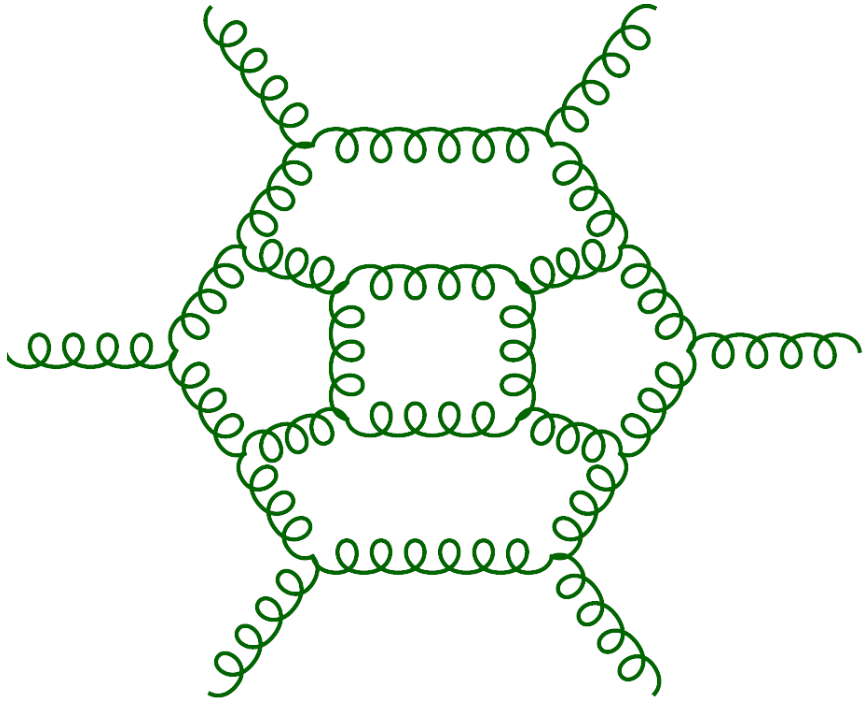
The Amplitude Bootstrap Reloaded



Lance Dixon (SLAC)

S. Caron-Huot, LD, M. von Hippel, A, McLeod, to appear

AMPLITUDES 2016
INTERNATIONAL CONFERENCE
STOCKHOLM · 4–8 JULY

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Planar (large N_c) N=4 SYM Properties

- Conformally invariant ($\beta = 0$)
- Uniform transcendental weight:
  "$\ln^{2L} x$" at L loops
- Perturbation theory has finite radius of convergence (no renormalons, no instantons)
- Amplitudes for *n*=4 or 5 gluons "trivial" to all loop orders
- Dual (super)conformal invariance for any *n*
- Amplitudes equivalent to Wilson loops
- Strong coupling → minimal area surfaces
- **Integrability + OPE → exact, nonperturbative predictions for near-collinear limit** Basso talk

Integrability a key

- **Not new:** For inverse-square law, $\mathbf{F}(r) = -\frac{k}{r^2}\hat{\mathbf{r}}$ the Laplace-Runge-Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$$

is conserved. Leads to non-precessing ellipses.

- Pauli used it to solve hydrogen atom.
- In a version of planar N=4 SYM with masses, LRL vector → dual conformal transformations

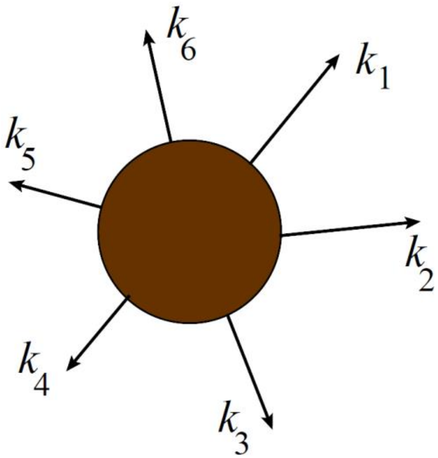
Caron-Huot, Henn, 1408.0296

"Planar N=4 SYM = Hydrogen Atom of 21st Century"

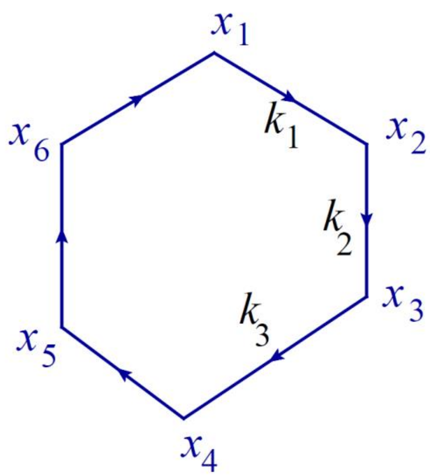
Goal: Solve Planar N=4 SYM

- Can we solve a relativistic 4d quantum field theory exactly in the coupling for dynamical (time-dependent) quantities for generic kinematics?
- Can already do so for operator anomalous dimensions and "nearby" quantities. Basso talk
- Once that's done, we'll get a more concrete picture of holography at finite coupling, including how gluons morph into strings, as a function of the kinematics and the coupling.

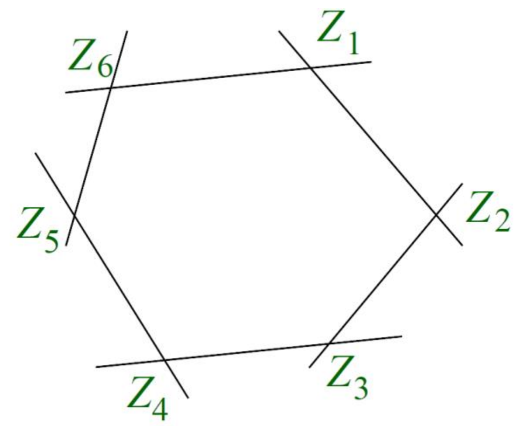
Amplitudes = Wilson loops



Spacetime



Dual Spacetime



Momentum Twistor Space

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Hodges, 0905.1473
Arkani-Hamed et al, 0907.5418, 1008.2958, 1212.5605
Adamo, Bullimore, Mason, Skinner, 1104.2890

Strong coupling = AdS_5 minimal surface

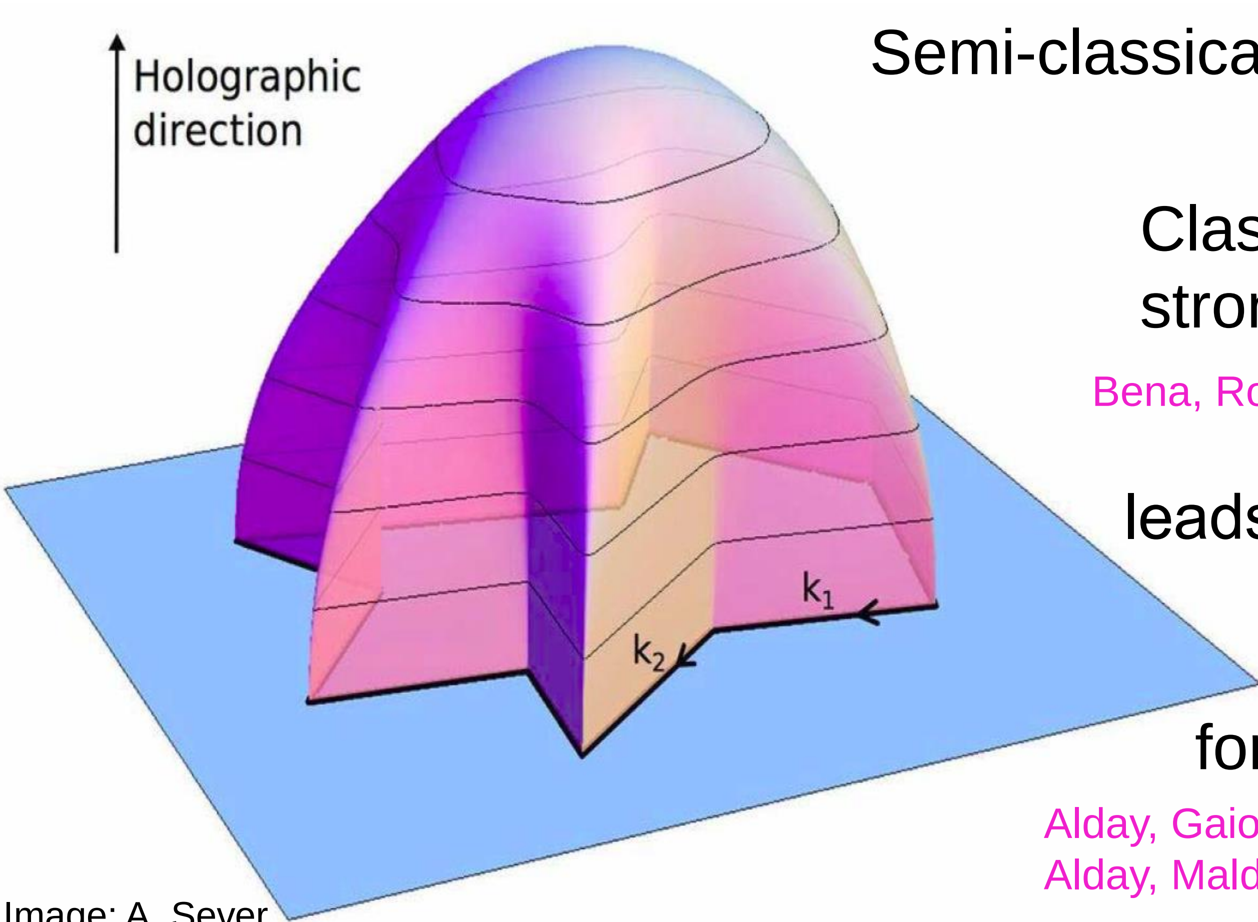


Image: A. Sever

Semi-classical string stretched tight:

Alday, Maldacena, 0705.0303

Classical integrability of strong-coupling σ model

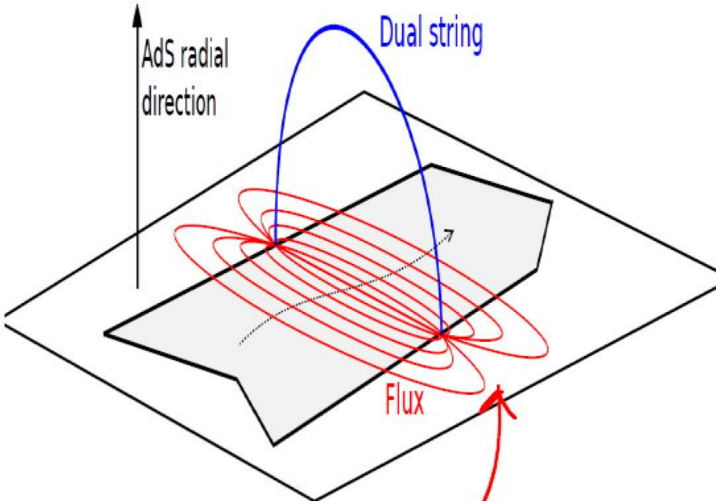
Bena, Roiban, Polchinski, hep-th/0305116

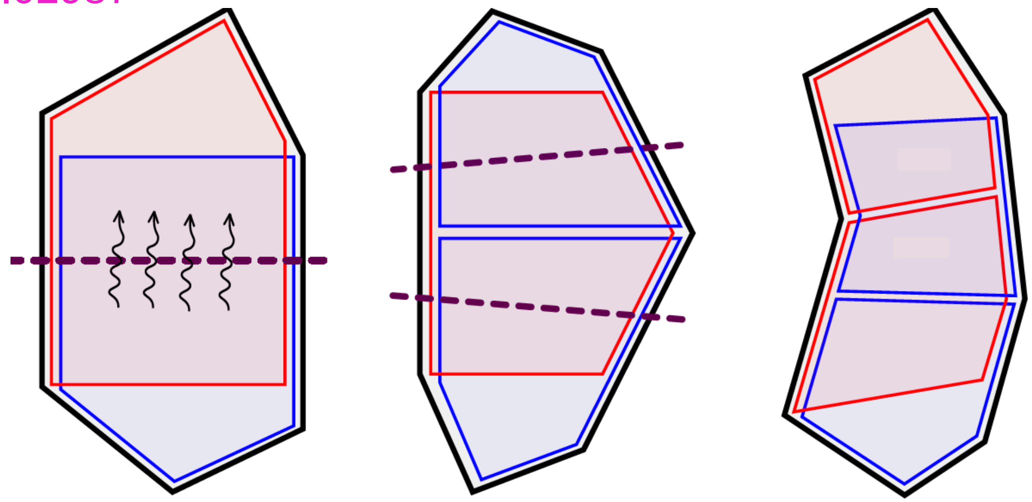
leads to "Thermodynamical Bubble Ansatz" and "Y-system" for minimal area problem

Alday, Gaiotto, Maldacena, 0911.4708,
Alday, Maldacena, Sever, Vieira, 1002.2459

Finite coupling = flux tubes

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Pentagonal "atoms" are integrable 2d S-matrices
- Quantum integrability → pentagons computable exactly as a function of the 't Hooft coupling
- Generate 4d S-matrix as an expansion (OPE) in the number of flux-tube excitations
- Expansion is around the kinematical limit where gluons are collinear

