A TOY MODEL OF BLACK HOLE COMPLEMENTARITY

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A toy model of black hole complementarity

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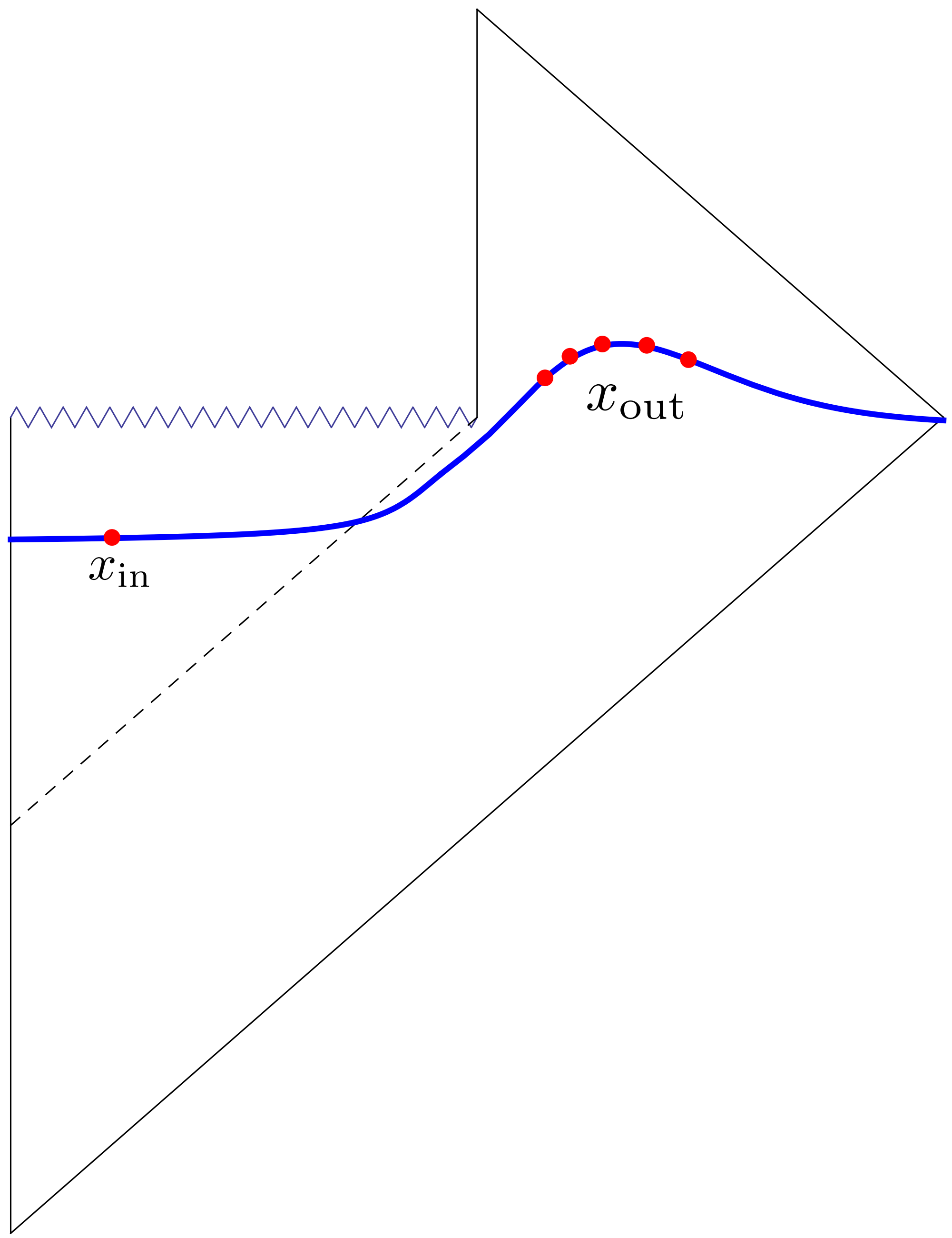
MOTIVATION

