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A new perspective on the Schwinger-Keldysh formalism

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[1510.02494], [1511.07809] + to appear

X. Dong, A. Lewkowycz, MR [1607.07506]

Motivation: non-equilibrium QFT dynamics

- What is the framework for a consistent Wilsonian treatment of low energy dynamics in mixed states of a QFT?

- There is a reasonably good phenomenological understanding, but the theoretical underpinnings are not yet fully understood.

- Entanglement of the system with some external reservoir/purifier is central to the discussion.

- There are many reasons to be interested in this question:

  - intrinsic interest from QFT and many-body physics standpoint.

  - dynamics of black holes via AdS/CFT.

  - cosmology.

Schwinger-Keldysh formalism

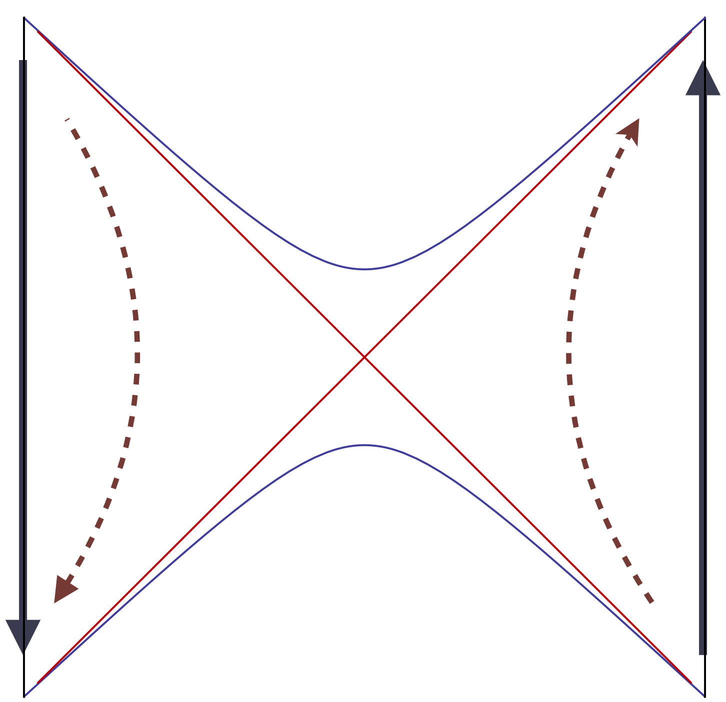
A microscopic perspective

- **Doubling**: Mixed states of a QFT can be purified by introducing an ancillary system. Focus on pure states in tensor product Hilbert space.
- Central to the Schwinger-Keldysh formalism developed to compute real time correlation functions in QFTs.

$$\mathcal{H}_{phys} \subset \mathcal{H}_R \otimes \mathcal{H}_L$$

$$\mathcal{H}_L$$

$$\langle\psi| \to e^{iHt}\langle\psi|$$



$$\mathcal{H}_R$$

$$|\psi\rangle \to e^{-iHt}\,|\psi\rangle$$

$$\rho = |\psi\rangle\langle\psi| \to e^{-iHt}\,|\psi\rangle\langle\psi|\,e^{iHt}$$