Weak coupling = gluons

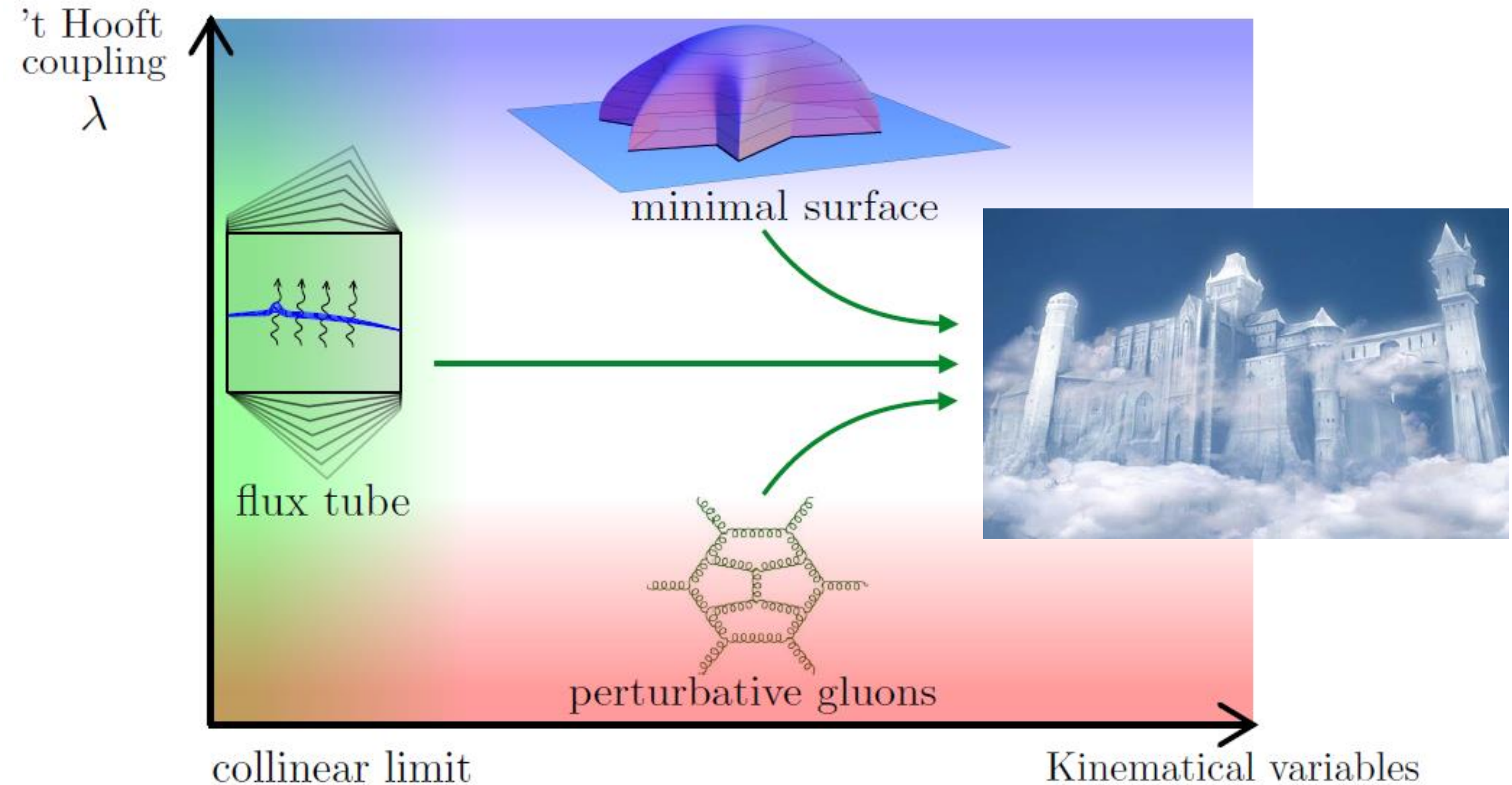
- Ordinary Feynman diagrams could be used, but very cumbersome.
- For hexagonal Wilson loop, tractable at 2 loops
  Del Duca, Duhr Smirnov, 0911.5332, 1003.1702
- Simplified to

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3} \left(L_4(x_i^+, x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i) \right)$$
$$- \frac{1}{8}\left(\sum_{i=1}^{3} \mathrm{Li}_2(1-1/u_i) \right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

using the "symbol" of an iterated integral
Goncharov, Spradlin, Volovich, Vergu, 1006.5703

Solving Planar N=4 SYM

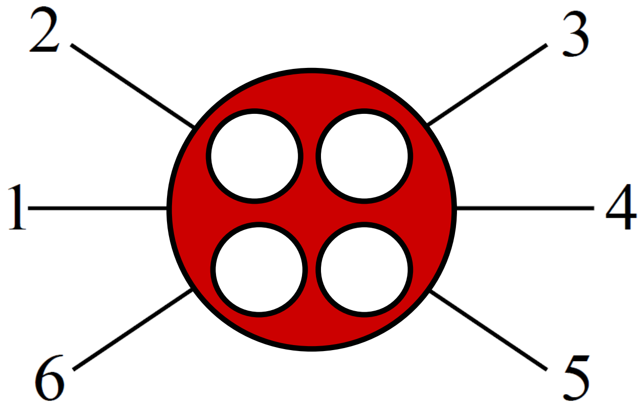
Images: A. Sever, N. Arkani-Hamed



Hexagon function bootstrap

LD, Drummond, Henn, 1108.4461, 1111.1704;
LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127;
Drummond, Papathanasiou, Spradlin, 1412.3763

Use analytical properties of perturbative amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops



First step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

Outline of program

1. Make ansatz for IR finite versions of 6 gluon scattering amplitudes as linear combination of "hexagon functions"
2. NEW: Steinmann constraints dramatically reduce the size of the ansatz at high loop orders!
3. Use precise "boundary value data" to fix constants in ansatz. Linear constraints, solve matrix for rational numbers.
4. Cross check.

- Works fantastically well for 6-gluon amplitude, first "nontrivial" amplitude in planar N=4 SYM

  → 5 loops for both MHV = (--++++) and NMHV = (---+++)

Boundary value data

- Precise information in many different limits (much more than we need):
- OPE limit Basso, Sever, Vieira (2013,…)
- Multi-Regge-limit Bartels, Lipatov, Sabio-Vera, Schnitzer (2008,…); Basso, Caron-Huot, Sever (2014)
- NMHV multi-particle- factorization limit Bern, Chalmers (1995); LD, von Hippel, 1408.1505; BSV, to appear
- Self-crossing limit Georgiou, 0904.4675; LD, Esterlis, 1602.02107

Global: Dual superconformal "Descent Equation" or $\bar{Q}$-equation Bullimore, Skinner; Caron-Huot, He (2011)

BDS Ansatz

Bern, LD, Smirnov, hep-th/0505205

- Captures all IR divergences of amplitude
- Also accounts for an anomaly in dual conformal invariance due to IR divergences
- **Fails for** *n = 6,7,…*
- But failure (remainder function) is dual conformally invariant

$$\mathcal{A}_n^{\rm BDS} \;=\; \mathcal{A}_n^{\rm tree} \times \exp\Big[\sum_{l=1}^{\infty} [\frac{\lambda}{8\pi^2}]^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon; s_{ij}) + C^{(l)} + \mathcal{O}(\epsilon) \Big)\Big]$$

constants, indep.of kinematics

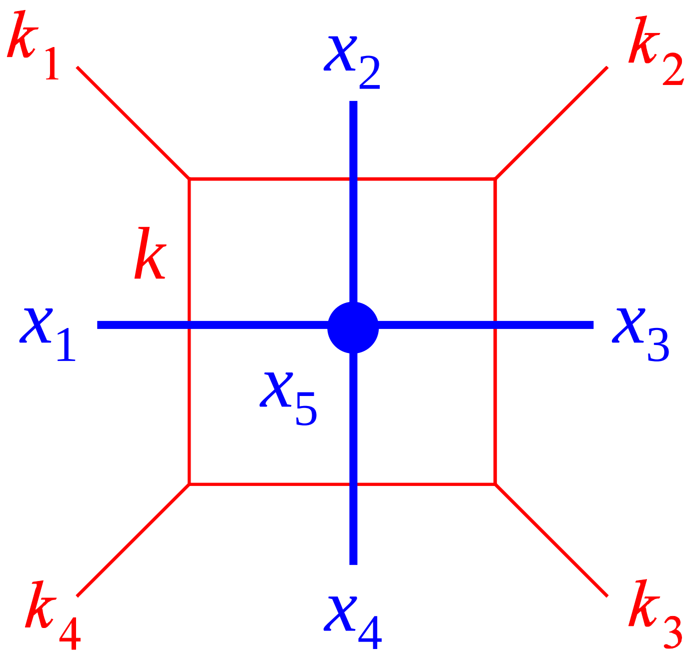
all kinematic dependence from 1-loop amplitude

Dual conformal invariance

Broadhurst (1993); Lipatov (1999); Drummond, Henn, Smirnov, Sokatchev, hep-th/0607160

Conformal symmetry acting in momentum space,
on dual or sector variables x_i

First seen in N=4 SYM planar amplitudes in the loop integrals



$$I = \int d^4k \frac{(k_1+k_2)^2(k_2+k_3)^2}{k^2(k-k_1)^2(k-k_1-k_2)^2(k+k_4)^2}$$

$$I = \int d^4x_5 \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

$k_1 = x_{12}$
$k_2 = x_{23}$
$k_3 = x_{34}$
$k_4 = x_{41}$
$k = x_{15}$

invariant under inversion:

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2}$$

$$x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}, \qquad \mathrm{d}^4x_i \to \frac{\mathrm{d}^4x_i}{x_i^8}$$

Dual conformal invariance (cont.)

- Amplitude fixed, up to functions of dual conformally invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2$$

- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for $n = 4,5$

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

$n = 6$ → precisely 3 ratios:

$$v = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

$$w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

Remainder function, starts at 2 loops

$$\mathcal{A}_6^{\mathrm{MHV}}(\epsilon; s_{ij}) = \mathcal{A}_6^{\mathrm{BDS}}(\epsilon; s_{ij}) \exp[R_6(u,v,w)]$$

BDS-like – better than BDS!

Consider
$$\frac{\mathcal{A}_6^{\rm BDS-like}}{\mathcal{A}_6^{\rm MHV\,(0)}} = \exp\left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon)\frac{1}{2}\hat{M}_6(L\epsilon) + C^{(L)}\right)\right]$$

where

$$\begin{aligned}\hat{M}_6(\epsilon) &= M_6^{\rm 1-loop} + Y(u,v,w)\\ &= \sum_{i=1}^{6}\left[-\frac{1}{\epsilon^2}\Big(1-\epsilon\ln(-s_{i,i+1})\Big) - \ln(-s_{i,i+1})\ln(-s_{i+1,i+2}) + \frac{1}{2}\ln(-s_{i,i+1})\ln(-s_{i+3,i+4})\right]\\ &\quad + 6\,\zeta_2\,,\end{aligned}$$

Alday, Gaiotto, Maldacena, 0911.4708

It contains all the IR poles, but **no 3-particle invariants**.

Here

$$Y(u,v,w) \equiv {\rm Li}_2(1-u) + {\rm Li}_2(1-v) + {\rm Li}_2(1-w) + \frac{1}{2}\left(\ln^2 u + \ln^2 v + \ln^2 w\right)$$

is the dual conformally invariant part of the one-loop amplitude.

BDS-like normalized amplitude

Define

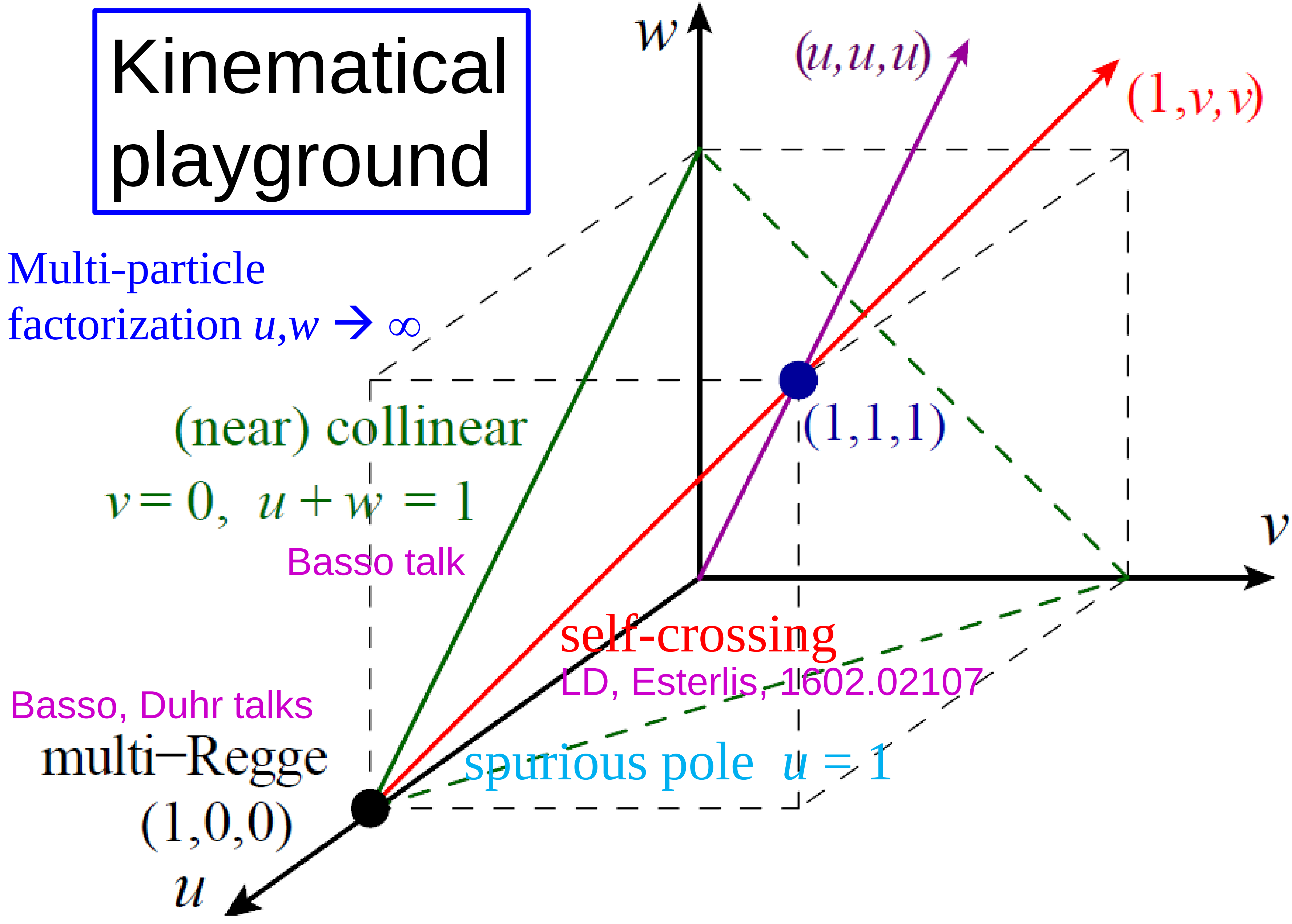
$$\frac{\mathcal{A}_6^{\text{MHV}}}{\mathcal{A}_6^{\text{BDS-like}}} \equiv \mathcal{E}(u,v,w) = \exp\left[R_6 - \frac{\gamma_K(a)}{8} Y\right]$$

where $a = \frac{\lambda}{8\pi^2}$ `t Hooft coupling

$\gamma_K(a) = 4 f_0(a)$ cusp anomalous dimension

No 3-particle invariants in denominator of $\mathcal{E}$
→ simpler analytic behavior