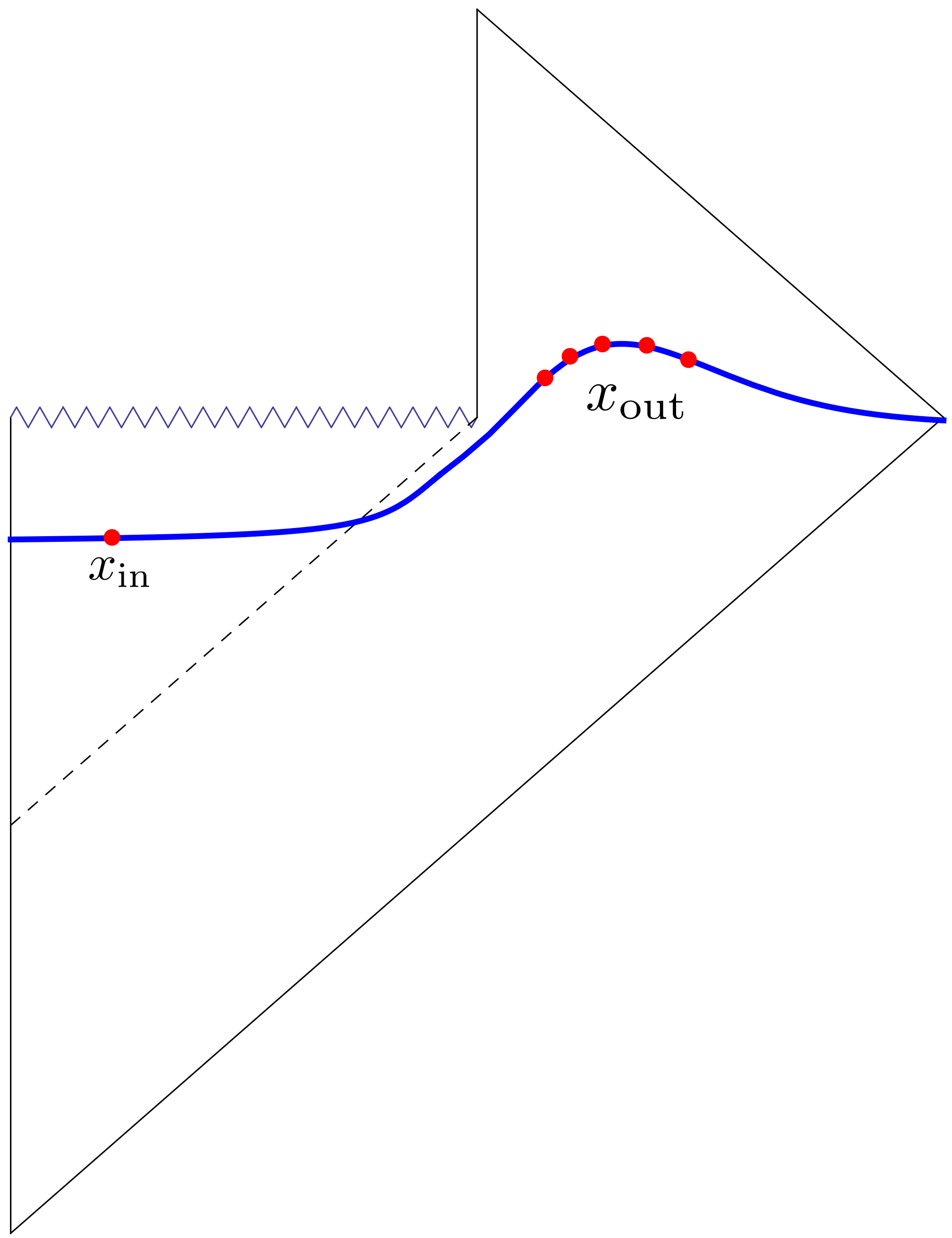
Key questions



COMPLEMENTARITY - BH AND BEYOND

- “complementarity” in the case of black holes is related to the idea that the Hilbert spaces of excitations inside and outside the horizon are NOT factorized.
- Degrees of freedom in the interior are highly scrambled copies of those in the exterior.

$$\phi(x_{in}) = P^{complex}(x_{out})$$



COMPLEMENTARITY AND LOCALITY

- This implies a radical loss of locality in correlators with $O(S_{BH})$ insertions.
- A possible resolution of black hole information paradox is based on this loss of locality (Papadodimas-Raju)
- However, it was not clear what should be the form of $P^{complex}$

- Furthermore, non-locality is believed to be an essential ingredient of quantum gravity. Therefore this very feature must definitely be realised even when there is no black hole.