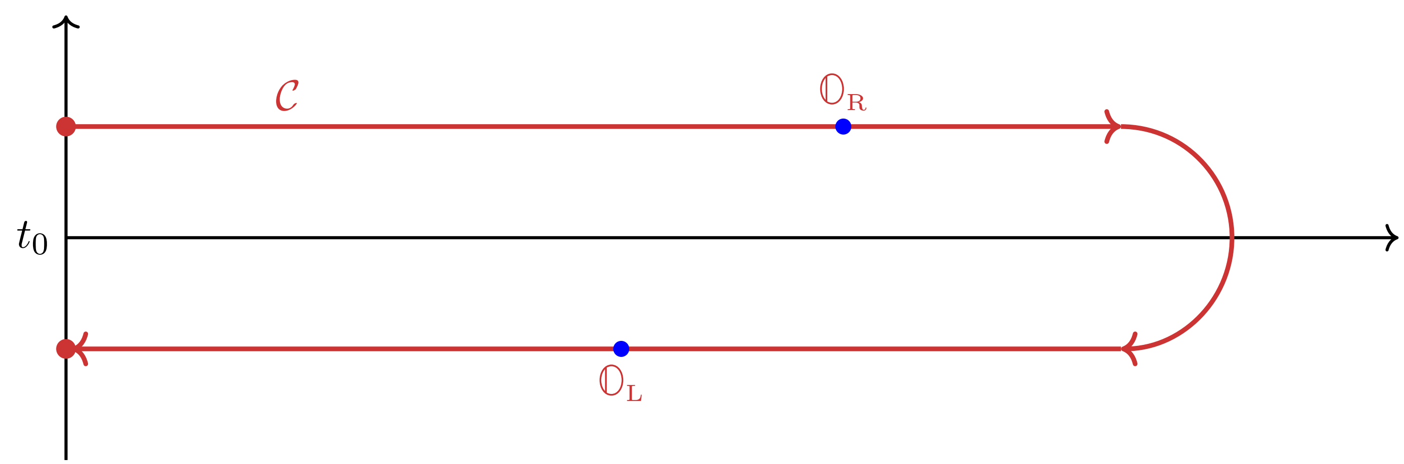
Schwinger-Keldysh formalism

- The Schwinger-Keldysh formalism computes time ordered correlation functions in a generic (mixed) state.

- We double the degrees of freedom to account for the operator nature of the density matrix or equivalently work with a closed time contour:

$$S_{SK} = S[\Phi_{\mathrm{R}}] - S[\Phi_{\mathrm{L}}]$$

$$\delta S_{SK} = \int d^d x \sqrt{-g}\,(\mathcal{J}_{\mathrm{R}}\,\mathbb{O}_{\mathrm{R}} - \mathcal{J}_{\mathrm{L}}\,\mathbb{O}_{\mathrm{L}})$$



Generating functional

$$\mathcal{Z}_{SK}[J_{\mathrm{R}}, J_{\mathrm{L}}] \equiv \mathrm{Tr}\Big\{U[J_{\mathrm{R}}]\ \hat{\rho}_{\mathrm{initial}}\ (U[J_{\mathrm{L}}])^{\dagger}\Big\}$$

Time ordered correlations

$$\mathrm{Tr}\Big(\ \hat{\rho}_{\mathrm{initial}}\ \bar{\mathcal{T}}\left(U^{\dagger}\mathbb{O}_{\mathrm{L}}U^{\dagger}\mathbb{O}_{\mathrm{L}}\ldots\right)\ \mathcal{T}\left(U\mathbb{O}_{\mathrm{R}}U\mathbb{O}_{\mathrm{R}}\ldots\right)\Big)$$

Topological limit

- Lorentz signature inner product in R-L basis from forward/backward evolution implies:

$$\mathcal{Z}_{SK}[\mathcal{J}_{\text{R}} = \mathcal{J}_{\text{L}} = \mathcal{J}] = \text{Tr}\{\hat{\rho}_{\text{initial}}\}$$

- Equal sources on L-R collapses to a theory of initial conditions.

$$\langle \mathcal{T}_{SK} \prod_k \left(\mathbb{O}_{\text{R}}^{(k)} - \mathbb{O}_{\text{L}}^{(k)} \right) \rangle \equiv \langle \mathcal{T}_{SK} \prod_k \mathbb{O}_{dif}^{(k)} \rangle = 0$$

- Keldysh (light-cone) basis $\qquad \mathbb{O}_{dif} \equiv \mathbb{O}_{\text{R}} - \mathbb{O}_{\text{L}}\ , \qquad \mathbb{O}_{av} \equiv \frac{1}{2}\left(\mathbb{O}_{\text{R}} + \mathbb{O}_{\text{L}}\right)$

- Rather remarkable statement, which is agnostic of microscopic dynamics.

- Furthermore, a *largest time equation* is satisfied; difference operators cannot be future-most in any correlation function.

