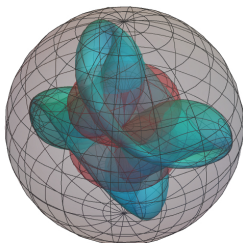


BMS Vacua and Black Holes

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Inward Bound

Conference on Black Holes and Emergent Spacetime
Nordita, Stockholm, 17-20 August 2016

Talk based on

- “Vacua of the gravitational field”,
G.Compère, Jiang Long, JHEP, arXiv :1601.04958
- “Classical static final state of collapse with
supertranslation memory,
G.Compère, Jiang Long, CQG, arXiv :1602.05197
- “Symplectic and Killing symmetries of AdS_3 gravity :
holographic vs boundary gravitons”,
G.Compère, Pu-Jian Mao, A. Seraj, M.M. Sheikh-Jabbari,
JHEP, arXiv :1511.06079

Intuition : AdS_3 (Einstein gravity)

Spectrum (Lorentzian signature) :

- AdS_3 , conical defects/excesses and the BTZ black holes
- Holographic gravitons \leftrightarrow Boundary stress-tensor

AdS_3 /Defects/BTZ with Virasoro hair :

$$ds^2 = \ell^2 \frac{dr^2}{r^2} - \left(r dx^+ - \ell^2 \frac{L_-(x^-) dx^-}{r} \right) \left(r dx^- - \ell^2 \frac{L_+(x^+) dx^+}{r} \right).$$

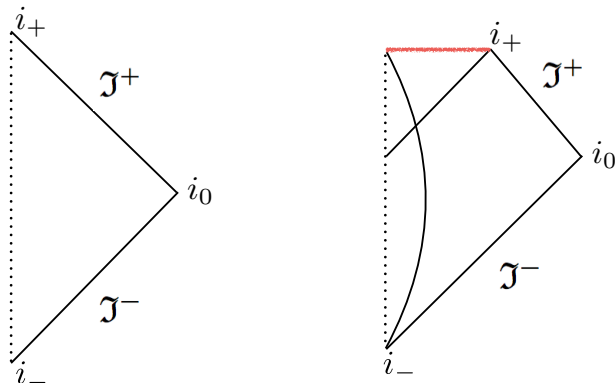
Two not so well-known properties :

- Two Virasoro charges $\mathcal{L}_n, \bar{\mathcal{L}}_n$ are conserved everywhere in the bulk (“symplectic symmetry”)
- AdS_3 with Virasoro hair admits a source for the Virasoro charges

Main objective : Describe Minkowski and Schwarzschild with BMS (soft) hair.

The essentials on the BMS group

The BMS group is the Asymptotic Symmetry Group of $4d$ Asymptotically flat spacetimes at past and future null infinity.



The BMS group contains the Poincaré group as a subgroup. It contains an infinite set of commuting *supertranslations*.

[Bondi, van der Burg, Metzner, 1962] [Sachs, 1962]

The BMS group is the group of diffeomorphisms which

- preserve physically motivated boundary conditions written in a convenient gauge
- are associated with non-trivial canonical charges

Physically motivated boundary conditions mean :

- admitting the Kerr metric and gravitational waves
- with finite and well-defined canonical charges
- with positive energy
- allowing to describe memory effects [Zeldovich, Polnarev, 1974]
[Christodoulou, 1991]
- allowing for small perturbations to decay (non-linear stability) [Christodoulou, Klainerman, 1993]
- allowing to describe a semi-classical S-matrix which obeys all known theorems [Weinberg, 1965] [Cachazo, Strominger, 2014]

The BMS algebra and its representation

$$BMS \simeq so(3, 1) \oplus \text{Supertranslations}$$

The supertranslation symmetry vectors read as

$$T(\theta, \phi)\partial_u + \dots$$

The lowest spherical harmonics of $T(\theta, \phi)$ correspond to translations, e.g. around Minkowski

$$\partial_z = -\cos\theta\partial_u + \cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta.$$

The higher spherical harmonics lead to supertranslations.