- e.g let us consider empty AdS - no black hole.

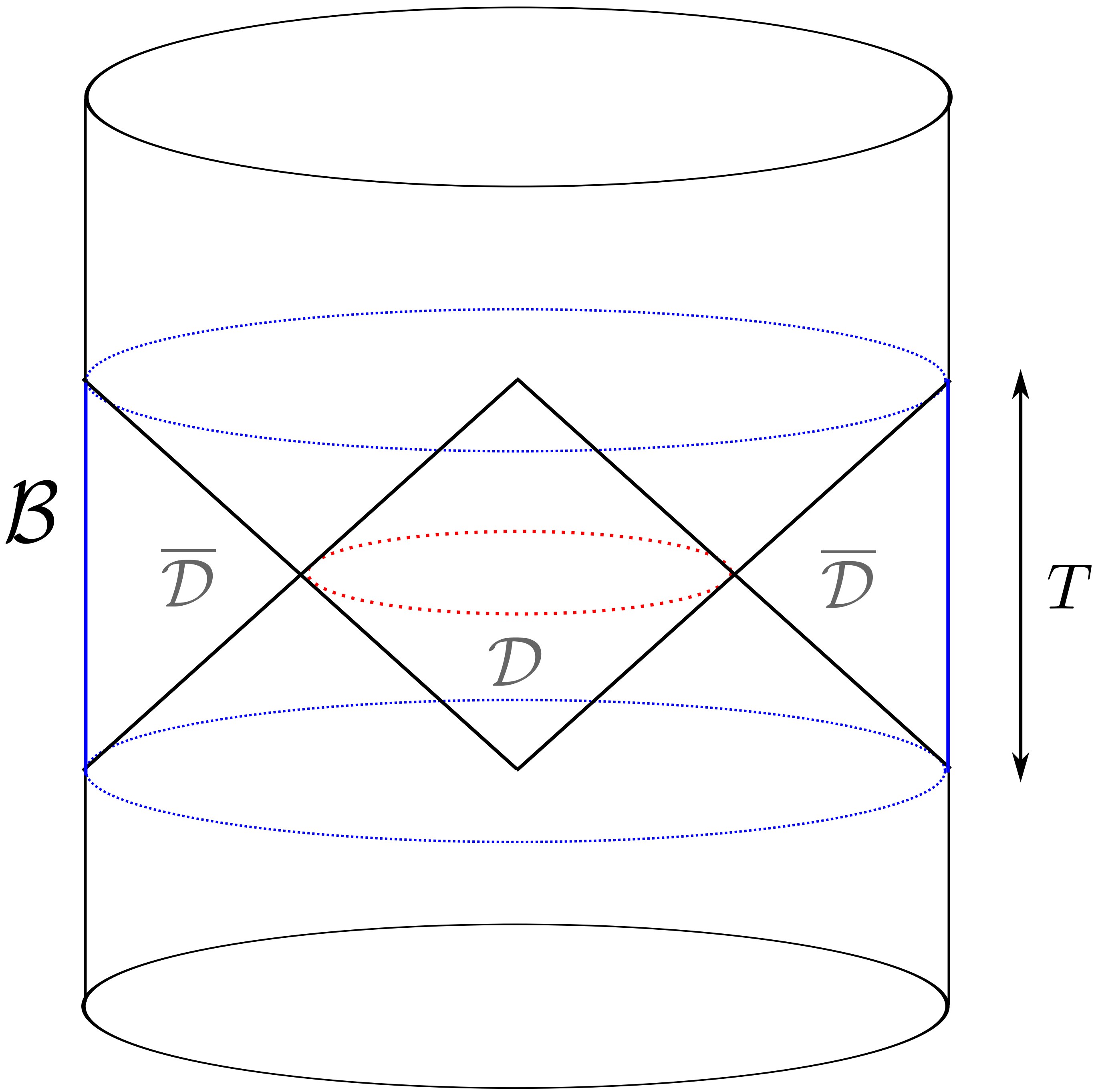
- Given $\psi(x_1)$ at a point x_1 in this AdS, is this possible to construct a complicated polynomial $\mathcal{P}$ with supports on points which are all space-like with respect to x_1 such that

$$\psi(x_1) = \mathcal{P}?$$

- There is a puzzle related to this (Harlow et al.).
- Consider a point at the centre of AdS.
- Bulk locality demands a local operator at that point should commute with all boundary operators.
- However, time-slice axiom in QFT implies that such an operator which commutes with all local operators on the boundary should be proportional to identity!!!!