The Schwinger-Keldysh quartet

- ✦ Difference operator correlation functions vanish because they are trivial elements of a BRST cohomology.
- ✦ There exists a pair of Grassmann odd charges which act on the doubled operator algebra.
- ✦ The SK theory is covariantly expressed in terms of a quartet of fields, which usual doubled formalism being a gauge fixed version (ghosts =0).

$$
\begin{tikzcd}
 & \mathbb{O}_{\mathrm{R}}, \mathbb{O}_{\mathrm{L}} \arrow[dl, "\mathcal{Q}_{SK}"'] \arrow[dr, "\overline{\mathcal{Q}}_{SK}"] & \\
\mathbb{O}_{G} \arrow[dr, "\overline{\mathcal{Q}}_{SK}"'] & & \mathbb{O}_{\overline{G}} \arrow[dl, "-\mathcal{Q}_{SK}"] \\
 & \mathbb{O}_{\mathrm{R}} - \mathbb{O}_{\mathrm{L}} &
\end{tikzcd}
$$

$$\mathcal{Q}_{SK}^2 = \overline{\mathcal{Q}}_{SK}^2 = \left[\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{SK}\right]_{\pm} = 0$$

The Superoperator algebra

- Useful to organize the doubled operator algebra into an operator superalgebra on which the Grassmann charges act as derivations.

$$\mathring{\mathbb{O}} = \mathbb{O}_{av} + \theta\, \mathbb{O}_{\overline{G}} + \bar{\theta}\, \mathbb{O}_{G} + \bar{\theta}\theta\, \mathbb{O}_{dif}$$

$$\mathbb{d}_{\text{SK}} \equiv \partial_{\bar{\theta}}\,, \qquad \bar{\mathbb{d}}_{\text{SK}} \equiv \partial_{\theta}$$

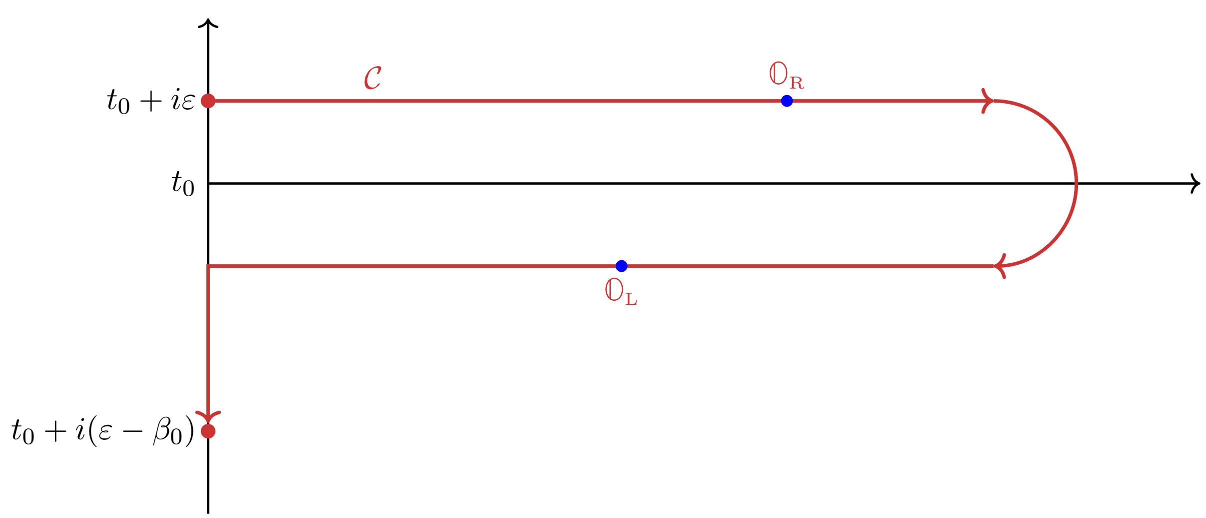
- This representation is useful to recover the time-ordering rules of the Schwinger-Keldysh construction directly from superspace.

$$\langle \mathring{\mathcal{T}}_{SK}\, \mathring{\mathbb{A}}_1\, \mathring{\mathbb{A}}_2 \cdots \mathring{\mathbb{A}}_n \rangle = \langle \mathring{\mathcal{T}}_{SK} \prod_{k=1}^{n} \left(\mathbb{A}^k_{av} + \theta_k\, \mathbb{A}^k_{\overline{G}} + \bar{\theta}_k\, \mathbb{A}^k_{G} + \bar{\theta}_k\, \theta_k\, \mathbb{A}^k_{dif} \right) \rangle\,.$$

- Requiring the supercorrelator to be supertranslation invariance, we obtain the standard Keldysh rules, modulo ambiguities for the partner ghost correlators.