BMS invariance is spontaneously broken. The boundary conditions lead to an **order parameter : the supertranslation field** $C(\theta, \phi)$ which shifts under supertranslations as

$$\delta_T C(\theta, \phi) = T(\theta, \phi).$$

The memory effects

The constraints of Einstein's equations imply at \mathfrak{I}^+ : [\[Frauendiener, 1992\]](#) [\[Geroch, Winicour, 1981\]](#) [\[Strominger, Zhiboedov, 2014\]](#)

$$-\frac{1}{4}\nabla^2(\nabla^2 + 2)(C|_{u_2} - C|_{u_1}) = m|_{u_2} - m|_{u_1} + \int_{u_2}^{u_1} du T_{uu},$$
$$T_{uu} \equiv \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r \rightarrow \infty} [r^2 T_{uu}^{matter}].$$

The supertranslation field $C(\theta, \phi)$ is effectively shifted by a supertranslation after the passage of radiation.

This conservation law leads to the memory effects : the geodesic deviation vector s^A , $A = \theta, \phi$ gets shifted

$$s^A|_{u_2} - s^A|_{u_1} \sim \frac{1}{r} \partial^A \partial_B (C|_{u_2} - C|_{u_1}) s^B$$

and leads to a finite relative displacement and a finite time shift of inertial detectors. [\[Zeldovich, Polnarev, 1974\]](#) [\[Christodoulou, 1991\]](#).

The extended BMS algebra

[Barnich, Troessaert, 2010]

$$\text{Ext BMS} \simeq \text{Superrotations}^* \oplus \text{Supertranslations}^*$$

where

$$\text{Superrotations}^* \simeq \text{Vir}^* \oplus \text{Vir}^*,$$

$$\text{Supertranslations}^* \simeq \text{Regular supert.} \oplus \text{Meromorphic supert.}$$

The Lorentz subalgebra

$$so(3, 1) \simeq sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}) \subset \text{Vir}^* \oplus \text{Vir}^*$$

is generated by global conformal transformations on the sphere. The rest of the algebra has generators which contain meromorphic functions, $\delta z = R^z(z)$, with poles on S^2 .

The extended BMS algebra : comments

The algebra is not realized as asymptotic symmetry algebra, at least in the standard sense :

- The Kerr black hole has infinite meromorphic supertranslation charges. [Barnich, Troessaert, 2010]
- Minkowski with superrotation hair has negative energy. [G.C., Long, 2016]

The superrotations still have a role to play :

- Superrotation charges are finite and can be non-trivial [Barnich, Troessaert, 2011] [Flanagan, Nichols, 2015] [G.C., Long, 2016]
- The subleading soft graviton theorem has been related to the Ward identity of the superrotation algebra [Kapec, Lysov, Pasterski, Strominger, 2014] [Campiglia, Laddha, 2015]

Building BMS vacua/black holes

Algorithm :

- Start with Minkowski/Schwarzschild spacetime.
- Write a generic change of coordinates $x^\mu \rightarrow x'^\mu$ which exponentiates the infinitesimal change of coordinates $x'^\mu = x^\mu + \xi^\mu + \dots$ where ξ^μ is a generic supertranslation + superrotation vector field at \mathfrak{I}^+
- Solve for the finite change of coordinates at each order in the asymptotic radial expansion such that $g_{\mu\nu} = \frac{\partial x^\beta}{\partial x'^\mu} \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\nu}$ fits in BMS gauge.
- Resum the infinite radial expansion.
- Rewrite the final metric in closed and beautiful form.

The result is the BMS orbit of Minkowski/Schwarzschild spacetime. It is the representation of the BMS group on the bulk metric.

The Poincaré vacua of Einstein gravity

The vacuum metric with supertranslation field only reads in BMS coordinates and static coordinates as

$$\begin{aligned} ds^2 &= -du^2 - 2dud(\sqrt{r^2 + U} + \frac{1}{2}(D^2 + 2)C) \\ &\quad + \left((r^2 + 2U)\gamma_{AB} + \sqrt{r^2 + U}C_{AB} \right) d\theta^A d\theta^B \\ &= -dt^2 + d\rho^2 + (\rho\gamma_{AC} - D_A D_C C - \gamma_{AC} C)\gamma^{CD} \times \\ &\quad (\rho\gamma_{DB} - D_D D_B C - \gamma_{DB} C) d\theta^A d\theta^B \end{aligned}$$

where $\theta^A = \theta, \phi$, $u = t - \rho$.