THE TOY MODEL

Without black holes



THE MODEL

- We consider empty AdS in global coordinates.

$$ds^2 = -\left(1+r^2\right)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1}^2$$

- We consider a time band in the CFT (de Boer et.al. Hole-o-graphy).
- If the length (T) of the time-band < the light-crossing time in AdS(π), it defines a causal diamond” in the centre of the AdS.
- Points on the diamond are space-like related to the points on the band.

THE MODEL

- The base of the diamond extends in the radial direction up to

$$r_d = \tan\left[\frac{\pi - T}{2}\right]$$

- As we increase the length of the time band, the diamond becomes smaller, disappears for $T \geq \pi$
- From the bulk point of view it looks like a spherical horizon at $r = r_d$
- The set up resembles a black hole. The interior of the diamond mimics "behind the horizon" region for an observer who is confined to move in the complementary annular region.

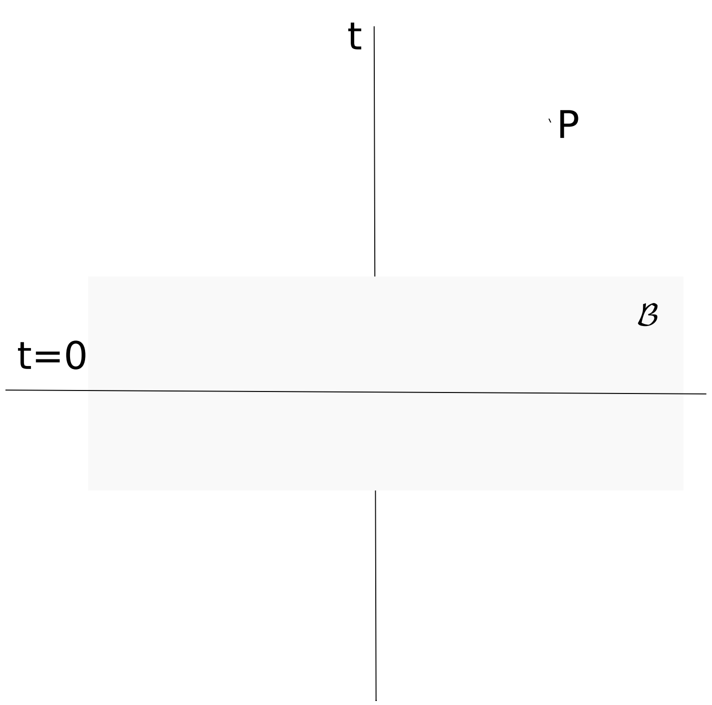
THE MODEL - BULK PICTURE

- If we do not have gravity, naively, one would expect a decomposition of the bulk Hilbert space: $\mathcal{H} = \mathcal{H}_{\mathcal{D}} \otimes \mathcal{H}_{\overline{\mathcal{D}}}$
- Equivalently, the algebra of operators, $\mathcal{A}(\mathcal{D})$ and $\mathcal{A}(\overline{\mathcal{D}})$ are well-defined and commuting.
- In presence of gravity, the situation is bit more complicated - no sharply defined local observables, issues of gauge invariance, Gauss law tails..
- These commuting algebras, however, do make sense but only in the large N limit - in this limit we expect to see some sort of factorisation of Hilbert space in two sectors - the interior of the diamond and the exterior.

THE MODEL - BOUNDARY PERSPECTIVE

- Our aim is to understand the bulk decomposition from the perspective of the algebra of operators $\mathcal{A}(\mathcal{B})$ in the time-band of the CFT.

- In usual QFT, we generally never talk about algebra of operators in a time-domain.