Kinematical playground



Basic bootstrap assumption

- MHV: $\mathcal{E}^{(L)}(u,v,w)$ is a linear combination of weight **2*L*** hexagon functions at any loop order ***L***
- NMHV: BDS-like normalized super-amplitude

$$\hat{\mathcal{P}}_{\rm NMHV} \equiv \frac{\mathcal{A}_{\rm NMHV}}{\mathcal{A}_{\rm MHV}^{\rm BDS-like}}$$

Drummond, Henn, Korchemsky, Sokatchev, 0807.1095;
LD, von Hippel, McLeod, 1509.08127

has expansion

$$\hat{\mathcal{P}}_{\rm NMHV} = \frac{1}{2}\Big[[(1)+(4)]E(u,v,w)+[(2)+(5)]E(v,w,u)+[(3)+(6)]E(w,u,v)$$
$$+[(1)-(4)]\tilde{E}(u,v,w)-[(2)-(5)]\tilde{E}(v,w,u)+[(3)-(6)]\tilde{E}(w,u,v)\Big]$$

Grassmann-containing dual superconformal invariants

$E,\tilde{E}$ = hexagon functions

Functional interlude

Chen; Goncharov; Brown; …

- Multiple polylogarithms, or n-fold iterated integrals, or weight n pure transcendental functions f.
- Define by derivatives: $$d\,f = \sum_{s_k \in \mathcal{S}} f^{s_k}\, d\ln s_k$$

S = finite set of rational expressions, "symbol letters", and

$$f^{s_k} \equiv \{n-1,1\} \text{ coproduct component}$$

are also pure functions, weight n-1

- Iterate: $d\,f^{s_k} \;\Rightarrow\; f^{s_j,s_k} \equiv \{n-2,1,1\} \text{ component}$

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

- Symbol = {1,1,…,1} component (maximally iterated)

Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Subsector of hexagon functions.
- Generalize classical polylogs, $\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t)$
- Define by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \qquad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Symbol letters: $\mathcal{S} = \{u, 1-u\}$

Hexagon symbol letters

- Momentum twistors $Z_i^A,\ i=1,2,\ldots,6$ transform simply under dual conformal transformations. Hodges, 0905.1473
- Construct 4-brackets $\varepsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D \equiv \langle ijkl \rangle$
- 15 projectively invariant combinations of 4-brackets can be factored into 9 basic ones:

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

$$u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle} \qquad 1-u = \frac{\langle 6135\rangle\langle 2346\rangle}{\langle 6134\rangle\langle 2356\rangle} \qquad y_u = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle}$$

+ cyclic

→ A_3 cluster algebra

Golden, Goncharov, Spradlin, Vergu, Volovich, 1305.1617;
Golden, Paulos, Spradlin, Volovich, 1401.6446; Golden, Spradlin, 1411.3289;
Harrington, Spradlin, 1512.07910; talk by M. Spradlin

Hexagon function symbol letters (cont.)

- y_i not independent of u_i :

$$y_u \equiv \frac{u - z_+}{u - z_-} \,, \ldots$$ where

$$z_\pm = \frac{1}{2}\Big[-1 + u + v + w \pm \sqrt{\Delta}\Big]$$
$$\Delta = (1 - u - v - w)^2 - 4uvw$$

- Function space graded by parity:

$$\begin{array}{rcl} i\sqrt{\Delta} & \leftrightarrow & -i\sqrt{\Delta} \\ z_+ & \leftrightarrow & z_- \\ y_i & \leftrightarrow & 1/y_i \\ u_i & \leftrightarrow & u_i \end{array}$$

Branch cut condition

- All massless particles → all branch cuts start at origin in

$$s_{i,i+1}, \quad s_{i,i+1,i+2}$$

→ Branch cuts all start from 0 or ∞ in

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad \text{or } v \text{ or } w$$

→ First symbol entry $\in \{u, v, w\}$ GMSV, 1102.0062

- Powerful constraint: At weight 8 (four loops) we would have 1,675,553 functions without it; exactly 6,916 with it. But this is still way too many!

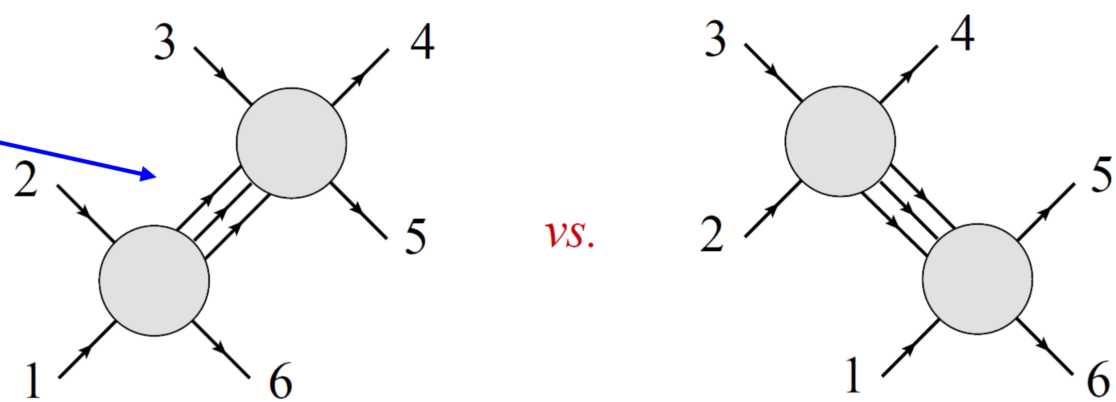
Steinmann relations

Steinmann, Helv. Phys. Acta (1960)

Bartels, Lipatov, Sabio Vera, 0802.2065

- Amplitudes should not have overlapping branch cuts.
- Cuts in 2-particle invariants subtle in generic kinematics
- Easiest to understand for cuts in 3-particle invariants using 3 → 3 scattering:

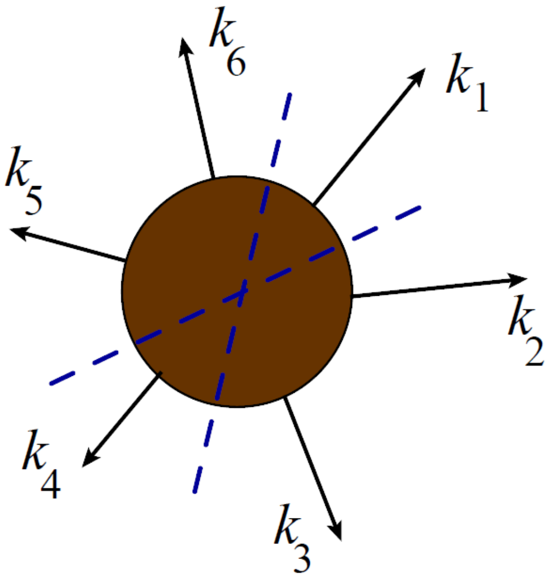
Intermediate particle flow in wrong direction for S_{234} discontinuity

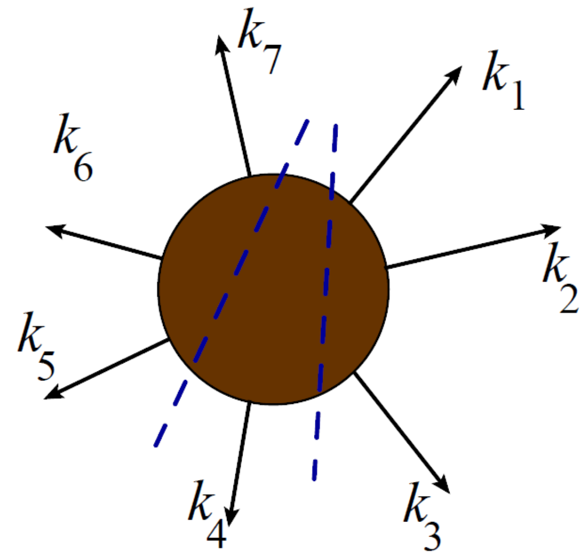


$$\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{345}}\mathcal{E}(u,v,w)\right] = 0$$

Steinmann relations (cont.)

- Amplitudes should not have overlapping branch cuts:





Not Allowed

Allowed

$$\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0$$

Steinmann relations (cont.)

$$\text{Disc}_{s_{234}}\left[\text{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0$$

+ cyclic conditions

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \qquad w = \frac{s_{61}s_{34}}{s_{345}s_{234}}$$

$\ln^2 u$ **NO** $\ln^2 \frac{uv}{w}$ **OK**

$$\frac{uv}{w} = \frac{s_{12}s_{23}s_{45}s_{56}}{s_{34}s_{61}s_{123}^2}$$

First two entries restricted to 6 out of 9:

$$\text{Li}_2(1-1/u) \quad \text{Li}_2(1-1/v) \quad \text{Li}_2(1-1/w)$$

$$\ln^2\frac{uv}{w} \quad \ln^2\frac{vw}{u} \quad \ln^2\frac{wu}{v} \quad \text{plus } \zeta_2$$

Analogous constraints for $n=7$ [Spradlin talk] using $A_7^{\text{BDS-like}}$

Iterative Construction of Steinmann hexagon functions

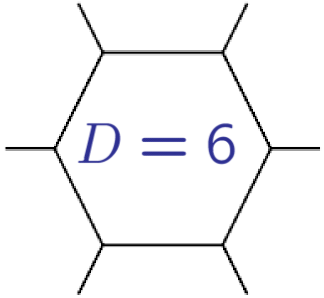
{n-1,1} coproduct F^x characterizes first derivatives, defines F up to overall constant (a multiple zeta value).

$$\left.\frac{\partial F}{\partial u}\right|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}} F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}} F^{y_w}$$

$$\frac{\partial \ln y_u}{\partial u} \nearrow$$

1. Insert general linear combinations for F^x
2. Apply "integrability" constraint that mixed-partial derivatives are equal
3. Stay in space of functions with good branch cuts and obeying Steinmann by imposing a few more "zeta-valued" conditions in each iteration.

The first true hexagon function



$$\Rightarrow \quad \tilde{\Phi}_6(u,v,w)$$

A real integral so it must be Steinmann

- Weight 3, totally symmetric in $\{u,v,w\}$ (secretly Li_3's)
- First parity odd function, so:

$$\tilde{\Phi}_6^u = \tilde{\Phi}_6^v = \tilde{\Phi}_6^w = \tilde{\Phi}_6^{1-u} = \tilde{\Phi}_6^{1-v} = \tilde{\Phi}_6^{1-w} = 0$$

- Only independent {2,1} coproduct:

$$\tilde{\Phi}_6^{y_u} = -\Omega^{(1)}(v,w,u) = -H_2^u - H_2^v - H_2^w - \ln v \, \ln w + 2\,\zeta_2$$

$$H_2^u = \mathrm{Li}_2(1-u)$$

- Encapsulates first order differential equation found earlier

LD, Drummond, Henn, 1104.2787

Back to physics

- enumerate all Steinmann hexagon functions with weight ***2L***
- write most general linear combination with unkown rational-number coefficients
- impose a series of physical constraints until all coefficients uniquely determined

Simple constraints on $\mathcal{E}$ or R_6

- S_3 permutation **symmetry** in $\{u, v, w\}$
- Even under "**parity**":

$$i\sqrt{\Delta} \leftrightarrow -i\sqrt{\Delta}$$
$$z_+ \leftrightarrow z_-$$
$$y_i \leftrightarrow 1/y_i$$

- R_6 vanishes in **collinear** limit ($R_6 \rightarrow R_5 = 0$)

$$v \rightarrow 0 \qquad u + w \rightarrow 1$$

Dual superconformal invariance

- Dual superconformal generator $\bar{Q}$ has anomaly due to virtual collinear singularities.
- Studying structure of anomaly, one can constrain the first derivatives of amplitudes
  → $\bar{Q}$ equation

Caron-Huot, 1105.5606; Bullimore, Skinner, 1112.1056,
Caron-Huot, He, 1112.1060

- General derivative leads to "source term" from $(n+1)$-point amplitude; however, for certain derivatives the source term vanishes, leading to homogeneous constraints

$\bar{Q}$ equation for MHV

- Constraint on first derivative of $\mathcal{E}$ has simple form
- In terms of the final entry of symbol, restricts to 6 of 9 possible letters:

$$\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w\right\}$$

- In terms of {n-1,1} coproducts, equivalent to:

$$\mathcal{E}^u + \mathcal{E}^{1-u} = \mathcal{E}^v + \mathcal{E}^{1-v} = \mathcal{E}^w + \mathcal{E}^{1-w} = 0$$

- Similar (but more intricate) constraints for NMHV 6-point [Caron-Huot], LD, von Hippel, 1509.08127

(MHV,NMHV): parameters left in $(\mathcal{E}^{(L)}, E^{(L)})$

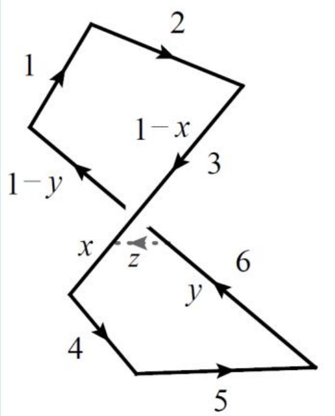
Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
0. Hexagon functions	(10,10)	(82,88)	(639,761)	(5153,6916)	?????
1. Steinmann	(7,7)	(37,39)	(174,190)	(758,839)	(3105,3434)
2. Symmetry	(3,5)	(11,24)	(44,106)	(174,451)	(???,???)
3. Final entry	(2,2)	(5,5)	(19,12)	(72,32)	(272,83)
4. Collinear limit	(0,0)	(0,0)	(1,1)	(3,5)	(9,15)
5. LL MRK	(0,0)	(0,0)	(0,0)	(1,1)	(3,4)
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(1,1)
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

(0,0) → amplitude uniquely determined

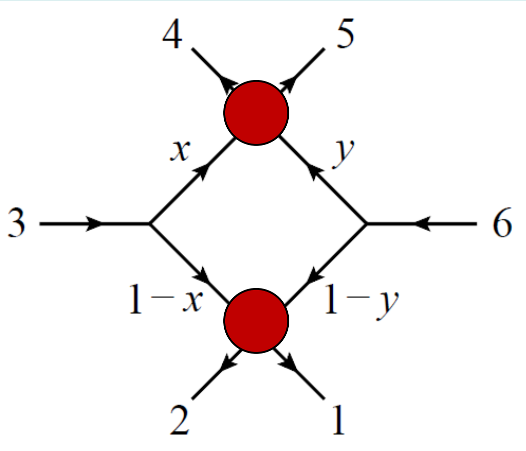
Next-to-final entry and NMHV spurious pole conditions are **impotent** after imposing Steinmann!!

