One-Loop Integrands from the Riemann Sphere

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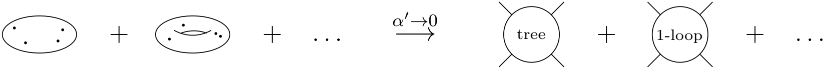
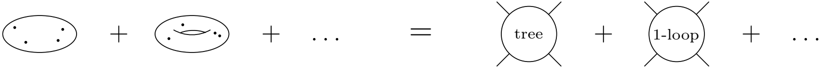
with Yvonne Geyer, Lionel Mason (Oxford), Piotr Tourkine (Cambridge)

Worldsheet Models of (Massless) QFTs

String theory: field theory is $\alpha' \to 0$



Worldsheet Models of QFTs: no α'



Modular integrals localised by scattering equations.

Worldsheet Models $\longleftrightarrow$ Scattering Equations

Old story

Twistor string theory [Witten 03] $\longrightarrow$ RSV formula [Roiban, Spradlin, Volovich 04]

$D = 4$. SYM, SUGRA. [Hodges, Cachazo, Geyer, Skinner, Mason 12]

Only tree level. [Berkovitz, Witten 04]

New story

Ambitwistor string theory [Mason, Skinner 13] $\longleftarrow$ CHY formulas [Cachazo, He, Yuan 13-14]

Any D. Many theories of massless particles.

This talk: Loop-level progress!

Outline

- Review of scattering equations and ambitwistor strings
- Ambitwistor strings at genus 1
- New formulas at one loop

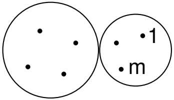
Scattering Equations

[Cachazo, He, Yuan '13]

Consider n massless particles, $k_i^2 = 0$, $i = 1, \ldots, n$

$$\boxed{E_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0, \quad \forall i}$$

- kinematic invariants $s_{ij} = 2\, k_i \cdot k_j \quad \longrightarrow \quad$ points $\sigma_i \in \mathbb{CP}^1$
- $\sum_i k_i = 0$: $SL(2,\mathbb{C})$ invariance, $\quad \sigma \to \dfrac{A\,\sigma + B}{C\,\sigma + D}$
- $(n-3)!$ solutions $\sigma_i^{(A)}$
- factorisation: $\quad (k_1 + \ldots + k_m)^2 \to 0 \quad$ gives $\quad \sigma_1, \ldots, \sigma_m \to \sigma_\star$



CHY Formulas

[Cachazo, He, Yuan '13]

Tree-level scattering amplitude: $\boxed{\mathcal{A} = \int d\mu\, \mathcal{I}}$

- measure is universal

$$\int d\mu = \int \frac{d^n\sigma}{\mathrm{vol}SL(2,\mathbb{C})} \prod_i{}' \delta(E_i) \qquad \Longrightarrow \qquad \mathcal{A} = \sum_{A=1}^{(n-3)!} \frac{\mathcal{I}}{J}\Big|_{\sigma=\sigma^{(A)}}$$

- $\mathcal{I}$ specifies the theory

$$\mathcal{I}^{\mathrm{YM}} = \mathrm{Pf}'M(\epsilon) \times \left(\frac{\mathrm{tr}(T^{a_1}T^{a_2}\cdots T^{a_n})}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}} + \text{non-cyclic perm} \right) \qquad \sigma_{rs} = \sigma_r - \sigma_s$$

$$\mathcal{I}^{\mathrm{Grav}} = \mathrm{Pf}'M(\epsilon) \times \mathrm{Pf}'M(\tilde{\epsilon}) \qquad \varepsilon_i^{\mu\nu} = \epsilon_i^{\mu}\tilde{\epsilon}_i^{\nu} \qquad \text{(graviton, dilaton, 2-form)}$$

$\Rightarrow$ Gravity $\sim$ YM^2 cf. Kawai-Lewellen-Tye relations
Bern-Carrasco-Johansson double copy

SE hard to solve, but no need for that.

[Dolan, Goddard, Cachazo, Gomez, Baadsgaard at al, Huang et al, Sogaard, Zhang, Cardona, Kalousios, Bjerrum-Bohr et al]

Geometry of Scattering Equations

Consider 1-form on $\mathbb{CP}^1$, $\boxed{P_\mu = d\sigma \sum_{i=1}^{n} \frac{k_{i\,\mu}}{\sigma - \sigma_i}}$ $SL(2,\mathbb{C})$ invariant

The scattering equations are $P^2(\sigma) = 0$.

Worldsheet model?

Mason, Skinner: target space is

ambitwistor space $=$ space of null geodesics of complexified spacetime.

