# The Large D Black Hole Membrane

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# References

- Talk based on

  ArXiv 1504.06613 S. Bhattacharyya, A. De, S.M, R. Mohan, A. Saha

  ArXiv 1511.03432 S. Bhattacharyya, M. Mandlik, S.M and S. Thakur

  1607.06475 Y. Dandekar, A. De, S. Mazumdar, S.M., A. Saha
- And ongoing work

  Currents Radiation and Thermodynamics

  S. Bhattacharyya, A. Mandal, M. Mandlik, U. Mehta, S.M., U. Sharma and S Thakur

  Scaling Limits

  Y. Dandekar, S. Mazumdar, S.M., A. Saha

  Membrane in more general backgrounds

  S. Bhattacharyya, P. Biswas, B. Chakrabarty, Y. Dandekar, S. Mazumdar, A. Saha
- Builds on observations and earlier work

  Several papers incl quasinormal modes Emparan, Suzuki, Tanabe (EST) et al
- Other related recent work:

  About 9 papers ArXiv 1504.06489...1605.08854 T, ST, EST + collaborators.

# Introduction

- Classically, an external observer is causally disconnected from the black hole interior and so can pretend that spacetime ends at the future event horizon.
- The condition of smoothness of the event horizon implies a relationship between canonical momenta (conserved currents) and the induced gauge field and metric on the horizon. With appropriate definitions the boundary conditions at the event horizon 'membrane' can be reworded in familiar material terms (e.g. Ohm's law).
- Unfortunately Einstein's equations are as difficult to solve just outside the event horizon as inside, so this elegant elimination does not buy much in practical terms.
- In this talk I describe a context in which one can do better; replace the entire curved black hole space time with a membrane. The context is large $D$.

# Thin black holes at large D

- The metric of a $D$ dimensional Schwarschild black hole boosted to velocity $u_M$ in Kerr Schild coordinates :

$$g_{MN} = \eta_{MN} + \frac{(dr_M - u_M)(dr_N - u_N)}{\left(\frac{r}{r_0}\right)^{D-3}},$$

$$u_M = \mathrm{const}, \quad u^2 = -1, \quad r^2 = x^M \mathcal{P}_{MN} x^N, \quad \mathcal{P}_{MN} = \eta_{MN} + u_M u_N$$

- Following EST note that the black hole reduces to flat space at any $r > r_0$ that is held fixed as $D \to \infty$.
- On the other hand if

$$r = r_0 \left(1 + \frac{R}{D-3}\right)$$

and $R$ held fixed as $D \to \infty$ then

$$g_{MN} = \eta_{MN} + e^{-R}(dr_M - u_M)(dr_N - u_N)$$

Thus the 'tail' of the black hole extends only thickness $\frac{r_0}{D}$.

# Collective Coordinate ansatz

- Now consider the more general ansatz metric

$$g^0_{MN} = \eta_{MN} + \frac{(n-u)_M (n-u)_N}{\rho^{D-3}}, \qquad (1)$$

where $\rho$ is an unspecified function in flat Minkowski space $u$ is a oneform 'velocity' field in flat space

$$n = \frac{d\rho}{\sqrt{\partial\rho^2}}, \quad u^2 = -1, \quad u.n = 0$$

- As above, the deviation of (1) from flat space is proportional to $e^{-D(\rho-1)}$. (1) is flat when $\rho - 1 \gg \frac{1}{D}$.
- Moreover it is easily checked that

$$n.n = \left(1 - \frac{1}{\rho^{D-3}}\right)$$

Thus the codimension one submanifold $\rho = 1$ is null. Its generators are tangent to $u^M$. Dissipative nature of equations will ensure that this surface is the event horizon.