Analytical behavior in new limits

- Self-crossing or "double parton scattering" limit
  Georgiou, 0904.4675; LD, Esterlis, 1602.02107



WL ← → Amp.

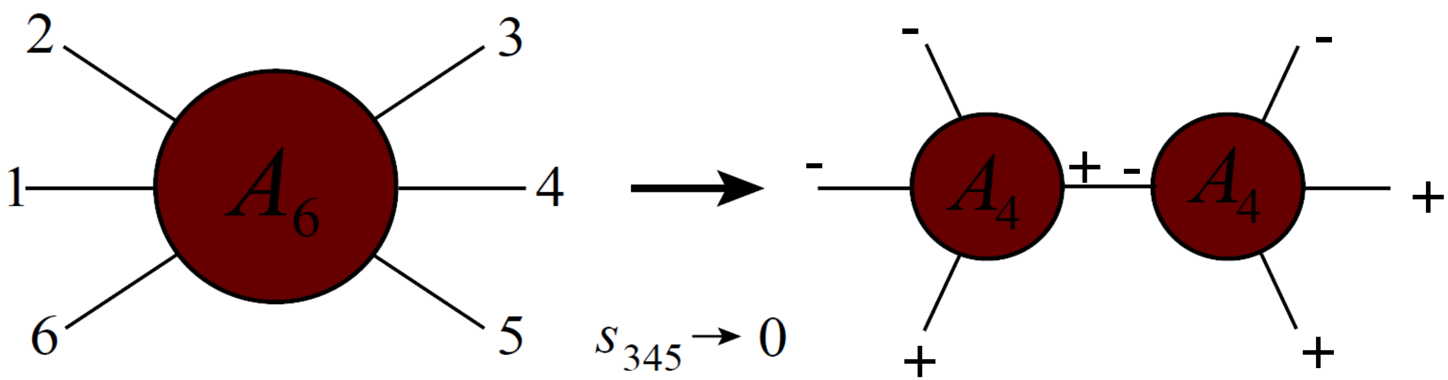


$u \to u\, e^{-2\pi i}$ then $(u,v,w) \to (1-\delta, v, v)$, $\delta \ll 1$

- Overlaps MRK limit when $v \to 0$
- In $\mathcal{E}(1-\delta, v, v)$, $\ln\delta$ terms independent of v
- Can derive using Wilson Loop RGE a la
  Korchemsky and Korchemskaya hep-ph/9409446

NMHV Multi-Particle Factorization

Bern, Chalmers, hep-ph/9503236; LD, von Hippel, 1408.1505



$$A_6^{\rm NMHV}(k_i) \xrightarrow{s_{345}\to 0} A_4(k_6,k_1,k_2,K)\,\frac{F_6(K^2,s_{i,i+1})}{K^2}\,A_4(-K,k_3,k_4,k_5)$$

Only interesting for NMHV: MHV tree has no pole $\quad \mathcal{A}_{\rm MHV}^{(0)} = i\,\frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \to \infty \qquad w = \frac{s_{61}s_{34}}{s_{345}s_{234}} \to \infty$$

$$u/w \text{ and } v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \text{fixed}$$

Multi-Particle Factorization (cont.)

$(1) = (4) \to \infty$, rest finite

→ look at $E(u,v,w)$

Or rather at $U(u,v,w) = \ln E(u,v,w)$

$$\frac{\mathcal{A}_{\mathrm{NMHV}}}{\mathcal{A}_{\mathrm{BDS-like}}} \approx e^{U}[(1) + (4)]$$

Factorization limit of U

$$U^{(1)}(u,v,w) = -\frac{1}{4}\ln^2(uw/v) - \zeta_2$$

$$U^{(2)}(u,v,w)|_{u,w\to\infty} = \frac{3}{4}\zeta_2\ln^2(uw/v) - \frac{1}{2}\zeta_3\ln(uw/v) + \frac{71}{8}\zeta_4$$

$$U^{(3)}(u,v,w)|_{u,w\to\infty} = \frac{1}{3}\zeta_3\ln^3(uw/v) - \frac{75}{8}\zeta_4\ln^2(uw/v) + (7\zeta_5 + 8\zeta_2\zeta_3)\ln(uw/v) - \frac{721}{8}\zeta_6 - 3(\zeta_3)^2$$

$$U^{(4)}(u,v,w)|_{u,w\to\infty} = \frac{1}{4}\zeta_4\ln^4(uw/v) - (4\zeta_5 + 3\zeta_2\zeta_3)\ln^3(uw/v) + \left(\frac{3769}{32}\zeta_6 + \frac{21}{4}\zeta_3^2\right)\ln^2(uw/v) - \left(\frac{785}{8}\zeta_7 + \frac{641}{4}\zeta_3\zeta_4 + \frac{191}{2}\zeta_2\zeta_5\right)\ln(uw/v) + \frac{62629}{64}\zeta_8 + \frac{133}{4}\zeta_2\zeta_3^2 + \frac{289}{4}\zeta_3\zeta_5$$

$$\frac{uw}{v} = \frac{s_{12}s_{34}}{s_{56}} \cdot \frac{s_{45}s_{61}}{s_{23}} \cdot \frac{1}{s_{345}^2}$$

Simple polynomial in $\ln(uw/v)$!

Sudakov logs due to on-shell intermediate state

Numerical results

- Plot perturbative coefficients on various lines and surfaces
- Instructive to take ratios of successive loop orders: $R_6^{(L)}/R_6^{(L-1)} = \bar{R}_6^{(L)}$
  - Planar N=4 SYM has no instantons and no renormalons.
  - Its perturbative expansion has a finite radius of convergence, 1/8
  - For "asymptotically large orders", $R_6^{(L)}/R_6^{(L-1)}$ should approach -8

At $(u,v,w) = (1,1,1)$, multiple zeta values

see e.g. talk by Green

$$R_6^{(2)}(1,1,1) = -(\zeta_2)^2 = -\frac{5}{2}\zeta_4$$

$$R_6^{(3)}(1,1,1) = \frac{413}{24}\zeta_6 + (\zeta_3)^2$$

First irreducible MZV

$$R_6^{(4)}(1,1,1) = -\frac{471}{4}\zeta_8 - \frac{3}{2}\zeta_2(\zeta_3)^2 - \frac{5}{2}\zeta_3\zeta_5 + \frac{3}{2}\zeta_{5,3}$$

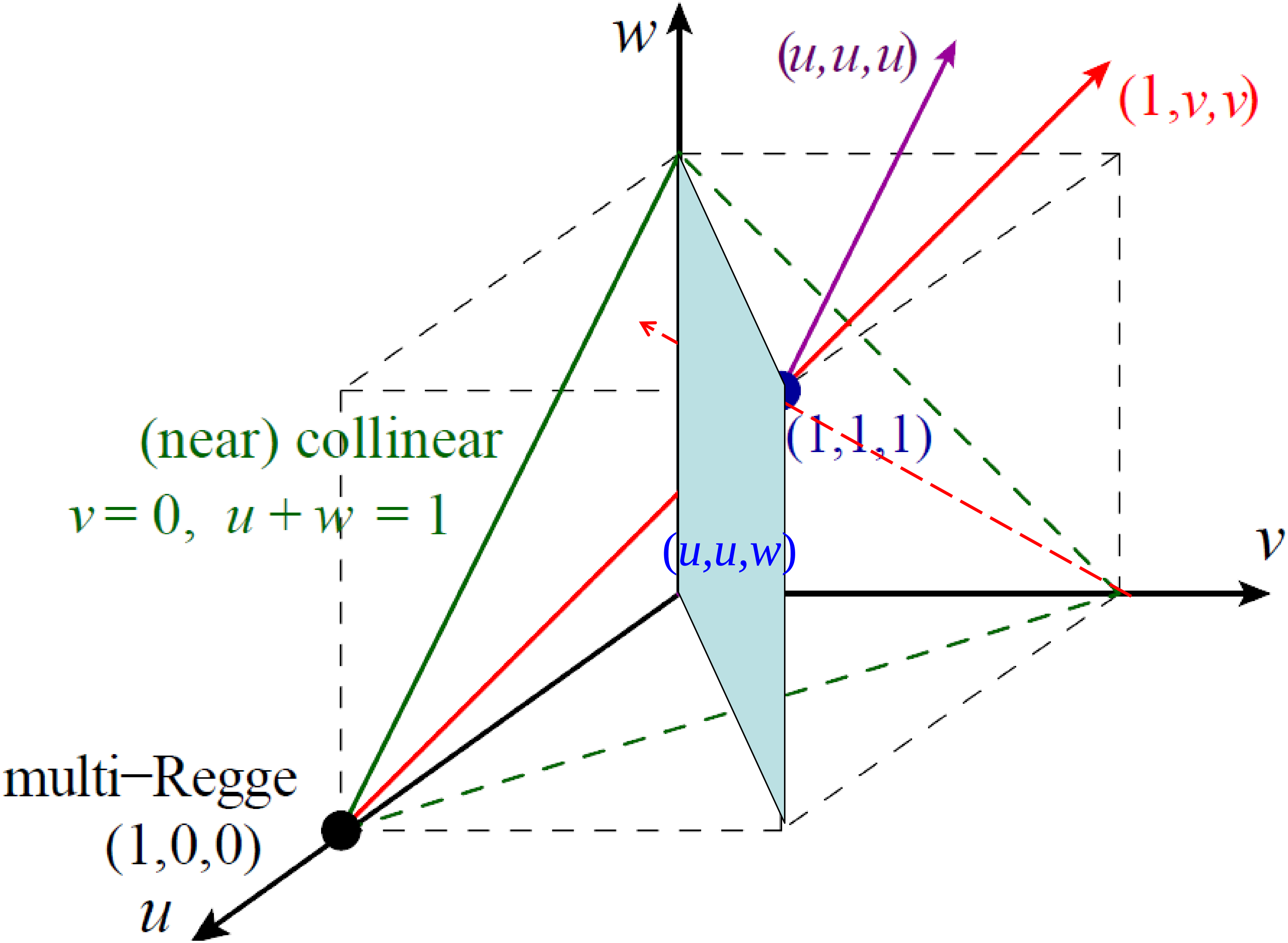
$$R_6^{(5)}(1,1,1) = \frac{8389}{10}\zeta_{10} + 12\zeta_2\zeta_3\zeta_5 + 17\zeta_4(\zeta_3)^2 - \frac{63}{2}\zeta_3\zeta_7 - \frac{111}{8}(\zeta_5)^2 - \frac{3}{2}\zeta_2\zeta_{5,3} - 6\zeta_{7,3}$$

Cusp anomalous dimension $\gamma_K(\lambda)$

- Known to all orders, Beisert, Eden, Staudacher [hep-th/0610251]
- Closely related to amplitude/Wilson loop
- Use as benchmark for approach to large orders:

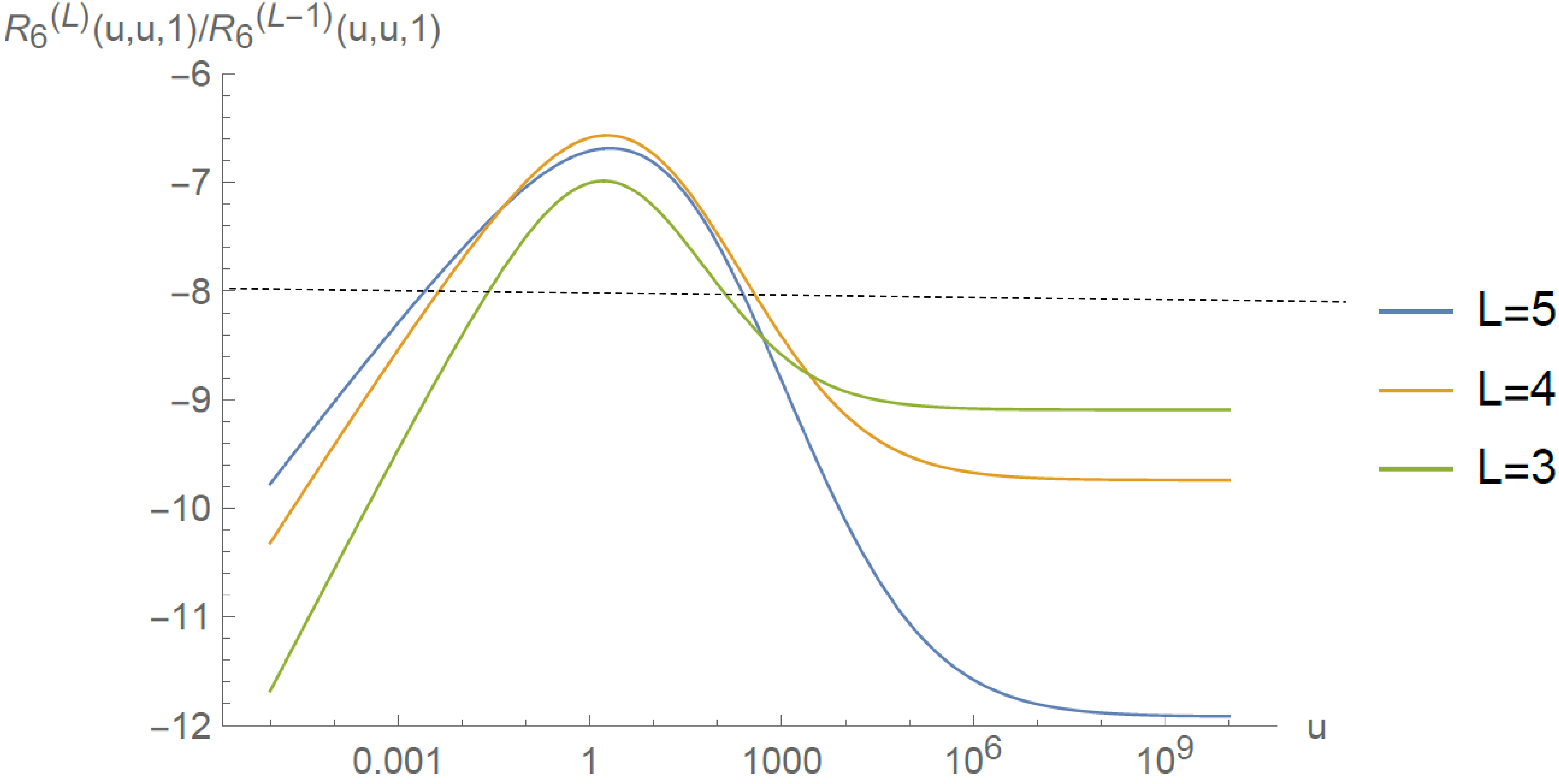
L	$\gamma_K^{(L)}/\gamma_K^{(L-1)}$	$\bar{R}_6^{(L)}(1,1,1)$	$\overline{\ln \mathcal{W}}_{\rm hex}^{(L)}(\frac{3}{4},\frac{3}{4},\frac{3}{4})$	$\overline{\ln \mathcal{W}}_{\rm hex}^{(L)}(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
2	-1.6449340	∞	-2.7697175	-2.8015275
3	-3.6188549	-7.0040885	-5.0036164	-5.1380714
4	-4.9211827	-6.5880519	-5.8860842	-6.0359857
5	-5.6547494	-6.7092373	-6.3453695	-6.4658887
6	-6.0801089	–	–	–
7	-6.3589220	–	–	–
8	-6.5608621	–	–	–