Under a supertranslation,

$$\delta_T C(\theta, \phi) = T(\theta, \phi).$$

It admits 10 Killing vectors. We checked that the 10 Poincaré charges are zero \Rightarrow Poincaré vacua.

All supertranslation charges are zero.

The Poincaré vacua of Einstein gravity

The vacua are non-trivial in the sense that they admit canonical charges : superrotation charges

$$\mathcal{Q}_R = -\frac{1}{4G} \int_S d^2\Omega R^A \left(\frac{1}{8} D_A (C_{EF} C^{EF}) + \frac{1}{2} C_{AB} D_E C^{EB} \right)$$

where $C_{AB} = -2D_A D_B C + \gamma_{AB} D^2 C$. [Barnich, Troessaert, 2011]

Even though the superrotation transformation $\delta z = R^z(z)$, $\delta \bar{z} = R^{\bar{z}}(\bar{z})$ admit poles, the superrotation charges are finite.

There is therefore an obstruction in the bulk at shrinking the surface of integration. Superrotation charge conservation in the bulk (symplectic symmetry) \Rightarrow Bulk source

Maybe our universe is patched with such vacua?

How to characterize the bulk sources?

Supertranslation horizon

The static coordinates (t, ρ, θ, ϕ) of the vacua

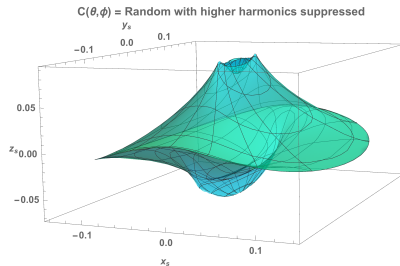
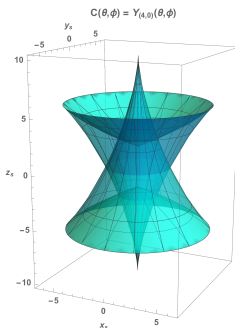
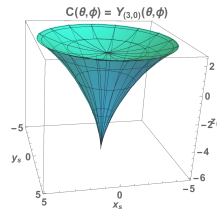
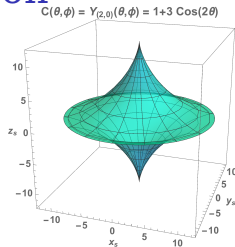
$$ds^2 = -dt^2 + d\rho^2 + g_{AB}d\theta^A d\theta^B,$$

break down where $\text{Det}(g_{AB}) = 0$.

This location $\rho = \rho_{SH}(\theta, \phi)$ defines the *supertranslation horizon*.

The source for superrotation should be at or beyond the supertranslation horizon.

Isometric embedding of the supertranslation horizon



Finite supertranslation diffeomorphism

$$ds^2 = -dt_s^2 + d\rho_s^2 + \gamma_{AB}d\theta_s^A d\theta_s^B = -dt^2 + d\rho^2 + g_{AB}d\theta^A d\theta^B$$

The finite diffeomorphism from Minkowski to the vacua is

$$\begin{aligned}t_s &= t + C_{(0,0)}, & (z \equiv \cot \frac{\theta}{2} e^{i\phi}) \\ \rho_s &= \sqrt{(\rho - C + C_{(0,0)})^2 + D_A C D^A C}, & (\text{Pythagoras' rule}) \\ z_s &= \frac{(z - \bar{z}^{-1})(\rho - C + C_{(0,0)}) + (z + \bar{z}^{-1})(\rho_s - z\partial_z C - \bar{z}\partial_{\bar{z}} C)}{2(\rho - C + C_{(0,0)}) + (1 + z\bar{z})(\bar{z}\partial_{\bar{z}} C - \bar{z}^{-1}\partial_z C)}.\end{aligned}$$

When C is a combination of the 4 lowest spherical harmonics, it is the change between spherical coordinates to other spherical coordinates at a translated origin.