# Einstein's equations on the ansatz

- Adopt philosophy of membrane paradigm: uninterested in interior. Moreover spacetime nontrivial only in thickness $\frac{1}{D}$ around $\rho = 1$. So can also forget about most of the exterior.
- Thus we focus entirely on the membrane region $\rho - 1 \sim \frac{1}{D}$.
- Evaluate Einstein's equations, $R_{MN} = 0$. Assume that $\rho$ and $u$ vary on length scale unity. $\frac{1}{\rho^{D-3}}$ nonetheless varies on length scale $1/D$. Consequently generically $R_{MN} = \mathcal{O}(D^2)$. However if

$$u = \text{const}, \quad \rho = \frac{r}{r_0}, \quad \text{then} \quad R_{MN} = 0$$

This fact can be used to show that when

$$\nabla^2 \left( \frac{1}{\rho^{D-3}} \right) = 0, \quad \nabla . u = 0, \quad \text{then} \quad R_{MN} = \mathcal{O}(D)$$

In other words velocity fields membrane shape are large $D$ collective coordinates.

# Perturbation theory

- Now consider the metric
$$g_{MN} = g^0_{MN} + \epsilon \frac{1}{D} g^1_{MN} \dots$$
- Where $g^1_{MN}$, like $g^0_{MN}$, is built out of $\frac{1}{\rho^{D-3}}$ along with $u_M$ and $\rho$ but is otherwise independent of $D$. $\epsilon$ is a counting parameter, eventually set to unity. Note
-
$$\begin{aligned} R_{MN} &= R_{MN}(g^0) + \epsilon \left( \frac{1}{D} \mathcal{O}(D^2) \right) \\ &= \mathcal{O}(D) + \epsilon \mathcal{O}(D) \end{aligned}$$
- Both terms above are of order $\frac{1}{D}$. 2nd term linear differential operator on $g^1_{MN}$. Requiring $R_{MN}$ vanishes at $\mathcal{O}(D)$ yields inhomogeneous linear differential equations for $g^1_{MN}$.

# Dynamical Equations

- Slices of constant $\rho$ give a foliation of spacetime. View Einstein's equations as evolving forward in $\rho$.
- Dynamical equations : inhomogeneous second order differential equations for $g^1_{MN}$. Classify modes by $SO(D-2)$ rotations orthogonal to $n$ and $u$.
- Equations for tensor decouple from vectors and scalars. Ordinary differential equations in $\rho$.
- Equation for vector decouples from scalar but mixes with the divergence of the tensor. Plugging in known tensor find 2nd order ordinary differential equation in $\rho$ with known source. Easily solved.
- Equation for scalars mixes with single divergence of vector and double divergence of tensor. Plugging in known solutions find 2nd order ordinary differential equation for scalars. Easily solved.

# Uniqueness of solution

- In order to actually solve we pick a gauge, demand $\rho = 1$ is the event horizon and $u^M$ tangent to its generators even at subleading order in $\frac{1}{D}$.
- Crucially we also demand that $g^1_{MN}$ deays for $\rho - 1 \gg \frac{1}{D}$ (i.e. to the exterior of the membrane region) and that the solution is regular at the event horizon $\rho = 1$.
- The solution for $g^1_{MN}$ turns out to be unique, completely explicit and and very simple. We find a low degree polynomial in $\frac{1}{\rho^{D-3}}$ and $\rho - 1$. Coefficients all local expressions constructed out of at most two derivatives of $u_M$ and the extrinsic curvature $K_{MN}$ of the $\rho = 1$ surface, viewed as a submanifold of flat space.
- Our solutions solve Einstein's equations to a given order in $\frac{1}{D}$ in the membrane region $\rho - 1 \sim \frac{1}{D}$. More generally they can be shown to well approximate the true solution provided $\rho - 1 \ll 1$.

# Constraint Equation

- Once the dynamical equations have been solved we need solve the contraint equations only on a single constant $\rho$ slice. The event horizon $\rho = 1$ most convenient choice. Infact $g^1_{MN}$ happens to drop out of the constraint equations evaluated on this slice, so the constraints can be evaluated on the metric $g^0_{MN}$. By explicit evaluation we find

$$\nabla . u = 0$$

$$\mathcal{P}^A_L \left[ u \cdot \nabla u_A - u^B K_{BA} - \frac{\nabla^2 u_A}{\mathcal{K}} + \frac{u^C K_{CB} K^B_A}{\mathcal{K}} + \frac{\nabla_A \mathcal{K}}{\mathcal{K}} \right] = 0$$

where $K_{AB}$ is the extrinsic curvature of the membrane and $\mathcal{K}$ is the trace of $K_{AB}$. Here $\mathcal{P}$ is the projector orthogonal to the velocity $u$ on the membrane.