- Hamiltonian evolution essentially associates with $\mathcal{A}(\mathcal{B})$, the entire set of operators in the CFT

$$\mathcal{O}(P) = e^{iHt_p}\mathcal{O}(0)e^{-iHt_p}$$

DEFINING SMALL ALGEBRA

- However, here we are dealing with CFTs having holographic dual - special class of operators - Generalised Free Fields.

- In a large N CFT, GFFs are low dimensional single trace operators (Heemskerk, Penedones, Polchinski, Sully; Papadodimas, El-Showk).

- Hence we have a natural hierarchy in the spectrum of local operators - simple (low-order polynomial of single trace operators) and complicated (polynomials of very high order).

DEFINING SMALL ALGEBRA

- Define

$$\mathcal{A}_{\text{small}}(\mathcal{B}) = \text{span of}\{\mathcal{O}_{\ell_1}(t_1), \mathcal{O}_{\ell_2}(t_2)\mathcal{O}_{\ell_3}(t_3), \ldots, \mathcal{O}_{\ell_4}(t_4)\mathcal{O}_{\ell_5}(t_5)\cdots\mathcal{O}_{\ell_{\mathcal{D}_m}}(t_{\mathcal{D}_m})\}$$

$$\mathcal{D}_m \ll N$$

- Although we would keep on calling it an algebra, it is not strictly an “algebra” due to the cut-off!

- However, this “edge effect” is not so important for low energy EFT experiments. However, this is important to realise the idea of “complementarity”.

DEFINING EFT HILBERT SPACE

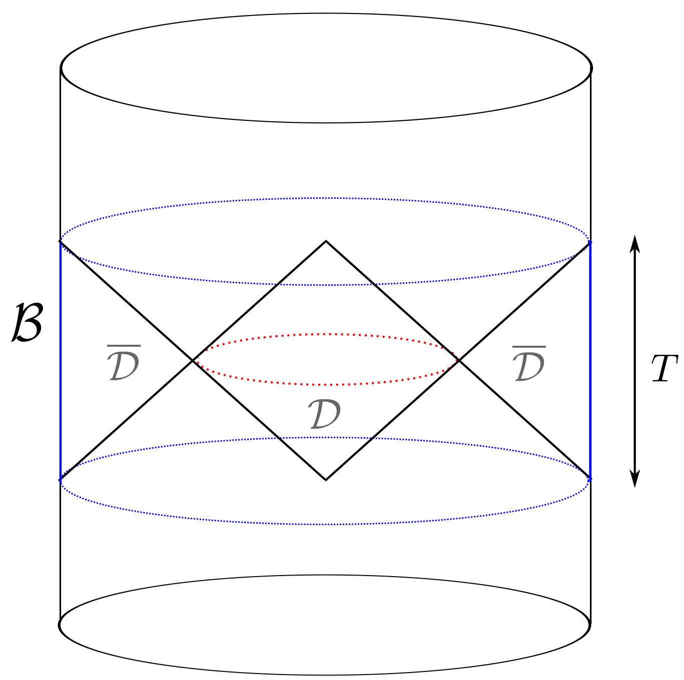
- Let us consider Hilbert space of EFT excitations on AdS. The bulk modes at large N can be reconstructed by acting with single trace operators in the time band of length π

$$\mathcal{H}_{\mathrm{EFT}} = \text{span of}\{\mathcal{O}_{\ell_1}(t_1)|0\rangle, \mathcal{O}_{\ell_2}(t_2)\mathcal{O}_{\ell_3}(t_3)|0\rangle, \ldots, \mathcal{O}_{\ell_4}(t_4)\mathcal{O}_{\ell_5}(t_5)\cdots\mathcal{O}_{\ell_{\mathcal{D}_m}}(t_{\mathcal{D}_m})|0\rangle\}$$

- However, remember, our time-band was shorter !

THE CLAIM

- First we will show that all effective field theory excitations, inside and outside of the diamond can be obtained from the full algebra of operators in the band.



- Excitations in $\overline{\mathcal{D}}$ can be related to $\mathcal{A}_{\text{small}}(\mathcal{B})$
- Excitations in $\mathcal{D}$ corresponds to "complicated" operators in the band.
- These complicated operators *approximately* commute with $\mathcal{A}_{\text{small}}(\mathcal{B})$ - approximate locality!