Supertranslation diffeomorphisms are generalization of “spatially translating the origin of coordinates”.

Vacua with superrotation field

The metric with supertranslation and superrotation fields can also be constructed from a combined finite supertranslation and superrotation diffeomorphism from Minkowski with in particular $z \rightarrow G(z) + O(r^{-1})$.

The Bondi mass decreases with retarded time u ,

$$\partial_u M = -\frac{1}{8} T^{AB} T_{AB}$$

where the traceless, divergence-free tensor is

$$T_{zz} = \frac{\partial_z^3 G}{\partial_z G} - \frac{3(\partial_z^2 G)^2}{2(\partial_z G)^2}$$

\Rightarrow Unbounded negative energy.

\Rightarrow Discard by imposing the Dirichlet boundary condition $T_{zz} = 0$.

\Rightarrow Only Lorentz transformations are asymptotic symmetries.

Schwarzschild with BMS hair

The final static ($J = 0$) metric after spherical gravitational collapse, if analytic, is diffeomorphic to the Schwarzschild metric. [No hair theorems]

[Carter, Hawking, Robinson, 1971-1975] [Chrusciel, Costa, 2008] [Alexakis, Ionescu, Klainerman, 2009]

A loophole of no hair theorems is that the diffeomorphism might be singular inside the event horizon, so the black hole can carry superrotation charges which characterize the classical vacuum.

Proposal : there might be a non-trivial supertranslation field $C(\theta, \phi)$ at the end of the collapse.

Two questions :

- What is the metric $g_{\mu\nu}(M, C(\theta, \phi))$?
- How does the supertranslation field $C(\theta, \phi)$ compares to the final mass M ?

The Schwarzschild metric

It admits Weyl conformally flat sections. This is manifest in isotropic coordinates $(t, \rho_s, \theta_s, \phi_s)$:

$$ds^2 = -\frac{\left(1 - \frac{M}{2\rho_s}\right)^2}{\left(1 + \frac{M}{2\rho_s}\right)^2} dt^2 + \left(1 + \frac{M}{2\rho_s}\right)^4 \left(d\rho_s^2 + \gamma_{AB} d\theta^A d\theta^B\right)$$

where

$$\gamma_{AB} d\theta^A d\theta^B = d\theta_s^2 + \sin^2 \theta_s d\phi_s^2,$$

$$\rho_s = \infty \text{ at spatial infinity}$$

$$\rho_s = \frac{M}{2} \text{ at the event horizon}$$

The Schwarzschild metric embedded in the BMS supertranslation vacuum

$$ds^2 = -\frac{\left(1 - \frac{M}{2\rho_s}\right)^2}{\left(1 + \frac{M}{2\rho_s}\right)^2} dt^2 + \left(1 + \frac{M}{2\rho_s}\right)^4 \left(d\rho^2 + g_{AB}d\theta^A d\theta^B\right)$$

where

$$\begin{aligned} g_{AB} &= (\rho\gamma_{AC} - D_A D_C C - \gamma_{AC} C)\gamma^{CD}(\rho\gamma_{DB} - D_D D_B C - \gamma_{DB} C) \\ \rho_s^2 &= (\rho - C)^2 + D_A C D^A C \end{aligned}$$

Remarks :

- When $C = 0$, this is Schwarzschild
- Obtained by finite supertranslation diffeomorphism
- The non-trivial Poincaré charges are just the energy M
- There are superrotation charges quadratic in C

A bound from weak cosmic censorship

This is a competition between the supertranslation horizon and the infinite redshift surface.

$$g_{tt} \sim \left(1 - \frac{M}{2\rho_s}\right)^2, \quad \rho_s^2 = (\rho - C)^2 + D_A C D^A C.$$

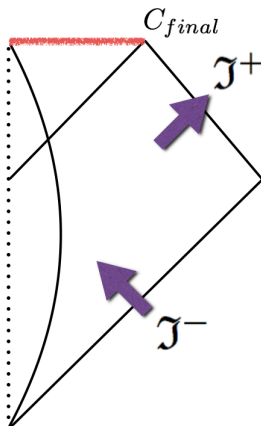
- When $C \ll M$, this is a slightly distorted Schwarzschild black hole
- When $D_A C D^A C > \frac{M^2}{4}$, there is no infinite redshift surface in the coordinate patch.