# Initial value problem from membranes

- We have $D-1$ membrane equations. Same as number of variables ($D-2$ velocities and one membrane shape). The membrane equations thus provide a dual autonomous description of black hole dynamics.
- Equations remeniscent of the hydrodynamics of incompressable fluid but on a dynamical surface.
- Contraint equations on the horizon also central to 'old' membrane pardigm. New element here: explicit construction metric in the vicinity of the event horizon in terms of collective coordinates. Transforms constraint into dynamical equations for a well posed initial value problem.
- Can replace all of the black hole spacetime - not just interior - with a non gravitational membrane that lives on a timelike submanifold of flat space. New power result of new parameter, $\frac{1}{D}$. Our discussion can be systematically generalized to arbitrary order in $\frac{1}{D}$.

# Membrane equations at subleading order

- At next subleading order we find

$$\nabla \cdot u = \frac{1}{2\mathcal{K}} \left( \nabla_{(A} u_{B)} \nabla_{(C} u_{D)} \mathcal{P}^{BC} \mathcal{P}^{AD} \right)$$

and

$$\left[ \frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u^B K_{BA} - u \cdot \nabla u_A \right] \mathcal{P}^A_C$$
$$\Bigg[ \left( -\frac{u^C K_{CB} K^B_A}{\mathcal{K}} \right) + \left( \frac{\nabla^2 \nabla^2 u_A}{\mathcal{K}^3} - \frac{u \cdot \nabla \mathcal{K} \nabla_A \mathcal{K}}{\mathcal{K}^3} - \frac{\nabla^B \mathcal{K} \nabla_B u_A}{\mathcal{K}^2} - 2\frac{K^{CD} \nabla_C \nabla_D u_A}{\mathcal{K}^2} \right)$$
$$\left( -\frac{\nabla_A \nabla^2 \mathcal{K}}{\mathcal{K}^3} + \frac{\nabla_A \left( K_{BC} K^{BC} \mathcal{K} \right)}{\mathcal{K}^3} \right) + 3\frac{(u \cdot K \cdot u)(u \cdot \nabla u_A)}{\mathcal{K}} - 3\frac{(u \cdot K \cdot u)(u^B K_{BA})}{\mathcal{K}}$$
$$6\frac{(u \cdot \nabla \mathcal{K})(u \cdot \nabla u_A)}{\mathcal{K}^2} + 6\frac{(u \cdot \nabla \mathcal{K})(u^B K_{BA})}{\mathcal{K}^2} + \frac{3}{(D-3)} u \cdot \nabla u_A - \frac{3}{(D-3)} u^B K_{BA} \Bigg] \mathcal{P}^A_C = 0$$

# Adding Charge

- The construction described above generalizes in a straightforward manner to the Einstein Maxwell system. Our collective coordinate construction is a simple generalization of the Reisnner Nordstorm solution in Kerr Schild coordinates. In addition to the shape and velocity field, our ansatz configurations depend on a charge density field $Q$.
- The leading order charged equations of motion are

$$\left(\frac{\nabla^2 u}{\mathcal{K}} - (1-Q^2)\frac{\nabla\mathcal{K}}{\mathcal{K}} + u\cdot K - (1+Q^2)(u\cdot\nabla)u\right)\cdot\mathcal{P} = 0,$$

$$\frac{\nabla^2 Q}{\mathcal{K}} - u\cdot\nabla Q - Q\left(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}} - u\cdot K\cdot u\right) = 0,$$

$$\nabla.u = 0$$

The extra'charge diffusion' equation governs the dynamics of the additional charge degree of freedom. Note the diffusive and convective terms.