RECONSTRUCTING THE EXTERIOR

- The local bulk fields in the complimentary annular region, $\overline{\mathcal{D}}$ can be reconstructed from the known bulk - boundary dictionary (HKLL prescription).
- One can write $\phi(t,r,\Omega) = \sum_{n,\ell} \mathcal{O}_{n,\ell} e^{-i2\pi nt/T} \zeta_{n,\ell}(r) Y_\ell(\Omega) + \mathrm{h.c}$ with

$$\mathcal{O}_{n,\ell} = \frac{1}{T}\int_0^T dt \int d^{d-1}\Omega\, \mathcal{O}(\tau,\Omega) e^{i2\pi nt/T} Y_\ell^*(\Omega) d^{d-1}\Omega dt$$

- Normalisable boundary condition fixes the radial mode

$$\zeta_{n,\ell}(r) = \left(\frac{r^2}{1+r^2}\right)^{\frac{\omega_n}{2}} r^{-\Delta}\, {}_2F_1\left(\frac{1}{2}(2-d-\ell-\omega_n+\Delta), \frac{1}{2}(\ell-\omega_n+\Delta); 1-\frac{d}{2}+\Delta; -\frac{1}{r^2}\right)$$

$$\omega_n \equiv \frac{2\pi n}{T}$$

- CAVEAT: Not possible to write down the mapping explicitly in position space. The Kernel suffers from divergence for large angular momentum modes!
- The same technical issue arises in the case of AdS-Rindler wedge and AdS black hole (Bousso, Freivogel, Leichenauer, Rosenhaus, Zukowski, Soo-Jong Rey)
- However, it is possible to treat the Kernel as a distribution function (Morrison, Papadodimas, Raju)- gives well defined correlation function.
- There are few more subtleties in this construction (work in progress : SB, Papadodimas, Raju, Samantray). For the time being let us ignore those.

RECONSTRUCTING THE INTERIOR

- Modulo these technical issues, we can in principle relate the local operators in the exterior of the diamond to simple CFT operators in the time band.

- The most natural question is then : can the same be adopted to reconstruct the excitation in the interior of the diamond from the CFT in the band?
  - If we restrict ourselves to only “simple” operators, we need to go beyond the band, while, restricting ourselves inside the band amounts to deal with operators “complicated” in nature.

CFT VACUUM AS A CYCLIC VECTOR

- However, one can generate the entire Hilbert space of effective field theory, $\mathcal{H}_{\mathrm{EFT}}$ by acting with elements of $\mathcal{A}_{\mathrm{small}}$

$$\mathcal{H}_{\mathrm{EFT}} \doteq \mathcal{A}_{\mathrm{small}}|0\rangle$$

- Equivalent statement: the set of states obtained by action of the small algebra on the CFT vacuum is *dense* in $\mathcal{H}_{\mathrm{EFT}}$

The CFT vacuum state is a cyclic vector for this Hilbert space with respect to small algebra

- This is a version of Reeh-Schlieder theorem in a restricted time-band - meaningful only when we have a large N CFT

CFT VACUUM AS A CYCLIC VECTOR : PROOF

- We would establish this result the free field limit $N \to \infty$
- Consider a state of the form

$$|\Psi\rangle = \int_0^{\pi} dt\, g(t)\, \mathcal{O}_\ell(t)\, |0\rangle$$

- We claim that any state like this can be (arbitrarily) well approximated by the state of the form

$$X_f|0\rangle = \int_0^{T} dt f(t)\mathcal{O}_\ell(t)|0\rangle$$

- Notice the difference in support of the functions, f and g

CFT VACUUM AS A CYCLIC VECTOR : PROOF

- The proof follows reductio ad absurdum.
- If there exists a bulk state which cannot be well approximated by a state of the said form, there must exist a non-vanishing $|\psi\rangle$ which would be orthogonal to all states of the form

$$\int_0^T dt f(t)\mathcal{O}_\ell(t)|0\rangle$$

- This implies

$$R(t) \equiv \langle 0|\mathcal{O}_\ell(t)|\Psi\rangle = 0 \qquad \forall t \in [0,T]$$

- This function is analytic in the lower half plane. Furthermore, this is identically zero in the time band. Then by the edge of the wedge theorem it vanishes everywhere! $\implies \big|\Psi\big\rangle = 0$ as a vector.