Thermal density matrices and KMS condition

- Thermal density matrices $\hat{\rho}_{T} = e^{-\beta\left(\widehat{\mathbb{H}} - \mu_{I}\,\widehat{\mathbb{Q}}^{I}\right)}$ define stationary equilibrium configurations.
- Correlation functions have analyticity properties which allows for a Euclidean (Matsubara) formulation, cf., $\mathcal{Z}_{T}(\beta, \mu_{I}) = \mathrm{Tr}\left(\hat{\rho}_{T}\right)$



$$\mathcal{Z}_{T}[\mathcal{J}_{\mathrm{R}}, \mathcal{J}_{\mathrm{L}}] = \mathrm{Tr}\left(U[\mathcal{J}_{\mathrm{R}}]\,\hat{\rho}_{T}\,(U[\mathcal{J}_{\mathrm{L}}])^{\dagger}\right)$$

- KMS condition asserts that the correlation functions are analytic in the time strip $0 < \Im(t) < \beta$.
- Equivalently within correlation functions, operators and their KMS conjugates (or thermal translates) are equivalent.

KMS conjugates & thermal sum rules

- To extract the physical content of the KMS condition, let us define the KMS conjugate operator:

$$\tilde{\mathbb{O}}_{\mathrm{L}}(t) = \mathbb{O}_{\mathrm{L}}(t - i\,\beta) = e^{-i\,\delta_{\boldsymbol{\beta}}}\,\mathbb{O}_{\mathrm{L}}$$

- One corollary of the KMS condition and the structure of the SK correlation functions discussed earlier is the sum rule

$$\langle \mathcal{T}_{SK} \prod_{k=1}^{n} \left(\mathbb{O}_{\mathrm{R}}^{(k)} - \tilde{\mathbb{O}}_{\mathrm{L}}^{(k)} \right) \rangle = 0 \qquad\qquad \langle \mathcal{T}_{SK} \prod_{i=1}^{n} \mathbb{O}_{ret} \rangle = 0$$

- The retarded operators are the thermal analogs of differences and satisfy a differential equation:

$$i\Delta_{\boldsymbol{\beta}} \mathbb{O}_{ret} = \mathbb{O}_{\mathrm{R}} - e^{-i\delta_{\boldsymbol{\beta}}}\,\mathbb{O}_{\mathrm{L}} \qquad\qquad i\Delta_{\boldsymbol{\beta}} \mathbb{O}_{ret} = 1 - e^{-i\delta_{\boldsymbol{\beta}}}$$

The KMS supercharge

- Deviation from the KMS condition is naturally measured by $\Delta_{\boldsymbol{\beta}}$.
- This must extend to as a KMS superoperator acting on the operator superalgebra: *a thermal supertranslation*

$$\mathcal{I}^{\text{KMS}}_{\mathring{\Lambda}} \equiv \mathring{\Lambda}\,\Delta_{\boldsymbol{\beta}} = \left(\Lambda + \bar{\theta}\,\Lambda_{\psi} + \theta\,\Lambda_{\bar{\psi}} + \bar{\theta}\theta\,\tilde{\Lambda}\right)\Delta_{\boldsymbol{\beta}}$$

- Together with the SK differentials these operations generate a (super) algebra

$$\left[\mathbb{d}_{\text{SK}}, \mathcal{I}^{\text{KMS}}_{\mathring{\Lambda}}\right]_{\pm} = \mathcal{I}^{\text{KMS}}_{\partial_{\bar{\theta}}\mathring{\Lambda}}\,, \qquad \left[\bar{\mathbb{d}}_{\text{SK}}, \mathcal{I}^{\text{KMS}}_{\mathring{\Lambda}}\right]_{\pm} = \mathcal{I}^{\text{KMS}}_{\partial_{\theta}\mathring{\Lambda}}\,, \qquad \left[\mathcal{I}^{\text{KMS}}_{\mathring{\Lambda}}, \mathcal{I}^{\text{KMS}}_{\mathring{\Lambda}'}\right]_{\pm} = \mathcal{I}^{\text{KMS}}_{(\mathring{\Lambda},\mathring{\Lambda}')_{\boldsymbol{\beta}}}$$

- Successive thermal supertranslations do not commute if we assume that transformation parameters vary spatio-temporally

$$(\mathring{\Lambda}, \mathring{\Lambda}')_{\boldsymbol{\beta}} \equiv \mathring{\Lambda}\,\Delta_{\boldsymbol{\beta}}\mathring{\Lambda}' - \mathring{\Lambda}'\,\Delta_{\boldsymbol{\beta}}\mathring{\Lambda}$$

The SK-KMS superalgebra

- The algebra of the SK and KMS differentials is a known topological algebra $\mathcal{N}_T = 2$ extended equivariant cohomology algebra.