↓ -8

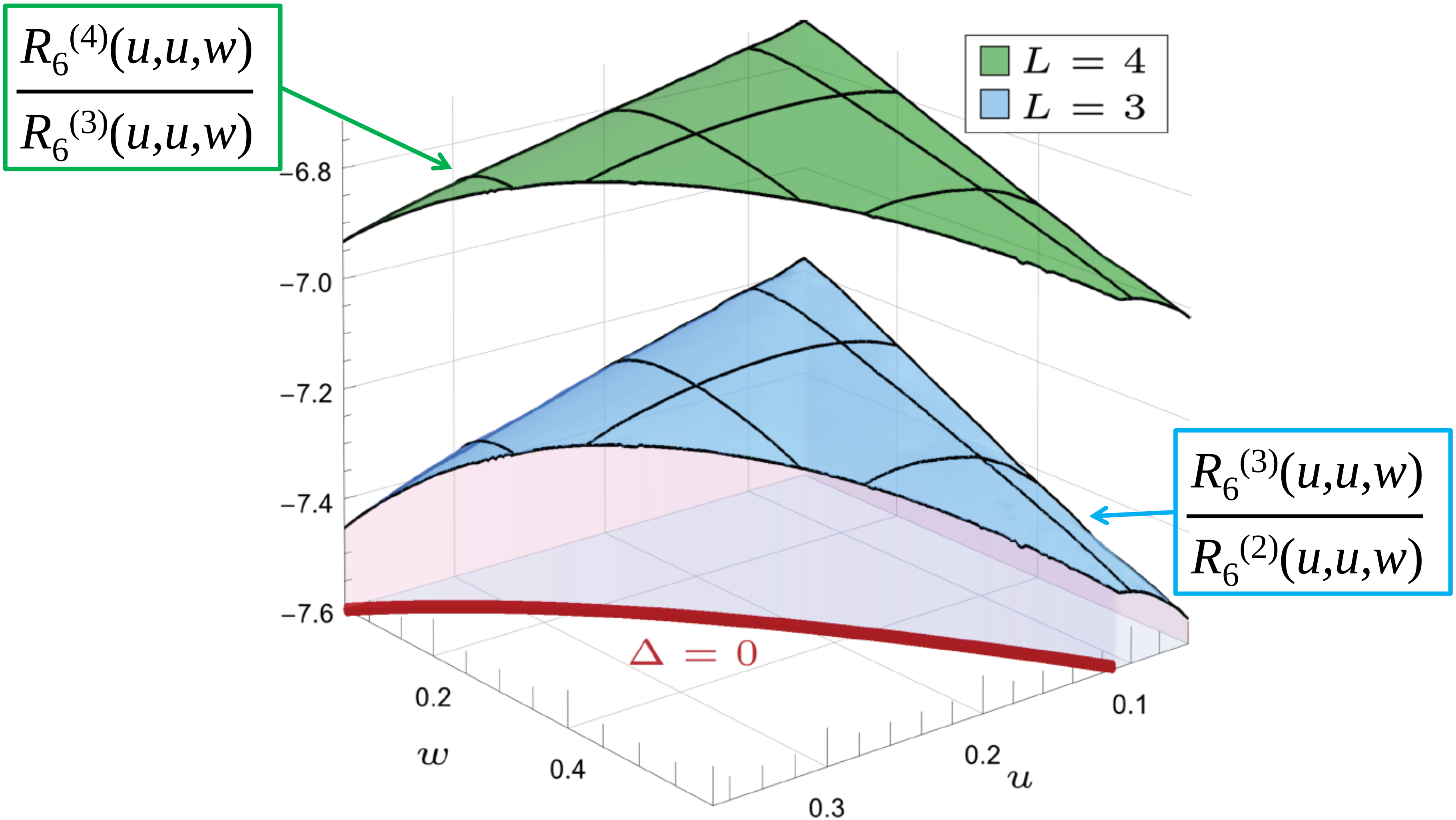


w
(u,u,u)
(1,v,v)
(near) collinear
v = 0, u + w = 1
(1,1,1)
(u,u,w)
v
multi−Regge
(1,0,0)
u

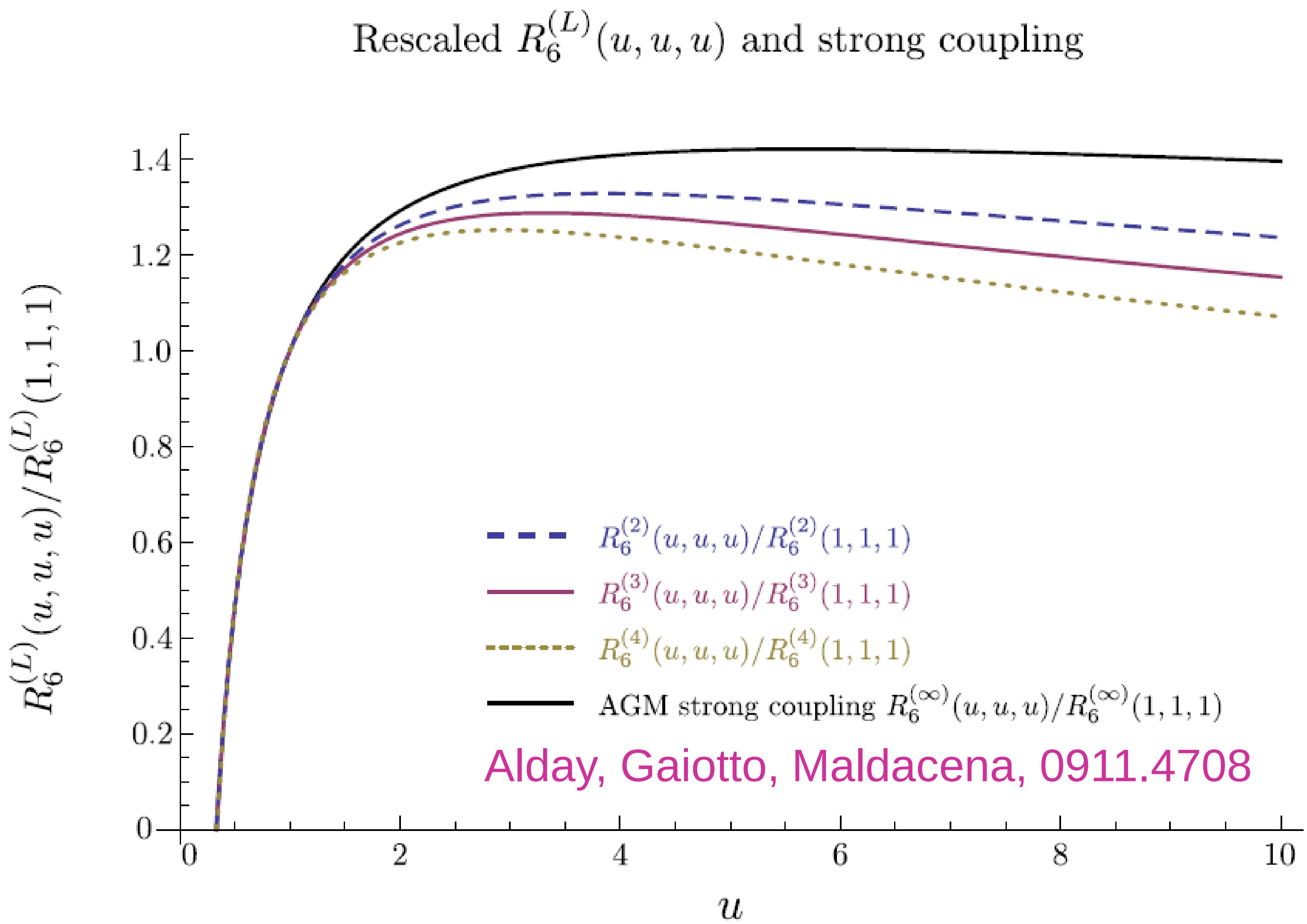
On $(u,u,1)$, everything collapses to HPLs of u



Ratio of successive loop orders extremely flat on (u,u,w)

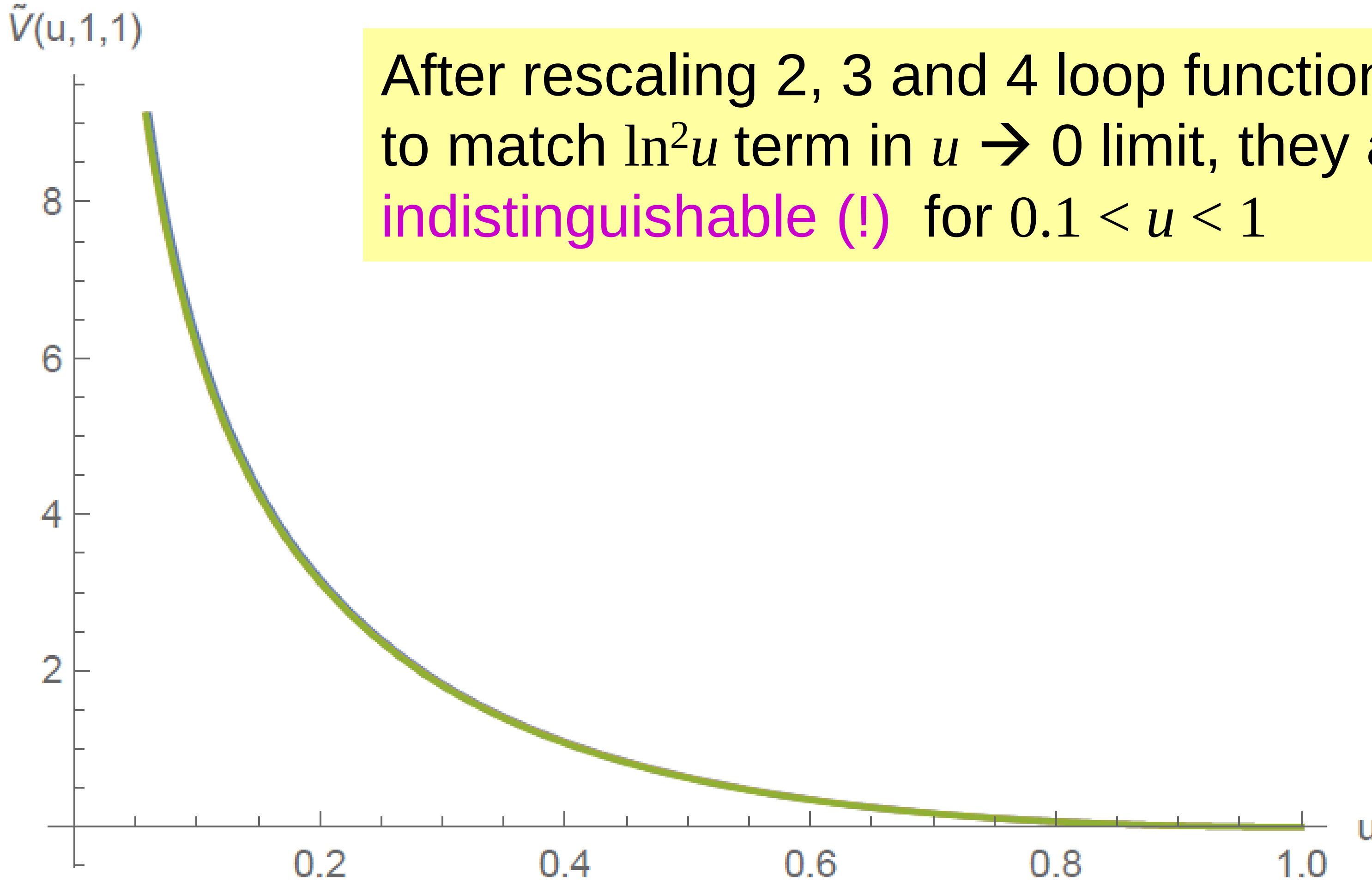


Not too far from -8, though not yet approaching -8 monotonically



(u,u,u) → cyclotomic polylogs (weak coupling)
$\arccos^2(1/4/u)$ (strong coupling)

Ratio function odd part $\tilde{V}(u,1,1)$



After rescaling 2, 3 and 4 loop functions to match $\ln^2 u$ term in $u \rightarrow 0$ limit, they are indistinguishable (!) for $0.1 < u < 1$

Beyond 6 gluons

- Cluster Algebras provide strong clues to "the right functions" Spradlin talk

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289

- Power seen particularly in symbol of 3-loop MHV 7-point amplitude
  Drummond, Papathanasiou, Spradlin 1412.3763
- Can turn such symbols into functions using same ideas discussed here.

Summary & Outlook

- Hexagon function ansatz → planar N=4 SYM amplitudes over full kinematical phase space, for 6 gluons, both MHV and NMHV, to high loop orders
- Steinmann + $\bar{Q}$ equation = powerful constraints → No need for loop-momentum integrands
- Only need very little additional information from multi-Regge (or OPE) limits
- Numerical and analytical results intriguing!
- → finite coupling for generic kinematics?
- → other theories? even QCD?