ON THE CHOICE OF NORM

- The proof is somewhat counter-intuitive. Naively, a general function in $[0, \pi]$ cannot be well-approximated by a function in $[0, T]$ if we choose the usual L^2 norm.
- However, the main point here is that the inner product in the Hilbert space itself induces a bilinear norm on the function space. Given two states

$$|\Psi_1\rangle = \int_0^\pi dt_1 g_1(t_1)\mathcal{O}_\ell(t_1)|0\rangle, \quad |\Psi_2\rangle = \int_0^\pi dt_2 g_2(t_2)\mathcal{O}_\ell(t_2)|0\rangle$$

- one can define

$$\langle\Psi_1|\Psi_2\rangle = \int_0^\pi dt_1 \int_0^\pi dt_2 g_1^*(t_1)G_\ell(t_1-t_2)g_2(t_2)$$

$$\langle 0|\mathcal{O}_\ell(t_1)\mathcal{O}_{\ell'}(t_2)|0\rangle \equiv G_\ell(t_1-t_2)\delta_{\ell\ell'}$$

ON THE CHOICE OF NORM

- In Fourier space

$$G_\ell(\omega) = \int dt e^{i\omega t} G_\ell(t) = \sum_{n=0}^{\infty} G_{n,\ell}\delta(\omega - \Delta - 2n - \ell)$$

$$G_{n,\ell} = \frac{\Gamma\left(\Delta + n + \ell\right)\Gamma\left(\Delta + n + 1 - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(n+1\right)\Gamma\left(\Delta\right)\Gamma\left(\Delta + 1 - \frac{d}{2}\right)\Gamma\left(\frac{d}{2} + n + \ell\right)}$$

- Note $G_{n,\ell} = 0, \forall n < 0$
- So essentially we need to match only the positive Fourier coefficients of the function, g and not the entire function.
- This allows our claim to hold!

CFT VACUUM AS A SEPARATING VECTOR

- Another key property that one can prove is that for we cannot *exactly* annihilate the vacuum by acting with simple operators in the band, i.e, elements of the algebra $\mathcal{A}_{\rm small}$

Vacuum is separating vector

- The proof is similar, uses the analyticity property of the norm of the state and the edge of the wedge theorem.
- Another related statement is that although one can approximate any EFT state *arbitrarily well* from the small algebra of simple operators in the band in CFT, an exact reproduction of the state is *not possible*!

CFT VACUUM AS A SEPARATING VECTOR : IMPLICATIONS

- The separating nature of vacuum is a statement about "entanglement" - it says the state in CFT is entangled relative to $\mathcal{A}_{\text{small}}$
- This further implies that the representation of $\mathcal{A}_{\text{small}}$ in $\mathcal{H}_{\text{EFT}}$ is reducible and has a non-trivial commutant.
- This commutant (roughly) corresponds to the operators in the interior of the diamond.
- Algebraically, it is possible to define a natural modular Hamiltonian for the time-band on the boundary using Tomita-Takesaki modular theory.

TOMITA–TAKESAKI MODULAR THEORY

- Define an anti-linear map

$$S\mathcal{A}|0\rangle = \mathcal{A}^\dagger|0\rangle$$

- with $S^2 = 1$
- Consider an operator $\tilde{\mathcal{A}} = S\mathcal{A}S^\dagger$
- This operators commutes with all elements $\mathcal{A}$ of the small algebra, however, not canonically normalised.
- Tomita-Takesaki : Modular Hamiltonian can be formally written as

$$H_{mod} = \log\left(S^\dagger S\right)$$

BULK ENTANGLEMENT AND MODULAR HAMILTONIAN

- Physically, from the bulk perspective, it translates into an entanglement between the interior of the diamond, $\mathcal{D}$ and its compliment, $\overline{\mathcal{D}}$
- A natural question is : what is the modular Hamiltonian in the bulk?
- Not very easy to find because in this case the horizon is not generated by Killing isometry. However, we expect that in the large N limit the modular Hamiltonian is quadratic and one can in principle expect to write a formal expression for it (in progress).