Vafa, Witten ’94
Dijkgraaf, Moore ‘96

$$\begin{tikzcd}
 & & \mathcal{I}^{\mathrm{KMS}}_{\bar\theta\theta} & & \\[-20pt]
 & & R_{_{KMS}} \arrow[dl, "\mathcal{Q}_{_{SK}}"] \arrow[dr, "\overline{\mathcal{Q}}_{_{SK}}"'] & & \\
\mathcal{I}^{\mathrm{KMS}}_{\theta} & -\mathcal{Q}_{_{KMS}} \arrow[dr, "\overline{\mathcal{Q}}_{_{SK}}"] & & \overline{\mathcal{Q}}_{_{KMS}} \arrow[dl, "-\mathcal{Q}_{_{SK}}"'] & \mathcal{I}^{\mathrm{KMS}}_{\bar\theta} \\
 & & P_{_{KMS}} & & \\[-20pt]
 & & \mathcal{I}^{\mathrm{KMS}}_{1} & &
\end{tikzcd}$$

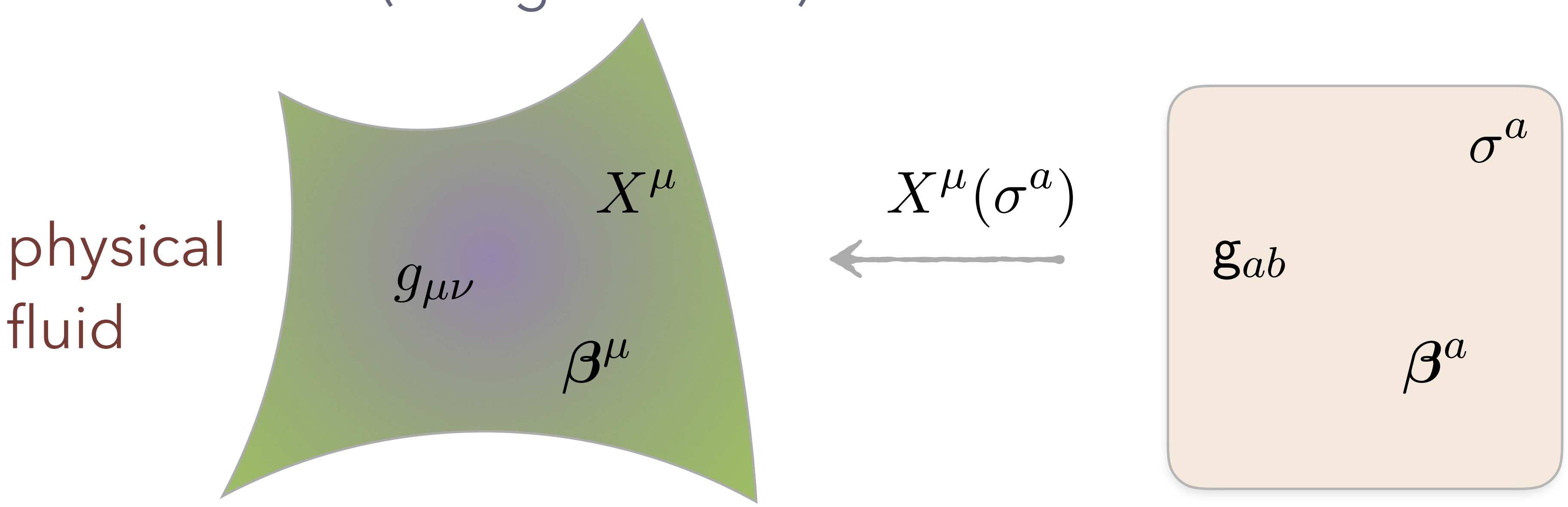
- The $\mathcal{N}_T = 1$ algebra is realized as the standard Weil algebra satisfied by the de Rham complex involving exterior derivatives, Lie derivative and interior contraction.

- The KMS symmetries act as local gauge symmetries: they generate diffeomorphisms along the Euclidean thermal circle.