# Quasinormal Modes about RN black holes

- Simplest solution: static spherical membrane. Dual to RN black hole. Linearizing the membrane equations about this

$$
\begin{aligned}
r_0\omega^r_{l=0} &= 0 \\
r_0\omega^r_l &= \frac{-i(l-1) \pm \sqrt{(l-1)(1-lQ_0^4)}}{1+Q_0^2} \quad (l \geq 1) \\
r_0\omega^Q_l &= -il \quad (l \geq 0) \\
r_0\omega^V_l &= \frac{-i(l-1)}{1+Q_0^2} \quad (l \geq 1)
\end{aligned}
\tag{2}
$$

- Note highly dissipative. Can compare with direct gravity analysis of QNMs at large $D$. Turns out two kinds of modes. Light, $\omega \sim \frac{1}{r_0}$. Heavy, $\omega \sim \frac{D}{r_0}$. Spectrum above in perfect agreement with light modes. Our membrane equations: nonlinear effective theory of light modes obtained after 'integrating out' the heavy stuff.

# Radiation and the Stess Tensor

- Would like to find the radiation emitted in the course of an arbitrary membrane motion. For instance consider a large $D$ version of a black hole collision.
- Initial state two sphrical membranes (or rotating membranes, see below). Non interacting till they collide. After than the two spheres merge into one. After this our membrane equations take over, describing the transition from a smoothed out version of two touching spheres to a larger single sphere.
- Question: what would a large $D$ LIGO experimentalist detect from such a collision? What radiation does a complicated membrane motion source?
- Because our construction of the metric dual to a membrane motion is valid only for $\rho - 1 \ll 1$ we cannot just read off the radiation by setting $\rho$ large in our formulae. Need to be cleverer.

# Stress Tensor

- Membrane spacetimes good when $\rho - 1 \ll 1$. However gravity effectively linear for $\rho - 1 \gg \frac{1}{D}$. Thus when

$$\frac{1}{D} \ll \rho \ll 1$$

  both approximations are good. Can use our membrane spacetimes to identify the effective linearized solution in overlap region, and then use linearized gravity to continue the solution to infinity to obtain radiation.
- Turns out that there is an elegant way to state the answer to the second part of this programme at large $D$.
- Given a linearized solution of gravity at large $D$ we evaluate its Brown York stress tensor on the membrane, and subtract from it a contribution that arises from the variation by a 'boundary counterterm' that can be computed order by order in $\frac{1}{D}$. By explicit computation

# Conservation of the Stress Tensor

- $$S = \int \sqrt{R} + 2\nabla^2 \left( \frac{1}{\sqrt{R}} \right) + \dots$$
  Where $R$ is the Ricci scalar of the induced metric on the membrane. Counterterm action appears to receive contributions at all order in $\frac{1}{D}$.
- Can abstractly show that this procedure yields a world volume stress tensor $T_{MN}$ on the membrane that is conserved on the membrane worldvolume viewed as a submanifold of flat space. Moreover $T_{MN} K^{MN} = 0$.
- It is easy to check that these two properties ensure that
  $$T^{st}_{MN} = T_{MN} \sqrt{(\partial\rho)^2} \delta(\rho - 1)$$
  is a conserved in space time. $T^{st}_{MN}$ is the effective source for gravitational radiation. Radiation obtained by convoluting against a retarded Greens function.

# Explicit form of Stress Tensor

$$
\begin{aligned}
8\pi T_{AB} = &\left(\frac{\mathcal{K}}{2}\right) u_A u_B - \left(\frac{\nabla_A u_B + \nabla_B u_A}{2}\right) + \left(\frac{1}{2}\right) K_{AB} \\
&+ \frac{1}{2}\left(u_A\left(\frac{\nabla_B \mathcal{K}}{\mathcal{K}} + \nabla^2 u_B\right) - u_B\left(\frac{\nabla_A \mathcal{K}}{\mathcal{K}} + \nabla^2 u_A\right)\right) \\
&- \mathcal{P}_{AB}\left(\frac{1}{2} u \cdot K \cdot u + \frac{1}{2}\frac{\mathcal{K}}{D} - \frac{K^{MN}\left(\nabla_M u_N + \nabla_N u_M\right)}{2\mathcal{K}}\right)
\end{aligned}
$$

