisation for MHV

- Four loops:

$$\mathcal{R}^{(4)}_{+\ldots+} = \frac{1}{6}\sum_{1\le i\le N-5}\log^3\tau_i\, g^{(3)}_{++}(\rho_i) + \frac{1}{2}\sum_{1\le i<j\le N-5}\left[\log^2\tau_i\,\log\tau_j\, g^{(2,1)}_{+++}(\rho_i,\rho_j) + \log\tau_i\,\log^2\tau_j\, g^{(1,2)}_{+++}(\rho_i,\rho_j)\right] + \sum_{1\le i<j<k\le N-5}\log\tau_i\,\log\tau_j\,\log\tau_k\, g^{(1,1,1)}_{++++}(\rho_i,\rho_j,\rho_k)\,.$$

- We have computed all MHV building blocks up to five loops.
  - ➡ Explicit analytic results for all MHV amplitudes in MRK at LLA up to five loops!

[Parallel work by Brödel and Spenger on 7-point amplitude through 5 loops, and by Bargheer on 7-point MHV symbol at three loops.]

Non-MHV amplitudes in MRK

Helicity flips
&
leading singularities

Helicity flips

- Master formula:

$$g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5}) = \frac{(-1)^{N+1}}{2}\left[\prod_{k=1}^{N-5}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar{z}_k}\right)^{n_k/2}\int_{-\infty}^{+\infty}\frac{d\nu_k}{2\pi}|z_k|^{2i\nu_k}E^{i_k}_{\nu_k n_k}\right]$$

$$\times\,\chi^{h_1}(\nu_1,n_1)\left[\prod_{j=1}^{N-6}C^{h_j}(\nu_j,n_j,\nu_{j+1},n_{j+1})\right]\chi^{-h_{N-5}}(\nu_{N-5},n_{N-5})\,.$$

- It is easy to flip helicities in FM space:

$$\begin{aligned}\mathcal{F}\left[\chi^+(\nu,n)\,F(\nu,n)\right] &\longrightarrow \mathcal{F}\left[\chi^-(\nu,n)\,F(\nu,n)\right]\\ &= \mathcal{F}\left[\chi^-(\nu,n)/\chi^+(\nu,n)\right]*\mathcal{F}\left[\chi^+(\nu,n)\,F(\nu,n)\right]\\ &= \mathcal{F}\left[\frac{i\nu+\frac{n}{2}}{i\nu-\frac{n}{2}}\right]*\mathcal{F}\left[\chi^+(\nu,n)\,F(\nu,n)\right]\,.\end{aligned}$$

➡ Helicity flip kernel:

$$\mathcal{H}(z)\equiv\mathcal{F}\left[\frac{i\nu+\frac{n}{2}}{i\nu-\frac{n}{2}}\right] = -\frac{z}{(1-z)^2}$$

- Flipping helicities on central emission blocks is similar.

Helicity flips

- Helicity flip kernel has a double pole: $\mathcal{H}(z) = -\frac{z}{(1-z)^2}$
  - ➡ Acts as a 'derivative' in residue computation.
  - ➡ Produces rational prefactors (leading singularities).
  - ➡ Non-MHV amplitudes are not pure functions!
- Factorisation theorem still holds for non-MHV amplitudes.
  - ➡ Unlike MHV, there is an infinite number building blocks already at two loops.
  - ➡ Irreducible building blocks: alternating helicities (-+-+-+…).
- Example:

$$\mathcal{R}^{(2)}_{-+\ldots} = \log\tau_1\, g^{(1)}_{-+}(\rho_1) + \sum_{j=2}^{N-5} \log\tau_j\, g^{(0,1)}_{-++}(\rho_1,\rho_j)$$

$$\mathcal{R}^{(2)}_{+-+\ldots} = \log\tau_1\, g^{(1,0)}_{+-+}(\rho_1,\rho_2) + \log\tau_2\, g^{(0,1)}_{+-+}(\rho_1,\rho_2) + \sum_{j=3}^{N-5} \log\tau_j\, g^{(0,0,1)}_{+-++}(\rho_1,\rho_2,\rho_j)$$

An algorithm

- All MRK amplitudes at LLA can be computed via a sequence of the following three elementary operations:
  - ➡ Flip helicity with helicity flip kernel:

$$g_{+++}^{(1,2)} \to g_{-++}^{(1,2)} = \mathcal{H}(z_1) * g_{+++}^{(1,2)}$$

  - ➡ Add particles with the same helicity without BFKL eigenvalue insertion, e.g.,

$$g_{-++}^{(1,2)} \to g_{---++}^{(0,0,1,2)} = g_{-++}^{(1,2)}$$

  - ➡ Increase loop number with BFKL eigenvalue:

$$g_{---++}^{(0,0,1,2)} \to g_{---++}^{(2,0,1,2)} = \mathcal{E}(z_1) * \mathcal{E}(z_1) * g_{---++}^{(0,0,1,2)}$$

- We have explicitly computed all non-MHV amplitudes up to four loops and eight legs.