Amplitudes 2018 @ SLAC/Stanford

Amplitudes 2016 at Nordita is part of a conference series that begun in 2009. Previous and upcoming conferences are:

- 2009: **IPPP Durham**
- 2010: **Queen Mary University of London**
- 2011: **University of Michigan, Ann Arbor**
- 2012: **DESY Hamburg**
- 2013: **Ringberg Castle**
- 2014: **IPhT Saclay**
- 2015: **ETH Zürich**
- 2017: **University of Edinburgh**

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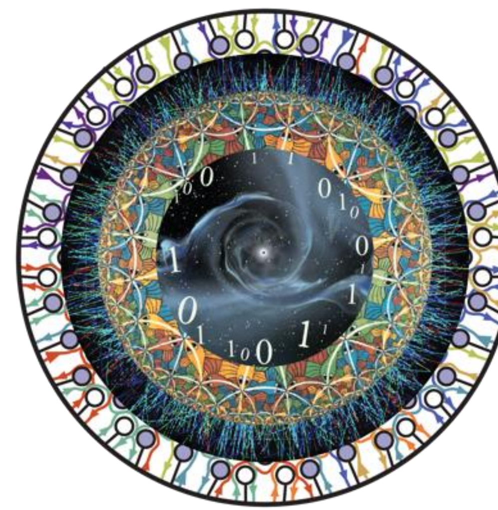
to be preceded by a school at QMAP (Center for Quantum Mathematics and Physics) @ UC Davis



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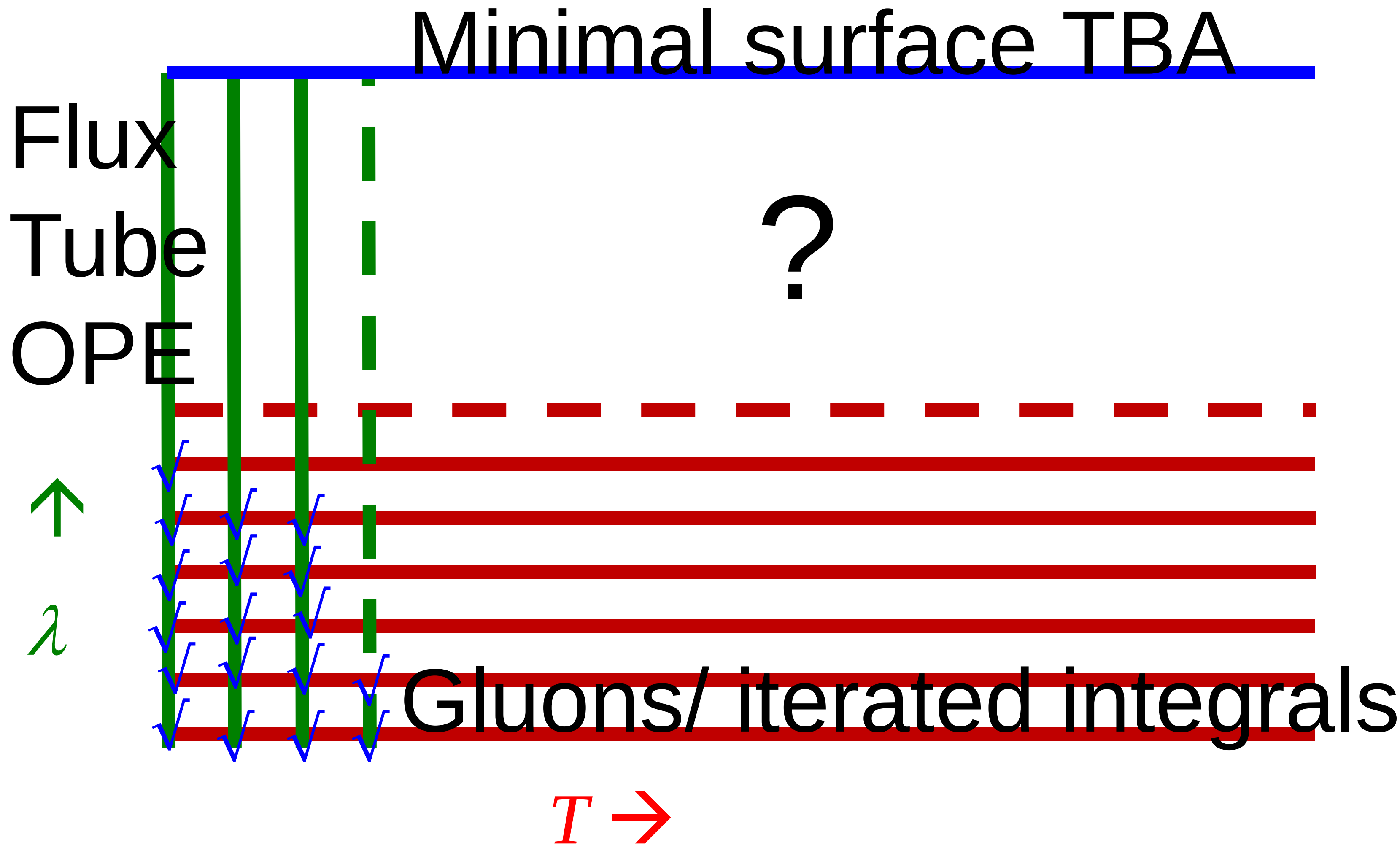
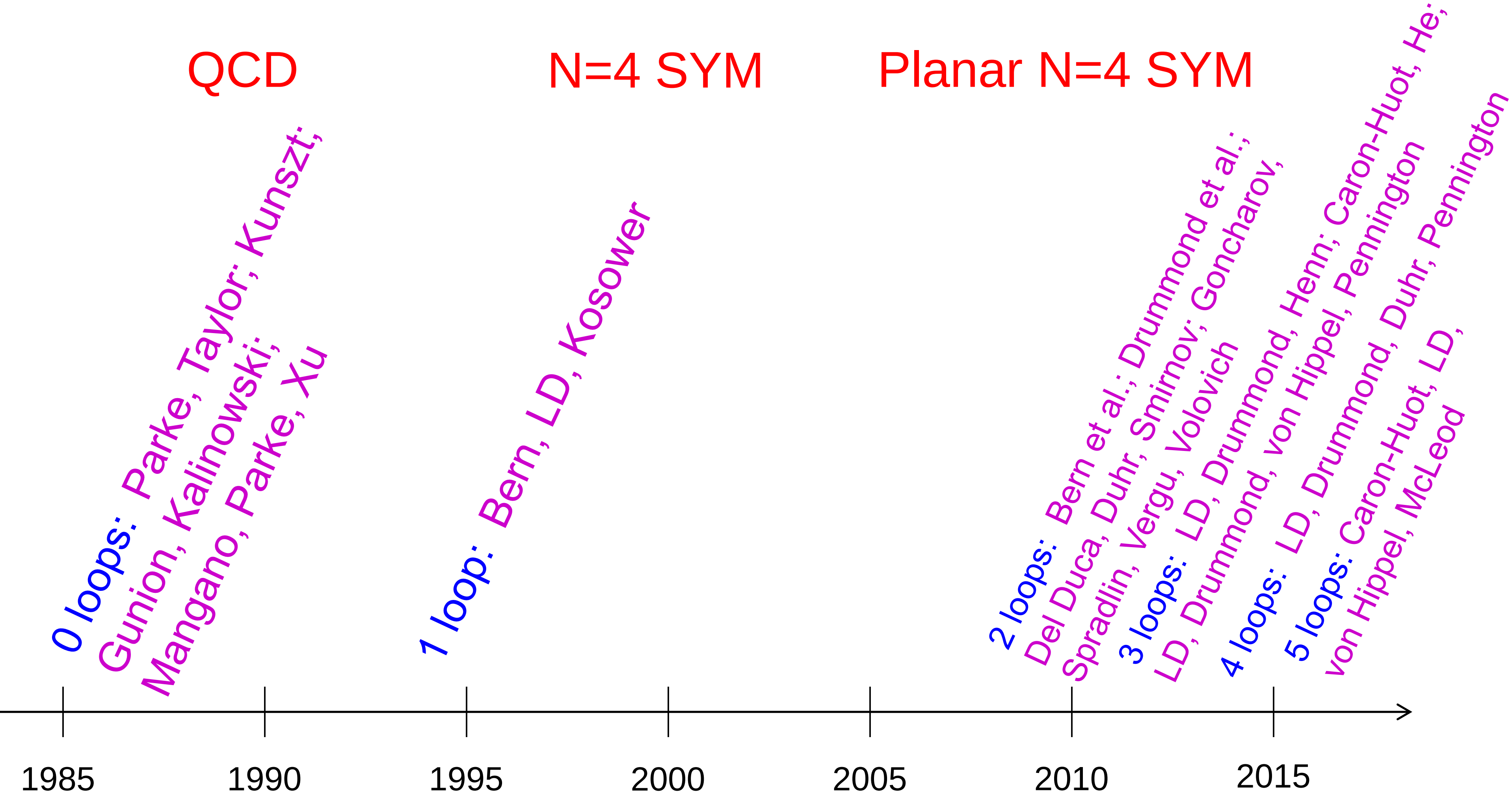
Extra Slides

Combining N=4 approaches



MHV (6-point) timeline



Iterative construction

$$\left.\frac{\partial F}{\partial u}\right|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}} F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}} F^{y_w}$$

- F weight n, from F^x weight n-1 (already classified)
- Just need to impose: 1. mixed-partials: $\dfrac{\partial^2 F}{\partial u_i \partial u_j} = \dfrac{\partial^2 F}{\partial u_j \partial u_i}, \qquad i \neq j$

$$\begin{aligned}
F^{u,v} &= F^{v,u} - F^{y_u,y_v} + F^{y_v,y_u}\,, \\
F^{v,w} &= F^{w,v} - F^{y_v,y_w} + F^{y_w,y_v}\,, \\
F^{w,u} &= F^{u,w} - F^{y_w,y_u} + F^{y_u,y_w}\,, \\
F^{1-u,1-v} &= F^{1-v,1-u} + F^{y_u,y_v} - F^{y_u,y_w} - F^{y_v,y_u} + F^{y_v,y_w} + F^{y_w,y_u} - F^{y_w,y_v} \\
F^{1-v,1-w} &= F^{1-w,1-v} + F^{y_v,y_w} - F^{y_v,y_u} - F^{y_w,y_v} + F^{y_w,y_u} + F^{y_u,y_v} - F^{y_u,y_w} \\
F^{1-w,1-u} &= F^{1-u,1-w} + F^{y_w,y_u} - F^{y_w,y_v} - F^{y_u,y_w} + F^{y_u,y_v} + F^{y_v,y_w} - F^{y_v,y_u} \\
F^{u,1-v} &= F^{1-v,u} + F^{y_u,y_w} - F^{y_w,y_u}\,, \\
F^{v,1-w} &= F^{1-w,v} + F^{y_v,y_u} - F^{y_u,y_v}\,, \\
F^{w,1-u} &= F^{1-u,w} + F^{y_w,y_v} - F^{y_v,y_w}\,, \\
F^{u,1-w} &= F^{1-w,u} + F^{y_u,y_v} - F^{y_v,y_u}\,, \\
F^{v,1-u} &= F^{1-u,v} + F^{y_v,y_w} - F^{y_w,y_v}\,, \\
F^{w,1-v} &= F^{1-v,w} + F^{y_w,y_u} - F^{y_u,y_w}\,,
\end{aligned}$$

$$\begin{aligned}
F^{u,y_u} &= F^{y_u,u}\,, \\
F^{v,y_v} &= F^{y_v,v}\,, \\
F^{w,y_w} &= F^{y_w,w}\,, \\
F^{u,y_w} &= F^{w,y_u} - F^{y_u,w} + F^{y_w,u}\,, \\
F^{v,y_u} &= F^{u,y_v} - F^{y_v,u} + F^{y_u,v}\,, \\
F^{w,y_v} &= F^{v,y_w} - F^{y_w,v} + F^{y_v,w}\,, \\
F^{1-v,y_v} &= F^{y_v,1-v} - F^{y_u,1-u} + F^{1-u,y_u} + F^{y_u,w} - F^{w,y_u} - F^{y_w,v} + F^{v,y_w} \\
F^{1-w,y_w} &= F^{y_w,1-w} - F^{y_v,1-v} + F^{1-v,y_v} + F^{y_v,u} - F^{u,y_v} - F^{y_u,w} + F^{w,y_u} \\
F^{1-u,y_u} &= F^{y_u,1-u} - F^{y_w,1-w} + F^{1-w,y_w} + F^{y_w,v} - F^{v,y_w} - F^{y_v,u} + F^{u,y_v} \\
F^{1-u,y_v} &= F^{y_v,1-u} + F^{y_v,w} - F^{w,y_v}\,, \\
F^{1-v,y_w} &= F^{y_w,1-v} + F^{y_w,u} - F^{u,y_w}\,, \\
F^{1-w,y_u} &= F^{y_u,1-w} + F^{y_u,v} - F^{v,y_u}\,, \\
F^{1-u,y_w} &= F^{y_w,1-u} + F^{y_w,v} - F^{v,y_w}\,, \\
F^{1-v,y_u} &= F^{y_u,1-v} + F^{y_u,w} - F^{w,y_u}\,, \\
F^{1-w,y_v} &= F^{y_v,1-w} + F^{y_v,u} - F^{u,y_v}\,.
\end{aligned}$$

- 2. No bad branch cuts: $F^{1-u_i}(y_i = 1, y_j, y_k) \;=\; 0$

T^1 OPE for NMHV: 1111 component

- Evaluate (i) prefactors →

$$\mathcal{P}^{(1111)}|_{T^1} = \frac{1}{2}\{V(u,v,w) + V(w,u,v) - \tilde{V}(u,v,w) + \tilde{V}(w,u,v)$$
$$+FT[\frac{1+S^4}{S(1+S^2)}V(v,w,u) - \frac{1-S^2}{S}V(u,v,w)]\}$$

$T = e^{-\tau}$
$S = e^{\sigma}$
$F = e^{i\phi}$

- BSV:

$$\mathcal{P}^{(1111)} = 1 + e^{i\phi-\tau}\int \frac{du}{2\pi}\mu(u)(h(u)-1)e^{ip(u)\sigma-\gamma(u)\tau} + e^{-i\phi-\tau}\int \frac{du}{2\pi}\mu(u)(\bar{h}(u)-1)e^{ip(u)\sigma-\gamma(u)\tau} + \ldots$$

$$h(u) = \frac{x^+(u)x^-(u)}{g^2}, \qquad \bar{h}(u) = \frac{g^2}{x^+(u)x^-(u)} \qquad x^{\pm}(u) = x(u \pm \tfrac{i}{2}) \qquad x(u) = \tfrac{1}{2}(u + \sqrt{u^2-4g^2})$$