COMPLEMENTARITY

Realised

INTERIOR OPERATORS AND PRECURSORS

- Can an operator in the interior of the diamond be expressed in terms of operators confined inside the time-band?
- Bulk EFT : CFT operators inside the time-band must commute with the operators inside the diamond to preserve locality (upto Gauss law tail).
- Time-slice axiom : if the CFT has operators representing the excitation in the interior of the diamond, these should as well be present in the time band.
- Puzzle?

INTERIOR OPERATORS AND PRECURSORS

- Not actually a puzzle : EFT is only “effective” !
- Resolution : The bulk locality is an emergent concept - the interior operators can be represented by complicated operators in the time-band. They commute with the simple operators in the band.
- However, if we go beyond the set of simple operators, it is indeed possible to construct bulk operators inside the diamond (precursors) directly from the operators in boundary time-band.
- If so, what is the minimal set we can include?

INTERIOR OPERATORS AND PRECURSORS

- For our case we find that the only extra operator we require outside the band is

$$P_0 = |0\rangle\langle 0| = \lim_{\alpha \to \infty} e^{-\alpha H}$$

- However it turns out to be sufficient to only consider approximation of the form

$$\mathcal{P}_{\alpha,p_c} = \sum_{p=0}^{p_c} \frac{(-1)^p (\alpha H)^p}{p!}$$

  - Typically, we need, for a good approximation,

$$\alpha = \ln(N) \qquad\qquad p_c = N\ln(N)$$

  - NO choice of cut off within our simple algebra!

INTERIOR OPERATORS AND PRECURSORS - IN 3 STEPS

- Step 1 : Bulk field at any point in AdS

$$\phi(t,r,\Omega) = \sum_n c_{n,\ell}\mathcal{O}_{n,\ell}e^{-i(2n+\ell+\Delta)t}Y_\ell(\Omega)\chi_{n,\ell}(r) + \text{h.c}$$

- Step 2: Each of these modes has a natural SHO form

$$\mathcal{O}_{n,\ell} = \sum_{\{p_{n_j,\ell_j}\}} \sqrt{p_{n,\ell}G_{n,\ell}}|p_{0,0}\ldots p_{n,\ell}-1\ldots\ldots\rangle\langle p_{0,0},\ldots p_{n,\ell}\ldots|$$

$$|p_{n_1,\ell_1}\ldots p_{n_j,\ell_j}\ldots\rangle = \prod_{j=0}^{\mathcal{D}_m}\left(\Gamma(p_{n_j,\ell_j}+1)G_{n_j,\ell_j}\right)^{-\frac{1}{2}}\left(\mathcal{O}_{n_j,\ell_j}\right)^{p_{n_j,\ell_j}}|0\rangle$$

- Step 3: (Known already) Each of the states can be *arbitrarily well approximated* by operators $X \in \mathcal{A}_{\text{small}}$

INTERIOR OPERATORS AND PRECURSORS : 1+2+3

- Putting all the ingredients together, we have our final formula for the precursor :

$$\boxed{\begin{aligned}\phi(t,r,\Omega) \doteq \sum_{n,\ell}\sum_{\{p_{n_j,\ell_j}\}} \sqrt{p_{n,\ell}G_{n,\ell}}\,X[p_{0,0}\ldots p_{n,\ell}-1\ldots]P_0X[p_{0,0},\ldots p_{n,\ell}\ldots]^\dagger \\ \times c_{n,\ell}\chi_{n,\ell}(r)e^{-i(\Delta+2n+\ell)t}Y_\ell(\Omega) + \text{h.c.}\end{aligned}}$$

SUMMARY

- “Complementarity” discussed in the context of black hole information paradox, is basically an artefact of a large-scale non-locality in quantum gravity.
- We showed explicitly that this phenomenon is evident in space-time where there is no black hole. We set up a calculable toy model in empty AdS to demonstrate this explicitly.
- Also discussed resolution of the puzzle regarding “emergent locality”in effective field theory. We argued that the correct way to understand this is to distinguish between simple and complicated operators in the time-band.

- We showed that while the region near the boundary is reconstructed by simple operators, to reconstruct the region near the centre, we do require complicated ones.
- The two sets commute *approximately* in simple experiments, however, the commutator is NOT zero as an operator equation.
- To demonstrate the very feature of quantum gravity we explicitly construct the “precursor”, i.e a specific “complicated” operator that probes the “centre of AdS”.

THANK YOU