Analytic structure

- Convolutions with helicity flip kernel preserve the weight.
  - ➡ Example:

$$\mathcal{R}_{-+\ldots+} = \mathcal{H}(z_1) * \mathcal{R}_{++\ldots+} \qquad \mathcal{H}(z) = -\frac{z}{(1-z)^2}$$

$$= z_1\partial_{z_1} \int \frac{d\bar{w}}{\bar{w}} \mathcal{R}_{++\ldots+}(w, z_2 \ldots, z_{N-5})$$

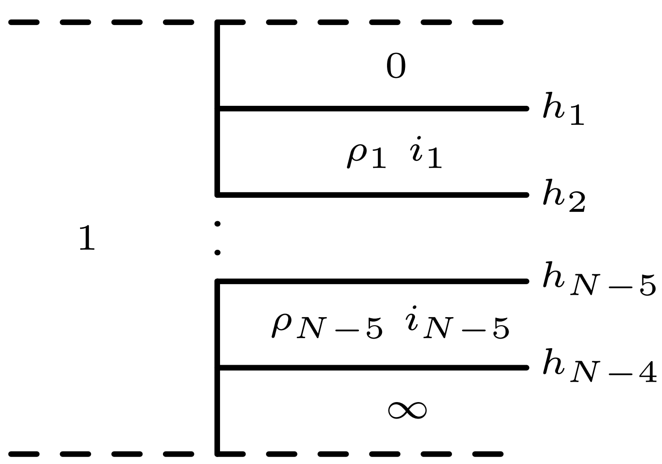
weight -1   weight +1

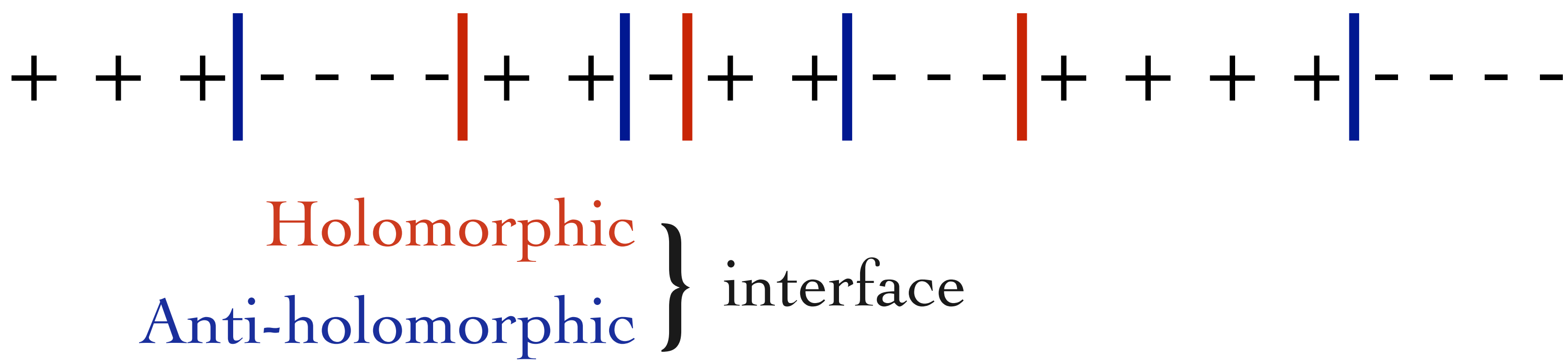
$$\mathcal{E}(z) = -\frac{z+\bar{z}}{2|1-z|^2}$$

- Convolutions with BFKL eigenvalue increase the weight by 1.
  - ➡ BFKL eigenvalues has simple holomorphic and anti-holomorphic poles!
  - ➡ Anti-holomorphic primitive increases the weight by 1.
  - ➡ Simple holomorphic pole does not lower the weight.

Leading singularities

- One can analyse the convolution integral and obtain an upper bound on leading singularities for a given helicity configuration!



