Renormalisation of entanglement entropy

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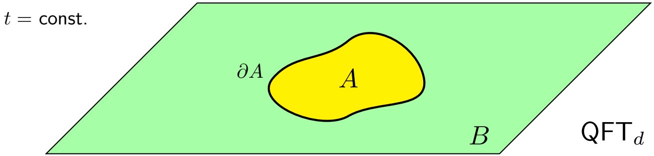
August 17, 2016

Introduction

- This talk will be about defining renormalised entanglement entropy, both holographically and in quantum field theory.

Introduction: Entanglement entropy



- Consider a spatial region A and a density matrix ρ.
- Define ρ_A as the reduced matrix obtained by tracing out all degrees of freedom outside region A.
- The associated von Neumann entropy is the entanglement entropy i.e. $S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$.

Properties of entanglement entropy

- Complementarity: S_A is equal to the entanglement entropy of the complement, S_B.
- UV divergences: in D spatial dimensions the leading UV divergence behaves as

$$S_A \sim \frac{\mathrm{Area}_{\partial A}}{\epsilon^{D-1}} + \cdots$$

where $\epsilon \ll 1$ is the UV cutoff. (Logarithmic in $D = 1$.)

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Regulator dependence

From a field theorist's perspective, strange to work with a regulated quantity!

- Non-universal divergences: power law divergences dependent on regularisation scheme (not seen with zeta function approach etc).
- Universal divergences: logarithmic divergences, related to conformal anomalies in CFTs.

Entanglement entropy in CMT

- Intrinsic UV cutoff: lattice spacing a.
- E.g. for ground state of quantum critical system described by 2d CFT

$$S_A = \frac{c}{3} \ln \left(\frac{l}{a} \right) + c'$$

  with c central charge, l length of interval and a lattice spacing.
- Usual to relate QFT computations to explicit calculations using eigenvalues of ρ_A:

$$S_A = - \sum_i \lambda_i \ln \lambda_i$$

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Entanglement entropy in QFT

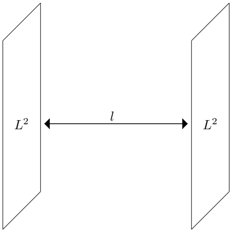
In a QFT, we usually define regulate divergences, introduce covariant counterterms and then renormalize by removing the regulator ($\epsilon \to 0$)..... can we do this for EE?

Reasons to define renormalized EE in QFT:

1. Finite part of EE is related to F quantity in odd-dimensional CFT.
2. Use of EE as an order parameter for phase transitions
3. Black hole physics (see Strominger's talk)

Previous approaches

Based on differentiating with respect to parameters:



- For a slab domain in a local QFT, divergences in S must be independent of l.
- Therefore

$$S_l \equiv \frac{\partial S}{\partial l}$$

is UV finite.

(e.g. Cardy and Calabrese; Casini and Huerta)

Geometry dependence

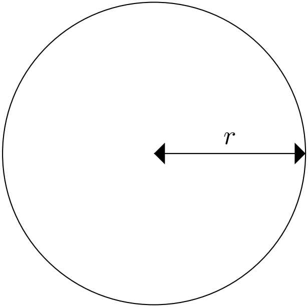
- For a spherical region of radius r, divergences in S depend on r.
- For a 3d CFT (disk region) since

$$S \sim \frac{r}{\epsilon} + \text{finite}$$

the combination

$$S(r) = \left(r \frac{\partial S}{\partial r} - S \right)$$

is UV finite. (Liu and Mezei)



Limitations of such approaches

Current interest in dependence of entanglement entropy on shape and theory but:

- No definition for generic shape entangling region.
- Relation to usual QFT renormalization is unclear.
- Renormalization scheme dependence is obscure.

Background subtraction versus renormalization

- Can also obtain finite result by subtracting reference background:
$$\Delta S = S_A - S_A^{\mathrm{ref}},$$
see Strominger for flat space example.
- Background subtraction is not renormalization in the usual QFT sense: counterterms, scheme dependence etc remain unclear.

References

- Marika Taylor and William Woodhead
  1. Renormalized entanglement entropy, 1604.06808
  2. The holographic F theorem, 1604.06809
  3. Renormalization of entanglement entropy in QFT, 1609.xxxxx
- Peter Jones and Marika Taylor
  1. Holographic renormalization of EE for non-conformal branes and asymptotically flat spacetimes, in progress.

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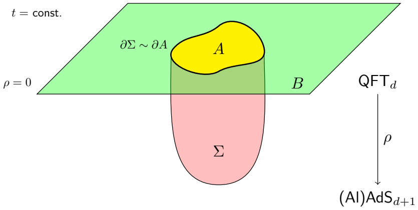
Outline

- **Holographic renormalization of entanglement entropy**
- General approach to renormalization

Holographic entanglement entropy

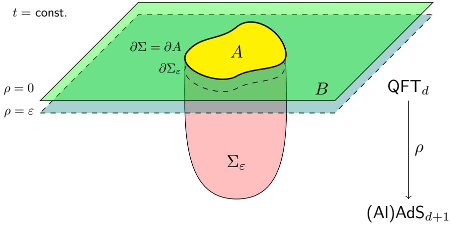
Entanglement entropy can be computed geometrically for field theories admitting a gravity dual in one higher dimension.