Application 1: Hydrodynamics

Fluid dynamics as a sigma model

- Hydrodynamics: low energy dynamics of conserved currents in near equilibrium situations.
- Dynamical variables for an effective action: Goldstone modes for spontaneously broken difference diffeomorphisms and difference gauge transformation (charge current).



$$\beta^\mu = \frac{u^\mu}{T}$$

- The SK-KMS algebra constrains the effective action and provides a guiding principle for the emergence of the low energy dynamics.

Brownian particles

- Dynamics of a Brownian particle immersed in a fluid is a simple starting point (generalizes to Brownian branes).

- Data for the worldvolume theory:
  - matter multiplet: a quadruplet $\mathring{X} = \{X, X_\psi, X_{\bar{\psi}}, \tilde{X}\}$
  - gauge multiplet: a dodecuplet $\mathring{\mathcal{A}} \equiv \mathring{\mathcal{A}}_t\, dt + \mathring{\mathcal{A}}_\theta\, d\theta + \mathring{\mathcal{A}}_{\bar{\theta}}\, d\bar{\theta}$

- Effective action: a full superspace integral built out of gauge covariant objects (covariant derivatives and field strengths)

$$S_{\text{B0}} = \int dt\, d\theta\, d\bar{\theta} \left\{ \frac{m}{2} \left(\mathring{\mathcal{D}}_t \mathring{X} \right)^2 - U(\mathring{X}) - i\, \nu\, \mathring{\mathcal{D}}_\theta \mathring{X} \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{X} \right\}$$

$$(\mathring{\Lambda}, \mathring{X})_{\boldsymbol{\beta}} = \mathring{\Lambda}\, \pounds_{\boldsymbol{\beta}} \mathring{X} = \mathring{\Lambda}\, \Delta_{\boldsymbol{\beta}} \mathring{X} = \mathring{\Lambda}\, \boldsymbol{\beta} \frac{d}{dt} \mathring{X} \qquad\qquad \mathring{\mathcal{D}}_I = \partial_I + [\mathring{\mathcal{A}}_I,\ \cdot\ ]$$

Langevin dynamics

- A BRST supersymmetric formulation of stochastic dynamics is well known and is usually derived by an analog of the familiar Faddeev-Popov trick.

Martin, Siggia, Rose '73

- The superspace formalism gives us a SK-KMS covariant presentation.
- Stochasticity/dissipation arises because of spontaneous CPT symmetry breaking.

$$\langle \mathring{\mathcal{F}}_{\theta\bar{\theta}} | \rangle = -i$$

- Useful moral: dissipation = ghost condensation.
- BRST supersymmetry + spontaneous CPT leads to Jarzynski relation which is a generalized fluctuation dissipation relation

$$S_{\text{B0}} \mapsto S_{\text{B0}} - i \, \langle \mathring{\mathcal{F}}_{\theta\bar{\theta}} | \rangle \, \boldsymbol{\beta} \, (\Delta G + W) \quad \Longrightarrow \quad \left\langle e^{-\beta W} \right\rangle = e^{-\beta\, \Delta G}$$

Dissipative hydrodynamic actions

- Working in superspace the symmetries suffice to constrain the terms that can appear in the worldvolume sigma model

$$S_{\rm wv} \equiv \int d^d\sigma\, \mathcal{L}_{\rm wv}\,, \qquad \mathcal{L}_{\rm wv} = \int d\theta\, d\bar{\theta}\, \frac{\sqrt{-\mathring{\mathrm{g}}}}{1+\boldsymbol{\beta}^e \mathring{\mathcal{A}}_e} \left(\mathring{\mathcal{L}} - \frac{i}{4}\, \mathring{\boldsymbol{\eta}}^{(ab)(cd)}\, \mathring{\mathcal{D}}_\theta \mathring{\mathrm{g}}_{ab}\, \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{\mathrm{g}}_{cd}\right)$$

- Integrating over superspace one ends up with a simple Lagrangian density that generalizes the adiabatic Lagrangian to include dissipation:

$$\mathcal{L}_{\rm wv} = \frac{\sqrt{-\mathrm{g}}}{1+\boldsymbol{\beta}^e \mathcal{A}_e} \left\{\frac{1}{2}\left[\mathbf{T}^{ab}_{\mathcal{L}} - \frac{i}{2}\boldsymbol{\eta}^{(ab)(cd)}\, (\mathcal{F}_{\theta\bar{\theta}}, \mathrm{g}_{cd})_{\boldsymbol{\beta}}\right] \tilde{\mathrm{g}}_{ab} - \mathbf{N}^a_{\mathcal{L}} \tilde{\mathcal{A}}_a \right.$$
$$\left. + \frac{i}{8}\left(\boldsymbol{\eta}^{(ab)(cd)} + \boldsymbol{\eta}^{(cd)(ab)}\right) \tilde{\mathrm{g}}_{ab}\, \tilde{\mathrm{g}}_{cd} + \ldots \right\},$$