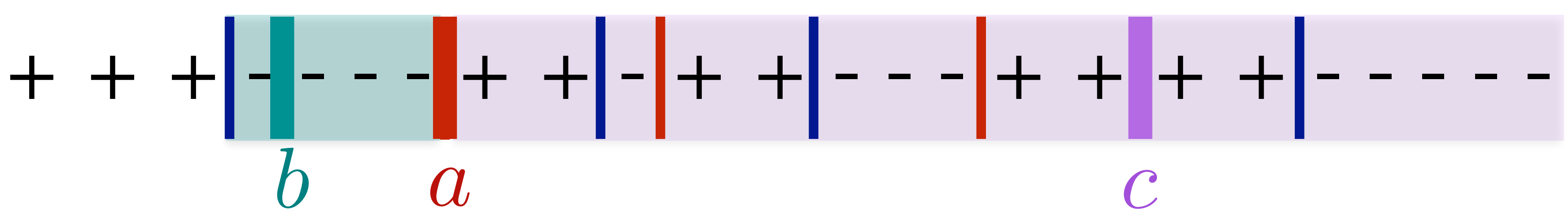


- Interfaces = places where helicity flips kernels were inserted.
- Rule for leading singularities:
  - ➡ At each interface a we can insert at most one (anti-) holomorphic cross ratio

$$R_{bac} = \frac{(\mathbf{x}_b - \mathbf{x}_a)(\mathbf{x}_c - \mathbf{x}_1)}{(\mathbf{x}_b - \mathbf{x}_c)(\mathbf{x}_a - \mathbf{x}_1)}$$

Leading singularities

- Rule for leading singularities:
  - ➡ At each interface a we can insert at most one (anti-) holomorphic cross ratio R_{bac}.
  - ➡ Ranges of b and c restricted as follows:



- Asymmetry reflects the fact that there are non linear relations among the cross ratios:

$$R_{23c} + R_{234}\, R_{4ac} = R_{23c}\, R_{4ac} + R_{234}\, R_{2ac}$$

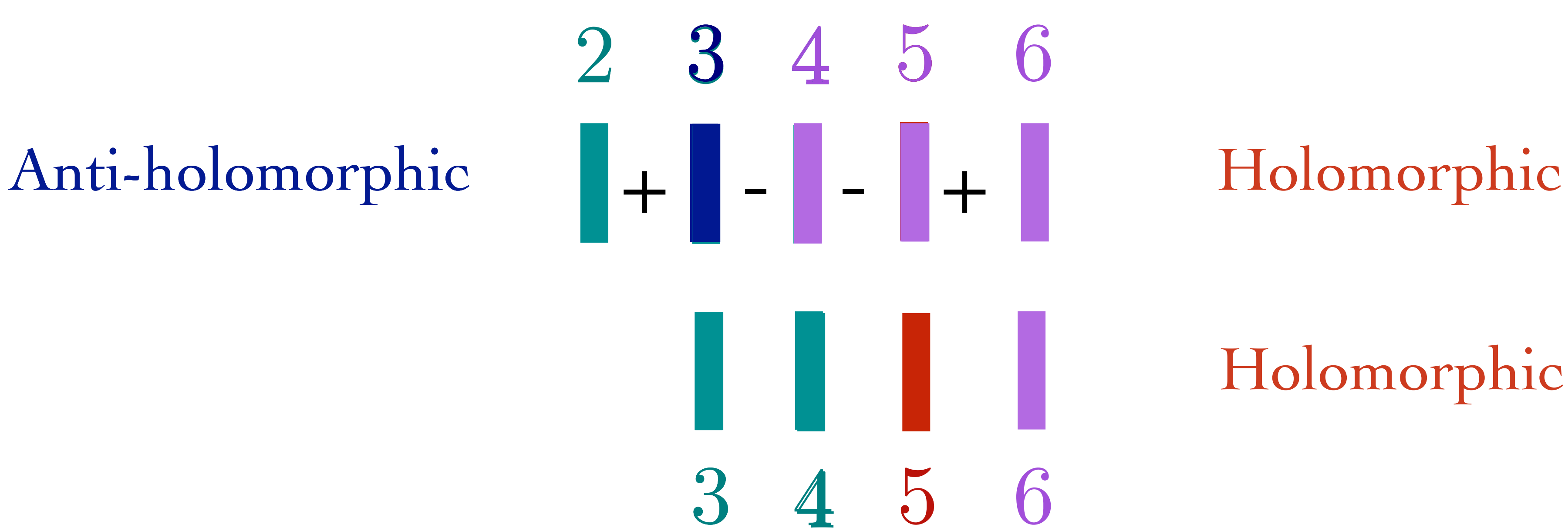
  - ➡ Effect can appear for the first time with 3 interfaces: (+-+-)
  - ➡ It does indeed appear!