Take (X^μ, P_ν) with $P^2 = 0$, $(X^\mu, P_\nu) \sim (X^\mu + \alpha\, P^\mu, P_\nu)$.

Strings in Ambitwistor Space

[Mason, Skinner 13]

Chiral complexification of worldline action for massless particle:

$$\boxed{S_B = \frac{1}{2\pi}\int_\Sigma P_\mu\,\bar\partial X^\mu - \frac{1}{2}\,e\,P^2}$$

- e enforces $P^2 = 0$. ambitwistor space $\checkmark$
- gauge freedom: $\delta X^\mu = \alpha P^\mu,\ \delta P_\nu = 0,\ \delta e = \bar\partial\alpha$.

Quantisation: $\mathcal{A} = \left\langle \prod_{i=1}^{n} V_i \right\rangle$ Fix $e = 0$, $V_i = \int_\Sigma \bar\delta(k\cdot P)\, e^{ik\cdot X} \dots$

X^μ integration is exact: $\bar\partial P_\mu = 2\pi i\, d\sigma \wedge d\bar\sigma \sum_i k_{i\,\mu}\,\delta^2(\sigma - \sigma_i)$

$$\Rightarrow \quad \text{on } \mathbb{CP}^1, \quad P_\mu = d\sigma \sum_i \frac{k_{i\,\mu}}{\sigma - \sigma_i} \quad \Rightarrow \quad \text{scattering equations}$$

No α'. Weight$(e^{ik\cdot X}) = 0$ $\rightarrow$ **massless spectrum**

Worldsheet Matter

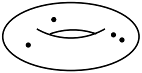
[Ohmori 15; Casali, Geyer, Mason, RM, Roehrig 15]

Combine with other chiral CFTs to reproduce CHY formulas:

$$S = S_B + S^\ell + S^r \quad \to \quad \mathcal{A} = \int d\mu\ \mathcal{I}^\ell\, \mathcal{I}^r$$

S^ℓ \ S^r	S_Ψ	S_{Ψ_1,Ψ_2}	$S^{(m')}_{\rho,\Psi}$	$S^{(N')}_{CS}$	$S^{(N')}_{CS_0}$
S_Ψ	E				
S_{Ψ_1,Ψ_2}	BI	Galileon			
$S^{(m)}_{\rho,\Psi}$	EM $U(1)^m$	DBI	EMS $U(1)^m \otimes U(1)^{m'}$		
$S^{(N)}_{CS,\Psi}$	EYM	gen. DBI	EYMS $SU(N)\otimes U(1)^{m'}$	EYMS $SU(N)\otimes SU(N')$	
$S^{(N)}_{CS}$	YM	NLSM	YMS $SU(N)\otimes U(1)^{m'}$	YMS $SU(N)\otimes SU(N')$	S $SU(N)\otimes SU(N')$

Ambitwistor String at Genus 1

Genus-1 worldsheet: torus  τ $\quad z \sim z + 2\pi \sim z + 2\pi\tau$

Ambitwistor string correlator: type II SUGRA [Adamo, Casali, Skinner 13]

$$\mathcal{A}^{(1)} = \int d^D\ell \, d\tau \, \bar{\delta}(P^2(z_0|\tau)) \left(\prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i|\tau)) \, dz_i \right) \mathcal{I}_\tau$$

- scattering equations on torus: $\bar{\delta}(\,\cdot\,) \;\Rightarrow\; \boxed{P^2(z|\tau) = 0}$
- $P_\mu = dz \left(\ell_\mu + \sum_i k_{i\,\mu} \dfrac{\theta_1'(z - z_i)}{\theta_1(z - z_i)} \right)$
- Modular invariance $\tau \sim \tau + 1 \sim -1/\tau$

How to evaluate?

[see also Cardona, Gomez 16]

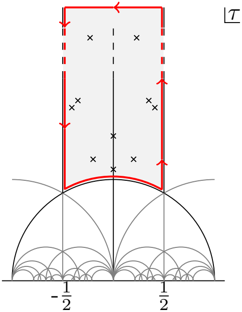
Contour argument

[Geyer, Mason, RM, Tourkine 15]

Integrand is rational, like tree level amplitude.

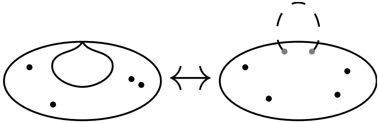
Torus similar to Riemann sphere?

Idea: apply residue theorem to localise on $\tau = i\infty$ $(q = e^{2\pi i\tau} = 0)$.



Use modular invariance:

only $q = 0$ contributes!

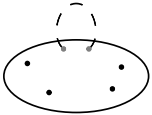


Nodal Riemann sphere!