- First term: leading order (order $D$). Is the stress tensor of dust with density $\mathcal{K}$. All remaining terms $\mathcal{O}(1)$. Second term shear viscosity. Will see below that $\frac{\eta}{s} = \frac{1}{4\pi}$. Second line can be absorbed into a redefinition of the velocity. Last line is like a field dependent surface tension. Discussion easily generalized to charge current. We find

$$
J^A = \left[Q\mathcal{K}u^A - \left(\frac{\mathcal{K}}{D}\right)\left(Q(u \cdot \partial)u_C + \partial_C Q\right)\mathcal{P}^{CA}\right]
$$

# Equation of motion from conservation

- As we have explained above, we could abstractly demonstrate the conservation of the stress tensor dual to any linearized solution of Einsten's equations. Interesting to see how it works in detail in the case of the membrane.
- By explicit computation we find that $u^A \nabla_B T^{AB} \propto \nabla . u$. And $\mathcal{P}_A^B \nabla_C T^{CB}$ is proportional to the other membrane equation of motion! In other words the membrane equations are simply the condition that the membrane stress tensor is conserved.
- Similar story for the charge current. Explicit formula. Conservation gives the new charge equation of motion

# Energy loss in radiation

- The membrane stress tensor presented above is $\mathcal{O}(D)$ and so is not small. One might thus incorrectly conclude that the energy lost in radiation is also substantial. This incorrect conclusion contradicts the conservation of the membrane energy and is also in tenson with the locality of membrane equations.
- Even though the stress tensor is substantial, in actuality he loss of energy in radiation is actually extremely small. In particular it is of order $\frac{1}{D^D}$ and so is non perturbatively small in the at large $D$.
- The explanation of this smallness lies not in the nature of Greens functions in a large number of dimensions as we now explain

# Greens functions at large $D$

- In order to study the structure of the retarded Green's function for the operator $\nabla^2$ in $D$ dimensions it turns out to be useful to work in Fourier space in time but coordinate space in the spatial coordinates.
- Let the Greens function take the form $G_\omega(r)e^{-i\omega t}$. Let $G_\omega(r) = \psi_\omega(r)/r^{-(D-3)/2}$. Away from $r = 0$ it is easy to check that $\psi$ obeys the equation

$$-\partial_r^2\psi_\omega + \frac{(D-4)(D-2)}{4r^2}\psi_\omega + \omega^2\psi_\omega = 0$$

- Effective Schrodinger problem with $\bar{h}^2/2m = 1$, $E = \omega$ and

$$V(r) = \frac{(D-4)(D-2)}{4r^2}$$

# Radiation

- The potential for the Schrodinger problem is positive and of order $\mathcal{O}(D^2)$ while the energy $\omega$ is of order unity. A mode of order unity at $r = r_0$ decays as it tunnels to $r = \frac{D}{2\omega}$ where it finally begins to propagate as radiation field of amplitude $\frac{1}{D^D}$.
- Restated, we have two kinds of light modes in black hole backgrounds: the light QNMs and light radiation far away from the black hole. The coupling between these two kinds of modes is nonperturbatively small at large $D$.
- At large $D$ the near horizon geometry of a Schwarschild black hole decouples from the outside, much as the near horizon geometry of $D3$ branes decouples from the outside at low energies. Our membrane equations are the analogues of the hydrodynamics of $\mathcal{N} = 4$ Yang Mills. Does there exist a quntum 'atomic' theory, the analogue of the $\mathcal{N} = 4$ Yang Mills Lagrangian?

# Entropy Current

- As black holes are thermodynamical objects, the black hole membrane should carry an entropy current in addition to its stress tensor and charge current. As the membrane equations are local we expect the second law of thermodynamics to operate in a local manner. Consequently the divergence of the entropy current should always be non negative.
- Our black hole membrane does indeed have such an entropy current. The area form on the event horizon defines an area form on the membrane, which measures the entropy carried by any part of the membrane. The entropy current is obtained by Hodge dualizing this form. The Hodge dual is taken w.r.t. the *flat space* induced metric on the membrane. The non negativity of divergence of this entropy current is then ensured by Hawking's area theorem.