- Quantities μ, p, γ meromorphic in rapidity u
- Evaluate u integral as (truncated) residue sum

See also Papathanasiou, 1310.5735

$\bar{Q}$ equation for NMHV

Caron-Huot, He, 1112.1060; S. Caron-Huot (2015)

$$\bar{Q}\hat{\mathcal{R}}_{6,1} = \frac{\gamma_K}{8}\int d^{2|3}\mathcal{Z}_7[\mathcal{R}_{7,2} - \hat{\mathcal{R}}_{6,1}\mathcal{R}^{\text{tree}}_{7,1}] + \text{ cyclic}$$

$$\bar{Q}^A_a = \sum_{i=1}^{n} \chi_i^A \frac{\partial}{\partial Z_i^a} \qquad \hat{\mathcal{R}}_{6,1} \equiv \frac{\mathcal{A}_{\text{NMHV}}}{\mathcal{A}_{\text{BDS-like}}}$$

prevents second (simpler) term from generating new "final entries"

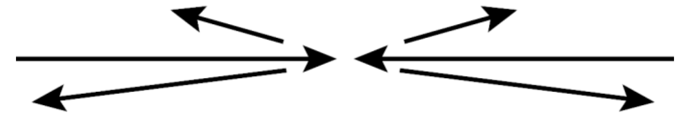
→ Only 18 out of 5 x 9 = 45 possible R-invariants x final entries:

$$(1)\, d\ln(uw/v), \quad (1)\, d\ln\left(\frac{(1-w)u}{w(1-u)y_v}\right),$$

$$[(2)+(5)+(3)+(6)]\, d\ln\left(\frac{v}{1-v}\right) + (1)\, d\ln\left(\frac{w}{y_u(1-w)}\right) + (4)\, d\ln\left(\frac{u}{y_w(1-u)}\right)$$

+ cyclic

2→4 multi-Regge limit



- Euclidean MRK limit vanishes
- To get nonzero result for physical region, first let

$u_1 \to u_1\, e^{-2\pi i}$, then $u_1 \to 1, \quad u_2, u_3 \to 0$

$$\frac{u_2}{1-u_1} \to \frac{1}{|1-z|^2} \qquad \frac{u_3}{1-u_1} \to \frac{|z|^2}{|1-z|^2}$$

$$R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u)\, [g_r^{(L)}(z,\bar{z}) + 2\pi i\, h_r^{(L)}(z,\bar{z})]$$

$g_r^{(L)}$ and $h_r^{(L)}$ all well understood by now;

Fadin, Lipatov, 1111.0782; LD, Duhr, Pennington, 1207.0186; Pennington, 1209.5357; Basso, Caron-Huot, Sever, 1407.3766; Broedel, Sprenger, 1512.04963

also NMHV behavior

Lipatov, Prygarin, Schnitzer, 1205.0186; LD, von Hippel, 1408.1505

MRK Master formulae

- MHV:

$$w = -z, \quad w^* = -\bar{z}$$

$$e^{R+i\pi\delta}|_{\rm MRK} = \cos \pi\omega_{ab} + i\,\frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} |w|^{2i\nu} \Phi_{\rm Reg}(\nu, n) \times \left(-\frac{1}{1-u}\frac{|1+w|^2}{|w|}\right)^{\omega(\nu,n)}$$

NLL: Fadin, Lipatov, 1111.0782;
Caron-Huot, 1309.6521

- NMHV:

$$\exp(R^{\rm NMHV} + i\pi\delta)|_{\rm MRK} = \mathcal{P} \exp(R^{\rm MHV} + i\pi\delta)$$

$$= \cos \pi\omega_{ab} - i\,\frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{d\nu}{(i\nu + \frac{n}{2})^2} |w|^{2i\nu} \times \Phi^{\rm NMHV}_{\rm Reg}(\nu, n) \left(-\frac{1}{1-u}\frac{|1+w|^2}{|w|}\right)^{\omega(\nu,n)}$$

LL: Lipatov, Prygarin, Schnitzer, 1205.0186

MRK limits agree with all-orders predictions

Basso, Caron-Huot, Sever 1407.3766

- BFKL eigenvalue:

$$E^{(1)}(\nu,n),\ \ E^{(2)}(\nu,n),\ \ E^{(3)}(\nu,n)$$

LL, NLL, NNLL, NNNLL

- Impact factors:

$$\Phi_{\text{Reg}}^{\text{(N)MHV},(1)}(\nu,n),\ \ \Phi_{\text{Reg}}^{\text{(N)MHV},(2)}(\nu,n),\ \ \Phi_{\text{Reg}}^{\text{(N)MHV},(3)}(\nu,n),\ \ \Phi_{\text{Reg}}^{\text{(N)MHV},(4)}(\nu,n)$$

- All zeta-valued linear combinations of:

derivatives of $\ln\Gamma(1 \pm i\nu + \frac{n}{2})$ $\quad \frac{i\nu}{\nu^2+\frac{n^2}{4}},\ \ \frac{n}{\nu^2+\frac{n^2}{4}}$

NMHV MRK limit

Like g, h for R_6:
Extract p, q from V, $\tilde{V}$
→ linear combinations of SVHPLs [Brown, 2004]

$$R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \, [g_r^{(L)}(w,w^*) + 2\pi i \, h_r^{(L)}(w,w^*)]$$

$$\mathcal{P}_{\rm MRK}^{(L)} = (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \Big[\frac{1}{1+w^*} \big(p_r^{(L)}(w,w^*) + 2\pi i \, q_r^{(L)}(w,w^*)\big) + \frac{w^*}{1+w^*} \big(p_r^{(L)}(w,w^*) + 2\pi i \, q_r^{(L)}(w,w^*)\big)\Big|_{(w,w^*)\to(\frac{1}{w},\frac{1}{w^*})} \Big] + \mathcal{O}(1-u)$$

- Then match p, q to master formula for factorization in Fourier-Mellin space

On the line $(u,u,1)$, everything collapses to HPLs of u.
In a linear representation, and a very compressed notation,

$$H_1^u H_{2,1}^u = H_1^u H_{0,1,1}^u = 3H_{0,1,1,1}^u + H_{1,0,1,1}^u \to 3h_7^{[4]} + h_{11}^{[4]}$$

2 and 3 loop answers:

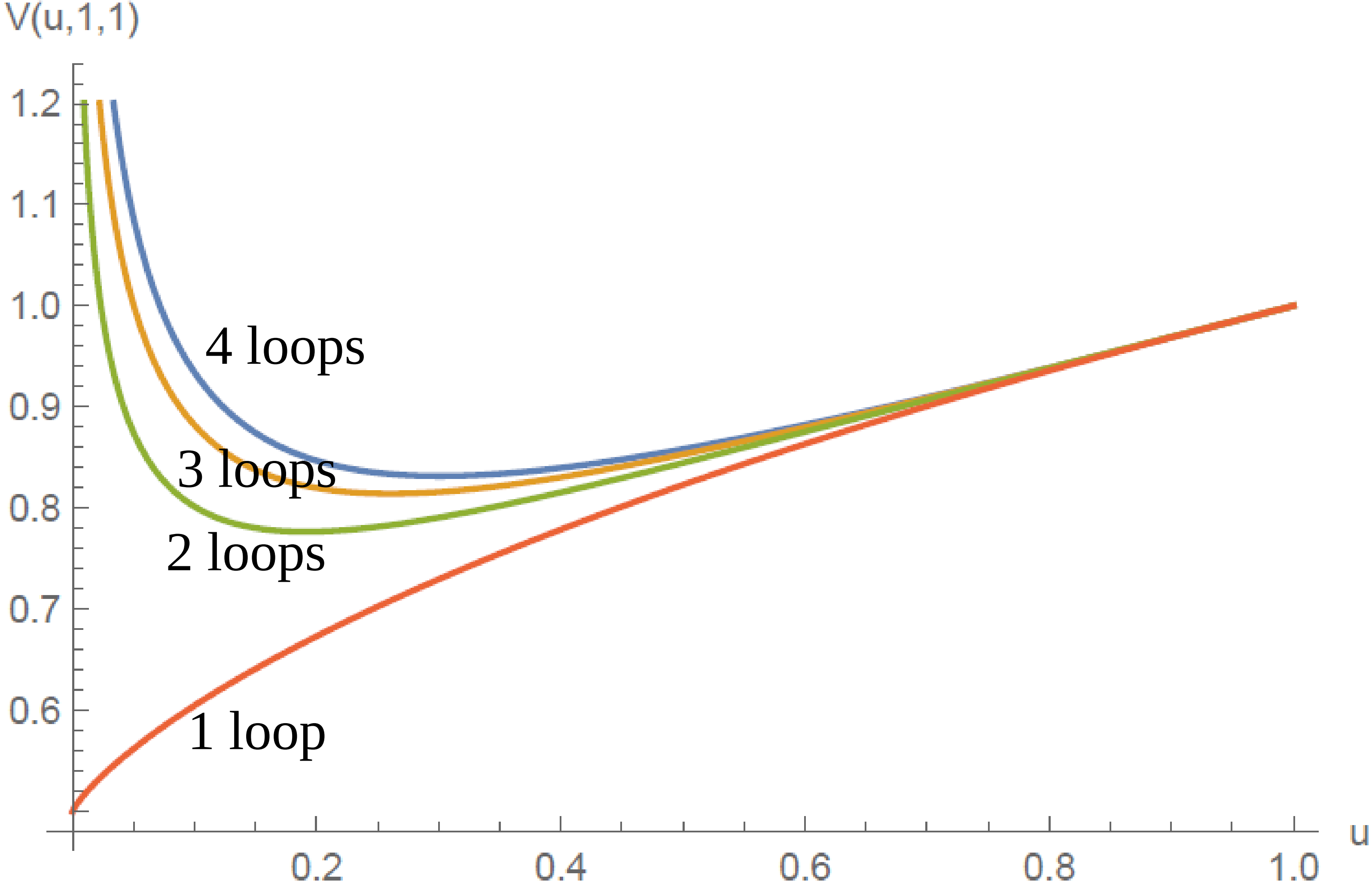
$$\begin{aligned}
R_6^{(2)}(u,u,1) &= h_1^{[4]} - h_3^{[4]} + h_9^{[4]} - h_{11}^{[4]} - \frac{5}{2}\zeta_4\,, \\
R_6^{(3)}(u,u,1) &= -3h_1^{[6]} + 5h_3^{[6]} + \frac{3}{2}h_5^{[6]} - \frac{9}{2}h_7^{[6]} - \frac{1}{2}h_9^{[6]} - \frac{3}{2}h_{11}^{[6]} - h_{13}^{[6]} - \frac{3}{2}h_{17}^{[6]} \\
&\quad + \frac{3}{2}h_{19}^{[6]} - \frac{1}{2}h_{21}^{[6]} - \frac{3}{2}h_{23}^{[6]} - 3h_{33}^{[6]} + 5h_{35}^{[6]} + \frac{3}{2}h_{37}^{[6]} - \frac{9}{2}h_{39}^{[6]} \\
&\quad - \frac{1}{2}h_{41}^{[6]} - \frac{3}{2}h_{43}^{[6]} - h_{45}^{[6]} - \frac{3}{2}h_{49}^{[6]} + \frac{3}{2}h_{51}^{[6]} - \frac{1}{2}h_{53}^{[6]} - \frac{3}{2}h_{55}^{[6]} \\
&\quad + \zeta_2\Big[-h_1^{[4]} + 3h_3^{[4]} + 2h_5^{[4]} - h_9^{[4]} + 3h_{11}^{[4]} + 2h_{13}^{[4]}\Big] \\
&\quad + \zeta_4\Big[-2h_1^{[2]} - 2h_3^{[2]}\Big] + \zeta_3^2 + \frac{413}{24}\zeta_6\,,
\end{aligned}$$