New One-Loop Formula

Take coords $\sigma = e^{2\pi i z}$ on $\mathbb{CP}^1$,

$$\mathcal{A}^{(1)} = -\int d^D\ell \, \frac{1}{\ell^2} \left(\prod_{i=2}^{n} \bar{\delta}(E_i) \, \frac{d\sigma_i}{\sigma_i^2} \right) \mathcal{I}_0$$



where

- $\mathcal{I}_0 = \mathcal{I}(q = 0)$. Details later.
- $P_\mu = \left(\frac{\ell_\mu}{\sigma - 0} - \frac{\ell_\mu}{\sigma - \infty} + \sum_i \frac{k_{i\,\mu}}{\sigma - \sigma_i} \right) d\sigma \qquad \sim$ tree-level for $n + 2$ pts

Scattering equations: $$\boxed{E_i = k_i \cdot P(\sigma_i) = \frac{k_i \cdot \ell}{\sigma_i} + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_{ij}} = 0}$$

Can be understood from forward limit of tree level. [HY, CHY 15] [Naculich 14]

Shifted Integrand

New formula: $$\mathcal{A}^{(1)} = -\int d^D\ell \, \frac{1}{\ell^2} \left(\prod_{i=2}^{n} \bar{\delta}(E_i) \, \frac{d\sigma_i}{\sigma_i^2} \right) \mathcal{I}_0$$

Puzzle! Only one $\frac{1}{\ell^2}$, rest depends only on $\ell \cdot K, \; \ell \cdot \epsilon \; \ldots$

Shifted Integrand

- use $\displaystyle \frac{1}{\prod_i D_i} = \sum_i \frac{1}{D_i \, \prod_{j \neq i}(D_j - D_i)}$
- shift each term $\displaystyle \frac{1}{D_i} \to \frac{1}{\ell^2}$

Example: $$\frac{1}{\ell^2(\ell+K)^2} = \frac{1}{\ell^2(2\,\ell \cdot K + K^2)} + \frac{1}{(\ell+K)^2(-2\,\ell \cdot K - K^2)}$$

$$\overset{\text{shift}}{\to} \frac{1}{\ell^2}\left[\frac{1}{2\,\ell \cdot K + K^2} + \frac{1}{-2\,\ell \cdot K + K^2}\right]$$

Good for obtaining loop integrands from trees!

[Baadsgaard et al 15] [CHY 15]

Yang-Mills and Gravity

Supersymmetry

SUGRA: take $q \to 0$ of torus correlator [Adamo, Casali, Skinner 13]

$$\mathcal{I}_0^{SUGRA} = \hat{\mathcal{I}}(\epsilon)\, \hat{\mathcal{I}}(\tilde{\epsilon}) \qquad \hat{\mathcal{I}} = 16\,(\mathrm{Pf}\, M_2 - \mathrm{Pf}\, M_3)\,|_{q^0} - 2\,\mathrm{Pf}\, M_3\,|_{q^{1/2}}$$

SYM: guess based on double copy to gravity.

$$\mathcal{I}_0^{SYM} = \hat{\mathcal{I}}(\epsilon)\, \mathcal{I}_{PT^{(1)}} \qquad \frac{1}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}} \rightsquigarrow \frac{\sigma_{+\ell,-\ell}}{\sigma_{+\ell,1}\,\sigma_{1,2}\,\sigma_{2,3}\cdots\,\sigma_{n,-\ell}} + \text{cyc}$$

No supersymmetry

Can identify bulding blocks in sum over spin structures.

In $D=4$, $\quad \text{vector} = \mathrm{Pf}\, M_3\,|_{q^{1/2}} + 2\,\mathrm{Pf}\, M_3\,|_{q^0}\,, \quad \text{scalar} = \mathrm{Pf}\, M_3\,|_{q^0}.$

$$\mathcal{I}_0 = (\,\text{vector}\,)\, \mathcal{I}_{\mathrm{PT}^{(1)}} \;= \text{pure Yang-Mills}$$

$$\mathcal{I}_0 = (\,\text{vector}\,)^2 \;= \text{NS-NS gravity (graviton-dilaton-axion)}$$

$$\mathcal{I}_0 = (\,\text{vector}\,)^2 - 2\,(\,\text{scalar}\,)^2 \;= \text{Einstein gravity}$$

Conclusion

- Scattering equations work at loop level.
- Geometric picture of the loop momentum.
- New formulas for gauge theory and gravity, with or without SUSY.
- Gravity $\sim$ YM^2 at one loop.

Some open questions

- Higher loops? Multiple degenerations. **Yvonne's talk**
- Relation to string theory? [Casali, Tourkine 16]
- What class of theories?
- Worldsheet models on nodal spheres?