We argued that weak cosmic censorship requires the bound

$$D_A C D^A C \leq \frac{M^2}{4}.$$

How much supertranslation hair?

What is the final value of $C(\theta, \phi)$?



It depends upon the fluxes and Bondi mass at \mathfrak{I}^+ and \mathfrak{I}^- and boundary conditions at spatial infinity.

How much supertranslation hair?

Assuming boundary conditions on radiation [Christodoulou, Klainerman, 1993], Einstein's equations give

$$\begin{aligned} & -\frac{1}{4}\nabla^2(\nabla^2 + 2)(C|_{final}(\theta, \phi) - \Delta C_\infty - C|_{in}(\pi - \theta, \phi + \pi)) \\ & = m|_{final} - m|_{in} + \int_{-\infty}^{+\infty} du T_{uu}(\theta, \phi) - \int_{-\infty}^{+\infty} dv T_{vv}(\pi - \theta, \phi + \pi) \end{aligned}$$

This is the global angle-dependent energy conservation law for asymptotically flat spacetimes. [Geroch, Winicour, 1980] [Strominger, Zhiboedov, 2014]

Assuming $\Delta C_\infty = 0$, spherically symmetric collapse of a null shell $\Rightarrow C|_{final} = 0$ (metric described by Vaidya metric).

How much supertranslation hair ?

Non-spherically symmetric collapse of a null shell is constrained by the null energy condition

$$T_{vv}(\theta, \phi) \geq 0.$$

Assuming all matter is incoming at $v = 0$,

$$T_{vv} = \left(\frac{M + M \sum P_{l,m} Y_{l,m}(\theta, \phi)}{4\pi r^2} + O(r^{-3}) \right) \delta(v)$$

we get the constraint

$$\sum P_{l,m} Y_{l,m}(\theta, \phi) \geq -1.$$

How much supertranslation hair?

In the ideal case (no outgoing radiation, no initial mass, only ingoing collapsing radiation) and assuming $\Delta C_\infty = 0$, the solution to the global energy conservation law is

$$C(\theta, \phi) = M \sum_{\ell \geq 2, m} \frac{4(-1)^\ell}{(\ell-1)\ell(\ell+1)(\ell+2)} P_{\ell, m} Y_{\ell, m}(\theta, \phi) = O(M^1)$$

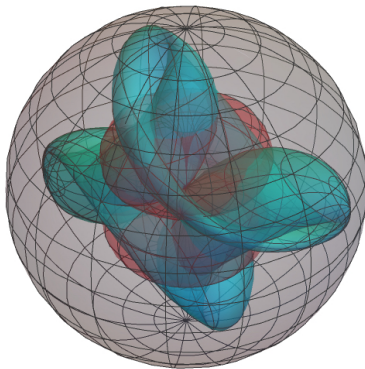
with the constraint

$$\sum P_{\ell, m} Y_{\ell, m}(\theta, \phi) \geq -1.$$

This leads to the bound (checked numerically) :

$$D_A C D^A C \leq \frac{M^2}{4}.$$

The event horizon contains the
supertranslation horizon



Summary

- We constructed the supertranslation orbit of Minkowski and identified the metric as Poincaré vacua with a source for superrotation charge which is hidden at/behind the supertranslation horizon.
- Superrotations do not belong to the asymptotic symmetry group since they would lead to unbounded negative energy but superrotation charges are well-defined.
- In the center-of-mass frame, supertranslations are spatial, except the zero mode (i.e. time translation).
- We proposed a new final state of collapse : the Schwarzschild black hole with supertranslation hair. The hair is soft, non-linear, and $O(\hbar^0)$. With one caveat (the boundary conditions at spatial infinity), the final hair is computable from past history of evolution and collapse and is $O(M)$.
- Much physics and maths remains to be understood.