- Holographic Ryu-Takayanagi (RT) prescription: area of codimension two minimal surface homologous to A

$$S_A = \frac{\mathcal{A}}{4G}$$

Area renormalization



- The natural UV cutoff is $\rho = \epsilon \ll 1$.
- One can regulate the area of the minimal surface and define a renormalized area using appropriately covariant counterterms.

Earlier work on renormalized minimal surfaces:
(Henningson/Skenderis; Graham/Witten; Gross et al)

Renormalized entanglement entropy

- The Ryu-Takayanagi functional is

$$S = \frac{1}{4G} \int_{\Sigma} d^{d-1}x \sqrt{\gamma}$$

- Use the equations for the minimal surface to expand the surface area asymptotically near the conformal boundary and regulate divergences.
- Covariant counterterms are

$$S_{\rm ct} \sim \int_{\partial\Sigma} d^{d-2}x \sqrt{h} \mathcal{L}(\mathcal{R}, \mathcal{K})$$

where $\mathcal{K}$ is the extrinsic curvature of $\partial\Sigma$ into $\rho = \epsilon$.

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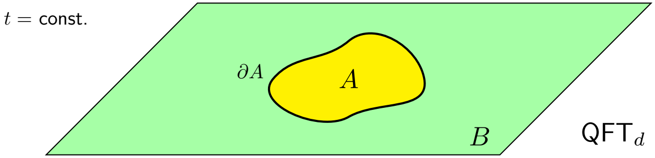
Extrinsic curvature of entangling region



- Counterterms can depend on intrinsic and extrinsic curvature of ∂A.
- Complementarity: for A and B to have the same renormalized entanglement entropy, we can include only terms which are even in the extrinsic curvature.

Results for 3d CFT

- The renormalized EE for an entangling surface in AdS_4 is

$$S_{\rm ren} = \frac{1}{4G}\int_{\Sigma} d^2x\sqrt{\gamma} - \frac{1}{4G}\int_{\partial\Sigma} dx\sqrt{h}(1 - c_s\mathcal{K})$$

  with $\partial\sigma$ the boundary of the minimal surface.
- Here $\mathcal{K}$ is the extrinsic curvature of the bounding curve.
- Complementarity implies that $c_s = 0$ (finite counterterm fixed to be zero).

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Disk entangling region

- Consider an entangling region which is a disk of radius r.

$$S_{\rm ren} = -\frac{\pi}{2G},$$

where G is dimensionless.
- This EE is related to the free energy on the S^3, the F quantity, by the CHM map: $S_{\rm ren} = -F$.
- Positivity of F implies negativity of $S_{\rm ren}$.

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Matching holographic renormalization schemes

- The renormalized onshell action for Euclidean AdS_4 indeed gives
$$F = \frac{\pi}{2G} = -S_{\rm ren}.$$
- Onshell action calculated using counterterms for $AlAdS_4$ manifolds (de Haro et al)
$$I_{\rm ct} = \frac{1}{8\pi G} \int d^3x \sqrt{-g}(K + 2 - \frac{R_g}{2})$$
There are no possible finite counterterms.

Generalizations of holographic procedure

Can generalize area renormalization of entangling surface to:

- RG flows
- Time dependent situations (HRT functional)
- Non AdS holography

Holographic RG flows

A holographic RG flow is described by:

- A "domain wall" geometry

$$ds^2 = dr^2 + \exp(2A(r))dx^\mu dx_\mu$$

- A set of scalar field profiles

$$\phi_a(r)$$

- First order equations of motion relating $A(r)$ and $\phi_a(r)$.

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RG flow of 3d field theory

Consider four dimensional bulk ($d = 3$), single scalar ϕ.

- Assume UV conformal, so potential can be expanded near boundary as

$$V = 6 - \sum_{n=2}^{\infty} \frac{\lambda_{(n)}}{n!} \phi^n$$

  with $\lambda_{(2)} = M^2 = \Delta(\Delta - 3)$.

- First order form of equations

$$\dot{A} = W \qquad \dot{\phi} = -2\partial_\phi W$$

  where V is a known expression quadratic in (fake) superpotential W.

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REE for relevant deformations

- We need the following counterterms in the REE:

$$S_{\rm ct} = -\frac{1}{4G}\int dx\sqrt{h}\left(1 + \frac{(3-\Delta)}{8(5-2\Delta)}\phi^2 + \cdots\right)$$

  where second term is needed for $\Delta > 5/2$.
- The counterterms can be expressed in terms of the superpotential

$$S_{\rm ct} = -\frac{1}{4G}\int dx\sqrt{h}Y(\phi)$$

  where

$$W(\phi)Y(\phi) + \frac{dW}{d\phi}\frac{dY}{d\phi} = 1.$$

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REE for RG flows

Can explore REE along RG flows and its relation to F quantity along RG flows (on spheres)....