Leading singularities

- Example:

$$
\begin{aligned}
g^{(i_1,i_2,i_3)}_{+--+}(\rho_1,\rho_2,\rho_3) &= \mathfrak{a}^{(i_1,i_2,i_3)}_{+--+}(\rho_1,\rho_2,\rho_3) + \overline{R}_{234}\,\mathfrak{b}^{(i_1,i_2,i_3)}_{1,+--+}(\rho_1,\rho_2,\rho_3) \\
&+ \overline{R}_{235}\,\mathfrak{b}^{(i_1,i_2,i_3)}_{2,+--+}(\rho_1,\rho_2,\rho_3) + R_{356}\,\mathfrak{b}^{(i_1,i_2,i_3)}_{3,+--+}(\rho_1,\rho_2,\rho_3) + R_{456}\,\mathfrak{b}^{(i_1,i_2,i_3)}_{4,+--+}(\rho_1,\rho_2,\rho_3) \\
&+ \overline{R}_{234}\,R_{356}\,\mathfrak{c}^{(i_1,i_2,i_3)}_{1,+--+}(\rho_1,\rho_2,\rho_3) + \overline{R}_{234}\,R_{456}\,\mathfrak{c}^{(i_1,i_2,i_3)}_{2,+--+}(\rho_1,\rho_2,\rho_3) \\
&+ \overline{R}_{235}\,R_{356}\,\mathfrak{c}^{(i_1,i_2,i_3)}_{3,+--+}(\rho_1,\rho_2,\rho_3) + \overline{R}_{235}\,R_{456}\,\mathfrak{c}^{(i_1,i_2,i_3)}_{4,+--+}(\rho_1,\rho_2,\rho_3)\ ,
\end{aligned}
$$

➡ $\overline{R}_{236}$ does not appear, because we have only two interfaces!



Analytic structure

- Pole structure of LS is R_{bac} is not random!

$$R_{bac} = \frac{(\mathbf{x}_b - \mathbf{x}_a)(\mathbf{x}_c - \mathbf{x}_1)}{(\mathbf{x}_b - \mathbf{x}_c)(\mathbf{x}_a - \mathbf{x}_1)}$$

- There is no pole in soft limit, because $b < a < c$.
  - ➡ No weight drop in soft limit!
- Soft limits of NMHV amplitudes can be MHV, i.e., pure.
  - ➡ Example:

$$g^{(i_1,i_2)}_{+-+} = \mathfrak{a}^{(i_1,i_2)}_{+-+} + \overline{R}_{234}\,\mathfrak{b}^{(i_1,i_2)}_{1,+-+} + R_{345}\,\mathfrak{b}^{(i_1,i_2)}_{2,+-+} + \overline{R}_{234}\,R_{345}\,\mathfrak{c}^{(i_1,i_2)}_{1,+-+}$$

  - ➡ In the limit all rational factors disappear:

$$\lim_{\mathbf{x}_b\to\mathbf{x}_a} R_{bac} = 0 \qquad\qquad \lim_{\mathbf{x}_c\to\mathbf{x}_a} R_{bac} = 1$$

Conclusion & Outlook

N-point scattering amplitudes in planar N=4 SYM in MRK are single-valued iterated integrals on $\mathfrak{M}_{0,N-2}$.

- Consequences:
  - ➡ Convolutions can be computed using Stokes' theorem.
  - ➡ Algorithmic construction of all MRK amplitudes at LLA.
  - ➡ Classification of leading singularities in MRK at LLA.
  - ➡ All amplitudes have uniform weight.
  - ➡ Explicit results for all MHV amplitudes up to 5 loops, and all non-MRK amplitude with up to 8 legs and 4 loops.

Conclusion & Outlook

- Generalisation beyond LLA?
  - ➡ Formalism is general!
  - ➡ BFKL eigenvalue & impact factor known to all orders from integrability.
  - ➡ Central emission block only known to LO.
  - ➡ Multi-Reggeon bound state exchanges? [See Basso's talk]
- Are there other cases where we can 'control' the geometry?
  - ➡ Amplitudes in 2D kinematics?
  - ➡ Limits where the cluster algebras are all of finite type?

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