4 loop answer →

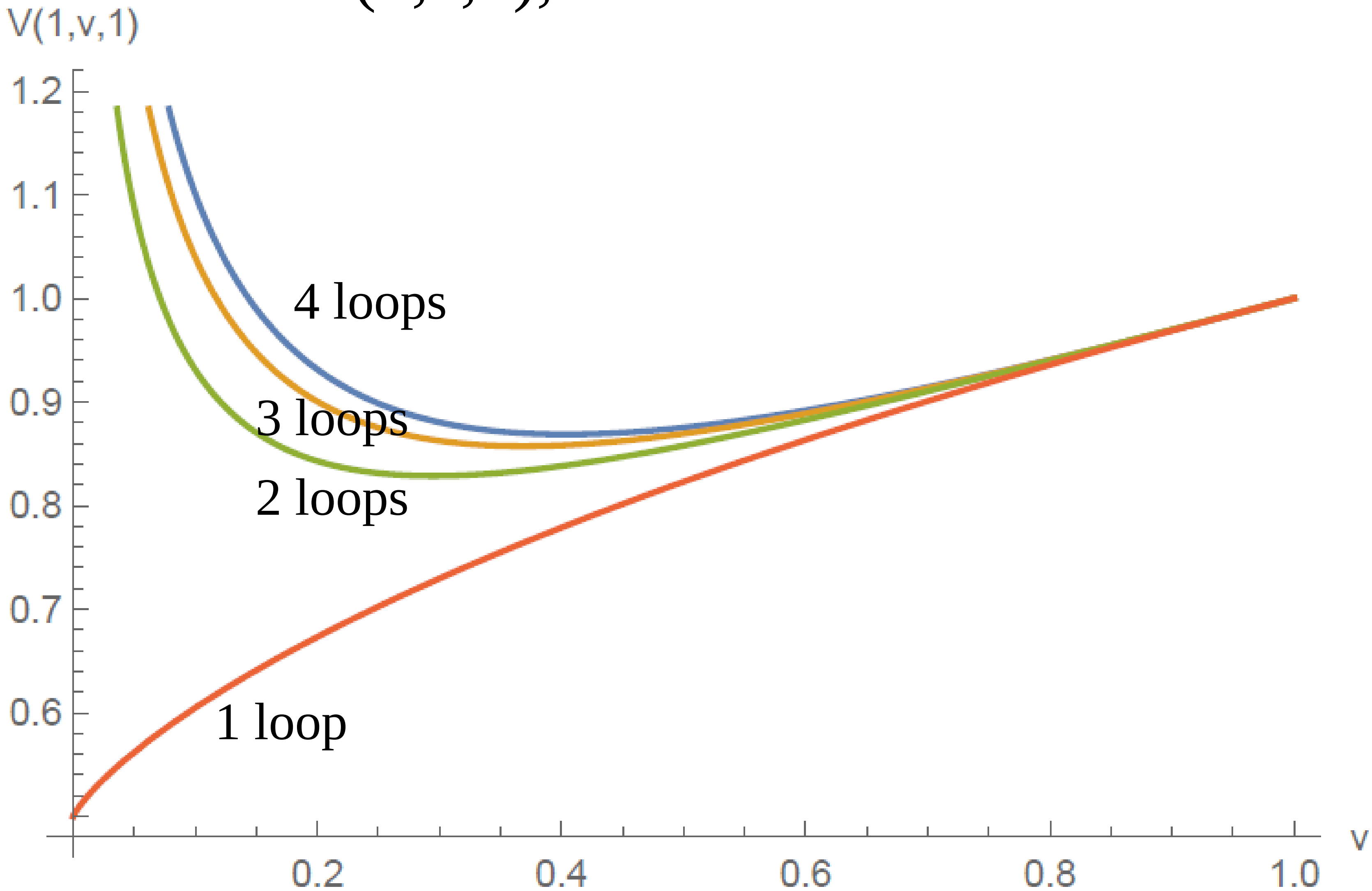
$$\begin{aligned}
R_6^{(4)}(u,u,1) = {}& 15h_1^{[8]} - 41h_3^{[8]} - \frac{31}{2}h_5^{[8]} + \frac{105}{2}h_7^{[8]} - \frac{7}{2}h_9^{[8]} + \frac{53}{2}h_{11}^{[8]} + 12h_{13}^{[8]} - 42h_{15}^{[8]} \\
& + \frac{5}{2}h_{17}^{[8]} + \frac{11}{2}h_{19}^{[8]} + \frac{9}{2}h_{21}^{[8]} - \frac{41}{2}h_{23}^{[8]} + h_{25}^{[8]} - 13h_{27}^{[8]} - 7h_{29}^{[8]} - 5h_{31}^{[8]} \\
& + 6h_{33}^{[8]} - 11h_{35}^{[8]} - 3h_{37}^{[8]} + 3h_{39}^{[8]} - 4h_{43}^{[8]} - 4h_{45}^{[8]} - 11h_{47}^{[8]} + \frac{3}{2}h_{49}^{[8]} - \frac{3}{2}h_{51}^{[8]} \\
& - 3h_{53}^{[8]} - 5h_{55}^{[8]} + \frac{3}{2}h_{57}^{[8]} - \frac{3}{2}h_{59}^{[8]} + 9h_{65}^{[8]} - 25h_{67}^{[8]} - 9h_{69}^{[8]} + 27h_{71}^{[8]} - 2h_{73}^{[8]} \\
& + 9h_{75}^{[8]} + 2h_{77}^{[8]} - 23h_{79}^{[8]} + 2h_{81}^{[8]} - h_{85}^{[8]} - 8h_{87}^{[8]} + 2h_{89}^{[8]} - 3h_{91}^{[8]} + \frac{5}{2}h_{97}^{[8]} \\
& - \frac{7}{2}h_{99}^{[8]} - \frac{1}{2}h_{101}^{[8]} + \frac{5}{2}h_{103}^{[8]} + \frac{1}{2}h_{105}^{[8]} + \frac{1}{2}h_{107}^{[8]} + \frac{1}{2}h_{109}^{[8]} - \frac{5}{2}h_{111}^{[8]} + 15h_{129}^{[8]} \\
& - 41h_{131}^{[8]} - \frac{31}{2}h_{133}^{[8]} + \frac{105}{2}h_{135}^{[8]} - \frac{7}{2}h_{137}^{[8]} + \frac{53}{2}h_{139}^{[8]} + 12h_{141}^{[8]} - 42h_{143}^{[8]} \\
& + \frac{5}{2}h_{145}^{[8]} + \frac{11}{2}h_{147}^{[8]} + \frac{9}{2}h_{149}^{[8]} - \frac{41}{2}h_{151}^{[8]} + h_{153}^{[8]} - 13h_{155}^{[8]} - 7h_{157}^{[8]} \\
& - 5h_{159}^{[8]} + 6h_{161}^{[8]} - 11h_{163}^{[8]} - 3h_{165}^{[8]} + 3h_{167}^{[8]} - 4h_{171}^{[8]} - 4h_{173}^{[8]} \\
& - 11h_{175}^{[8]} + \frac{3}{2}h_{177}^{[8]} - \frac{3}{2}h_{179}^{[8]} - 3h_{181}^{[8]} - 5h_{183}^{[8]} + \frac{3}{2}h_{185}^{[8]} - \frac{3}{2}h_{187}^{[8]} \\
& + 9h_{193}^{[8]} - 25h_{195}^{[8]} - 9h_{197}^{[8]} + 27h_{199}^{[8]} - 2h_{201}^{[8]} + 9h_{203}^{[8]} + 2h_{205}^{[8]} - 23h_{207}^{[8]} \\
& + 2h_{209}^{[8]} - h_{213}^{[8]} - 8h_{215}^{[8]} + 2h_{217}^{[8]} - 3h_{219}^{[8]} + \frac{5}{2}h_{225}^{[8]} - \frac{7}{2}h_{227}^{[8]} - \frac{1}{2}h_{229}^{[8]} \\
& + \frac{5}{2}h_{231}^{[8]} + \frac{1}{2}h_{233}^{[8]} + \frac{1}{2}h_{235}^{[8]} + \frac{1}{2}h_{237}^{[8]} - \frac{5}{2}h_{239}^{[8]} \\
& + \zeta_2\Big[2h_1^{[6]} - 14h_3^{[6]} - \frac{15}{2}h_5^{[6]} + \frac{37}{2}h_7^{[6]} - \frac{5}{2}h_9^{[6]} + \frac{25}{2}h_{11}^{[6]} + 7h_{13}^{[6]} - \frac{1}{2}h_{17}^{[6]} \\
& \qquad + \frac{5}{2}h_{19}^{[6]} + \frac{7}{2}h_{21}^{[6]} + \frac{9}{2}h_{23}^{[6]} - 3h_{25}^{[6]} + 3h_{27}^{[6]} + 2h_{33}^{[6]} - 14h_{35}^{[6]} - \frac{15}{2}h_{37}^{[6]} \\
& \qquad + \frac{37}{2}h_{39}^{[6]} - \frac{5}{2}h_{41}^{[6]} + \frac{25}{2}h_{43}^{[6]} + 7h_{45}^{[6]} - \frac{1}{2}h_{49}^{[6]} + \frac{5}{2}h_{51}^{[6]} + \frac{7}{2}h_{53}^{[6]} \\
& \qquad + \frac{9}{2}h_{55}^{[6]} - 3h_{57}^{[6]} + 3h_{59}^{[6]}\Big] \\
& + \zeta_4\Big[\frac{15}{2}h_1^{[4]} - \frac{55}{2}h_3^{[4]} - \frac{41}{2}h_5^{[4]} + \frac{15}{2}h_9^{[4]} - \frac{55}{2}h_{11}^{[4]} - \frac{41}{2}h_{13}^{[4]}\Big] \\
& + \Big(\zeta_2\zeta_3 - \frac{5}{2}\zeta_5\Big)\Big[h_3^{[3]} + h_7^{[3]}\Big] - \Big(\zeta_3^2 - \frac{73}{4}\zeta_6\Big)\Big[h_1^{[2]} + h_3^{[2]}\Big] \\
& - \frac{3}{2}\zeta_2\zeta_3^2 - \frac{5}{2}\zeta_3\zeta_5 - \frac{471}{4}\zeta_8 + \frac{3}{2}\zeta_{5,3}\,.
\end{aligned}$$

5 loop answer is several pages

$V(u,1,1)$, normalized at $u = 1$

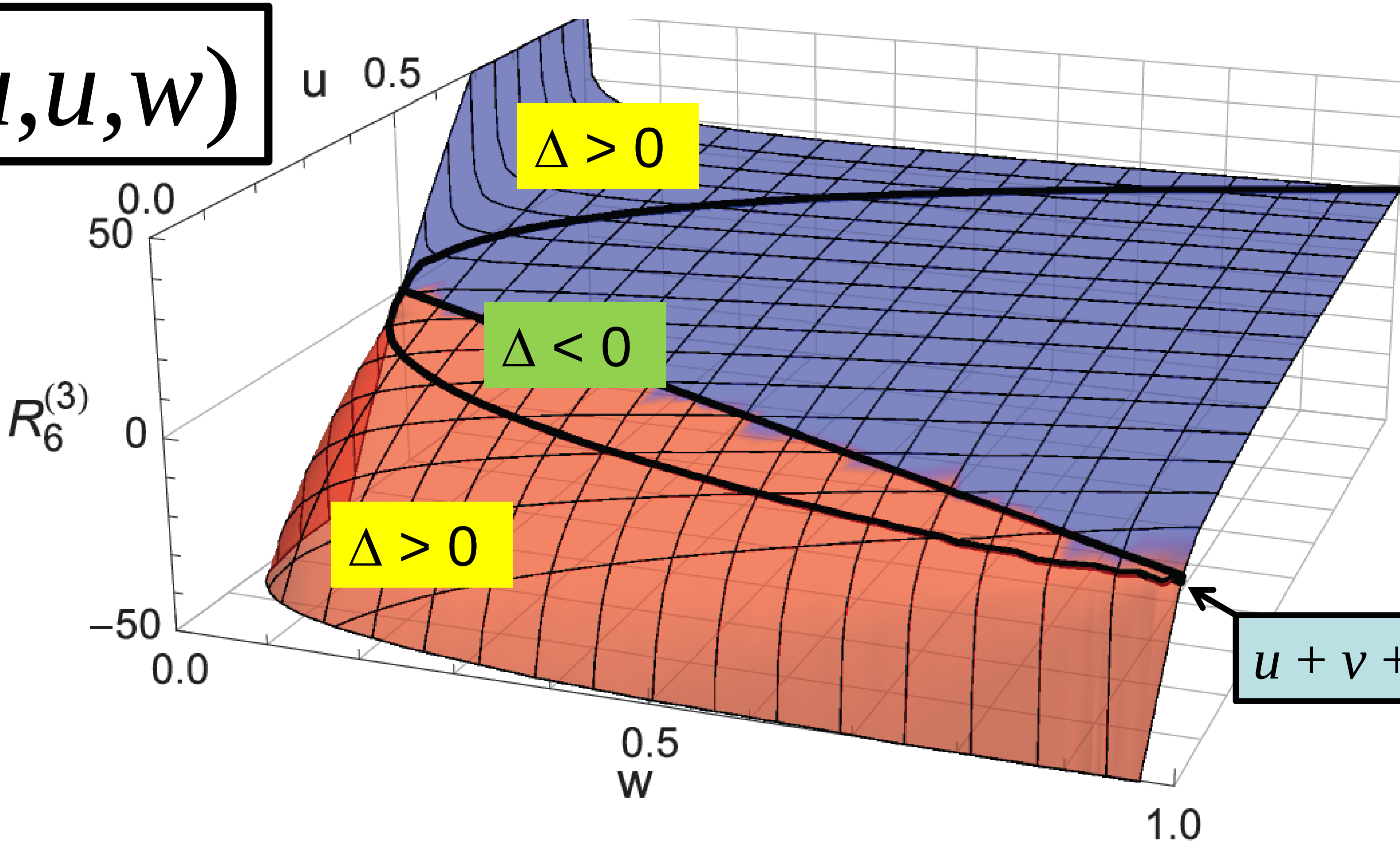


$V(1,v,1)$, normalized at $v = 1$



$R_6^{(3)}$ sign stable within $\Delta > 0$ regions

(u,u,w)



$u + v + w = 1$

But not of uniform sign in “MHV positive region at 4 loops; other quantities might be

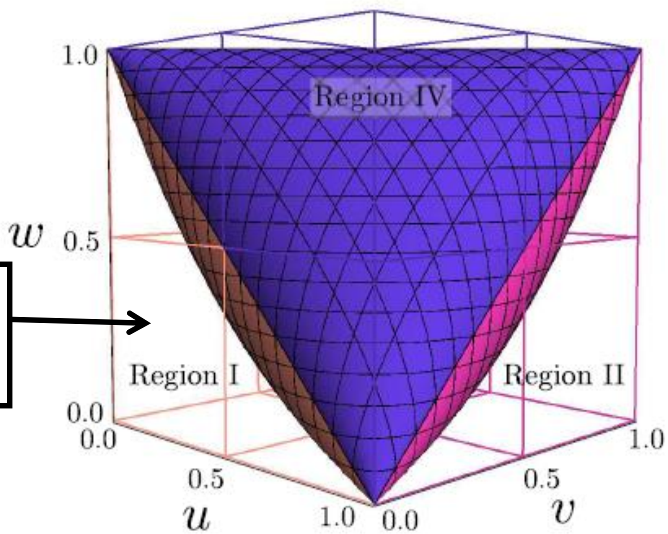
A menagerie of functions

1. HPLs: One variable, symbol letters $\{u, 1-u\}$. Near-collinear limit, lines $(u,u,1)$, $(u,1,1)$
2. Cyclotomic Polylogarithms [Ablinger, Blumlein, Schneider, 1105.6063]: One variable, letters $\{y_u, 1+y_u, 1+y_u+y_u^2\}$. For line (u,u,u).
3. SVHPLs [F. Brown, 2004]: Two variables, letters $\{z, 1-z, \bar{z}, 1-\bar{z}\}$. First entry/single-valuedness constraint (real analytic function in z plane). Multi-Regge limit.
4. Full hexagon functions. Three variables, symbol letters $\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$, branch-cut condition

Hexagon functions are multiple polylogarithms in y_i

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$

$$\text{Region I}: \quad \begin{cases} \Delta > 0\,, \quad 0 < u_i < 1\,, \quad \text{and} \quad u+v+w<1, \\ 0 < y_i < 1\,. \end{cases}$$



$$\mathcal{G} = \left\{ G(\vec{w};y_u) | w_i \in \{0,1\} \right\} \cup \left\{ G(\vec{w};y_v) \middle| w_i \in \left\{0,1,\frac{1}{y_u}\right\} \right\} \cup \left\{ G(\vec{w};y_w) \middle| w_i \in \left\{0,1,\frac{1}{y_u},\frac{1}{y_v},\frac{1}{y_u y_v}\right\} \right\}$$

- Useful for analytics and for numerics for $\Delta > 0$

  GiNaC implementation: Vollinga, Weinzierl, hep-th/0410259