# Entropy Current

- Using our explicit construction of the metric dual to any membrane motion we find
- $$J^S_M = \left( I + \mathcal{O}\left(\frac{1}{D^2}\right)\right) \frac{u^M}{4}$$
- At leading nontrivial order

$$\nabla . J^S = \frac{\nabla . u}{4} == \frac{1}{8\mathcal{K}} \left( \nabla_{(A} u_{B)} \nabla_{(C} u_{D)} \mathcal{P}^{BC} \mathcal{P}^{AD} \right) + \dots$$

# Stationary solutions

- Entropy production must vanish on stationary solutions. It follows that $\sigma_{MN}$ vanishes on such solutions. Recall that $\nabla.u$ also vanishes. It can be shown that a veolcity field has these properties if and only if it is proportional to a killing vector on the manifold on which it lives.
- In the simplest solution the membrane has a unique killing vector $\partial_t$. Easy to demonstrate that the lowest order membrane equation reduces to $\mathcal{K} = const$ in agreement with a direct analysis by EST of static solutions.
- Another simple situation: the manifold preserves some axial symmetries. In this case the velocity field has to be that of rigid rotations. Plugging this into the membrane equations we once again recover the equation $\mathcal{K} = \propto \gamma$ of EST ($\gamma = \frac{1}{\sqrt{1-v^2}}$). Easy to explicitly solve.
- Generalizes to charge. $Q \propto \gamma$. Can construct charged rotating solutions.

# Scaling solutions

- Gregorry Laflamme instability at scale $\sim \frac{1}{\sqrt{D}}$ at large $D$. Over the last year EST have developed an effective large $D$ theory for study of GL dynamics. Relationship to our membrane?
- Answer. Consider flat space in the coordinates

$$ds^2 = -dt^2 + \frac{1}{D}dz^2 + dr^2 + r^2 d\Omega_{D-3}^2.$$

  Consider the $SO(D-2)$ preserving membrane configurations of the form $r = r_0 + \frac{1}{\sqrt{D}}\delta r$ with $\delta r = \delta r(z,t)$ and $u^z = u^z(z,t)$. Plug this into the membrane equations. At leading order in $1/D$ find a nonlinear set of equations for $\delta r$ and $u^z$. After appropriate variable change identical to the equations of EST.
- In other words EST's effective equations = special scaling limit of membrane equations. Analogeous to the Navier Stokes limit of the relativistic fluid dynamics.

# General backgrounds

- So far considered black holes in flat space. What about black holes in, for instance, the background of a gravitational wave?
- Answer extremely simple. At leading order in $1/D$ the membrane equations are changed only by covariantization. Should allow one to compute the 'polarizability' etc of the black hole (see Barak's talk) but even for $\omega$ large compared to the black hole size as long as $\omega$ is small compared to $r_0/D$. Should also allow one to make contact with the blackfold approach with similar remarks.
- Should be easy - and would be interesting - to do explicit computations along these lines.

# Conclusions

- We have demonstrated that the near horizon geometry of charged and uncharged black holes decouples from asymptotic infinity at large $D$. At the classical level the decoupled theory is governed by a set of equations that describe the propagation of a membrane in flat space.
- The degrees of freeedom of this membrane are its shape and a velocity and a charge density. The membrane carries a conserved stress tensor and charge current whose explicit form we have detrmined at low orders. Membrane equatons of motion are simply the staement of conservation of these currents and appear to define a well posed non gravitational initial value problem.
- Radiation reflects the failiure of decoupling and occurs at order $1/D^D$. The explicit form of radiation fields is obtained by coupling the membrane stress tensor and charge current to the linearized exterior metric and gauge fields in the usual manner.

# Future Directions?

- Have studied black holes in spaces that are flat away from the membrbane region. Would be interesting to study response of membranes to distortions of spacetime (e.g. gravitational waves). Also useful to generalize to gravity with a cosmological constant. In progress (see refs)
- It would be interesting to perform a structural analysis of the constraints on membrane equations that follow from the requirement that they carry a conserved entropy current. Perhaps this analysis could shed light on the still mysterious second law of thermodynamics in higher derivative gravity.
- Could be interesting to use the membrane to study the large $D$ versions of complicated gravitational phenomena. E.g. black hole collsisons. $D = 4$?
- Could the membrane equations derived presented above turn out to be the hydrodynamical equations for a consistent quantum theory?