Outline

- Holographic renormalization of entanglement entropy
- **General approach to renormalization**

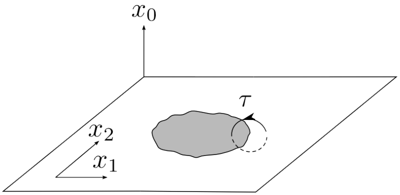
Definition of REE

The holographic area renormalization of minimal surfaces looks hard to connect with QFT renormalization....

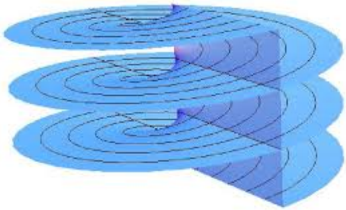
Replica trick

- Entanglement entropy is often computed using the replica trick:
$$S = -n\partial_n \left[\log Z(n) - n \log Z(1)\right]_{n=1}$$
where $Z(1)$ is the usual partition function and $Z(n)$ is the partition function on the replica space in which a circle coordinate has periodicity $2\pi n$.
- Holographically $\log Z(n)$ is computed by the renormalised onshell action $I(n)$ for a geometry with a conical singularity. (Lewkowycz and Maldacena)

Replica trick



3d field theory: on replica space τ has periodicity $2\pi n$.



Visualisation of $n = 3$ replica space.

Lewkowycz-Maldacena derivation

- Bulk term in onshell action is

$$I(n) = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{g} R_n$$

- Working perturbatively in $(n-1)$, the Ricci scalar is (Solodukhin)

$$R_n = R + 4\pi(n-1)\delta_\Sigma + \cdots$$

where δ_Σ is localised on the codimension two conical singularity.

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Lewkowycz-Maldacena derivation

- Leading contributions cancel in replica formula, leaving

$$S = \frac{\partial_n (n-1)_{n=1}}{4G} \int d^{d+1}x \sqrt{g} \delta_\Sigma = \frac{1}{4G} \int_\Sigma d^{d-1}x \sqrt{\gamma},$$

i.e. the Ryu-Takayanagi formula.

The renormalized entanglement entropy can be calculated using the replica trick on the renormalized onshell action.

Holographic renormalization

- For example, for an asymptotically locally AdS_4 spacetime the action counterterms are

$$I_{\rm ct} = \frac{1}{8\pi G} \int d^3x \sqrt{g} (K + 2 - \frac{\mathcal{R}}{2})$$

with $\mathcal{R}$ the curvature of the boundary metric.

- For the replica space

$$\mathcal{R}_n = \mathcal{R} + 4\pi(n-1)\delta_{\partial\Sigma} + \mathcal{O}(n-1)^2$$

where $\partial\Sigma$ is the boundary of the entangling surface.

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Holographic renormalization

- Applying the replica formula to the counterterms then leads to exactly

$$S_{\rm ct} = -\frac{1}{4G}\int_{\partial\Sigma} dx\sqrt{\gamma}$$

i.e. our counterterm localized on the boundary of the entangling surface.

Holographic renormalization

- Procedure works for (AdS) Einstein gravity in any dimension:

$$S_{\rm ct} = -\frac{1}{4(D-1)G}\int_{\partial\Sigma} d^{D-1}x\sqrt{\gamma}\left(1 + \frac{1}{(D-1)(D-3)}\mathcal{K}^2 + \cdots\right)$$

  as well as theories with matter and higher derivative theories such as Gauss-Bonnet.

- Direct matching of renormalization scheme for EE with that of action!

General definition using replica method

- We can **define** renormalized entanglement entropy as:

$$S_{\rm ren} = -n\partial_n \left[\log Z_{\rm ren}(n) - n \log Z_{\rm ren}(1)\right]_{n=1}$$

where the partition functions are renormalized by **any** appropriate method.

QFT renormalization

- Holographic regulator corresponds to explicit cutoff:

$$\log Z \sim a_0 \frac{V_d}{\epsilon^d} + \frac{a_2}{\epsilon^{d-2}} \int \sqrt{g} d^d x \mathcal{R} + \cdots$$

  with EE counterterms inherited from curvature terms.
- Zeta functions, dimensional regularisation etc have different "non-universal" divergent terms and hence counterterms.
- Scheme dependence inherited from partition function.

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Weyl transformations of REE

- Under a Weyl transformation $\delta g_{ij} = 2\sigma g_{ij}$ the CFT partition function transforms as

$$\delta(\log Z) = \sigma \int d^d x \sqrt{g} \langle T^i_i \rangle$$

i.e. in terms of the conformal anomaly $\langle T^i_i \rangle$.

- Using the replica trick this implies Weyl transformation of REE e.g. $d = 2$

$$\delta S_{\text{ren}} = -\frac{c}{6}\sigma$$

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Conclusions and outlook

Renormalized entanglement entropy is useful in field theory applications of entanglement entropy.

To explore further:

- Applications of REE to phase transitions, time dependent situations etc.
- Holographic definition of renormalized EE for more general asymptotics (including flat space?).

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