Class LT Lagrangian

Noise fluctuations

see also Kovtun, Moore, Romatschke '13; Crossley, Glorioso, Liu '15

- Dissipative dynamics again spontaneously breaks CPT, with the KMS field strength picking up a vev (ghost condensate).

Application 2: Covariant entanglement entropy

Background

- Given the boundary region $\mathcal{A}$ the prescription to compute entanglement holographically involves finding a bulk extremal surface $\mathcal{E}_\mathcal{A}$ which is anchored on $\partial\mathcal{A}$ and is homologous to $\mathcal{A}$.

Ryu, Takayanagi '06

$$S_\mathcal{A} = \frac{\text{Area}(\mathcal{E}_\mathcal{A})}{4G_N}$$

Hubeny, MR, Takayanagi '07

- Time independent proposal of RT derived by finding bulk dual of replica construction and invoking the saddle point expression.

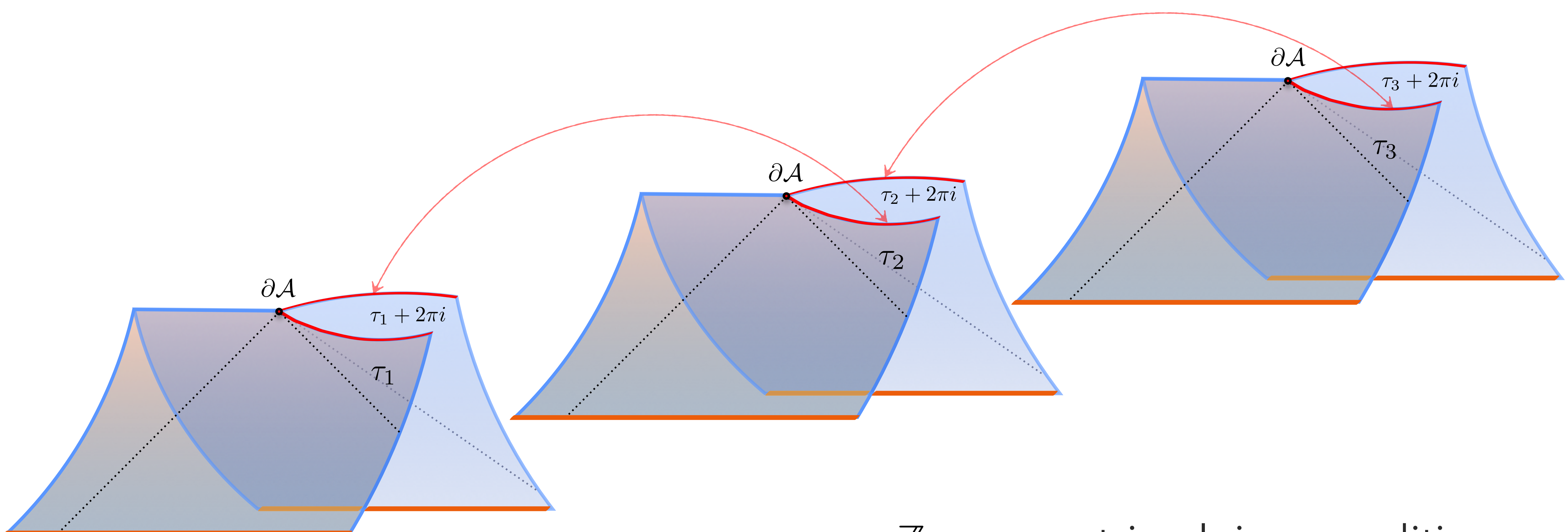
Lewkowycz, Maldacena '13

- Deriving the covariant prescription requires us to consider generalization of Schwinger-Keldysh contours to implement replica for consistency with causality.

- An implementation of the bulk Schwinger-Keldysh replica leads to the proposal.

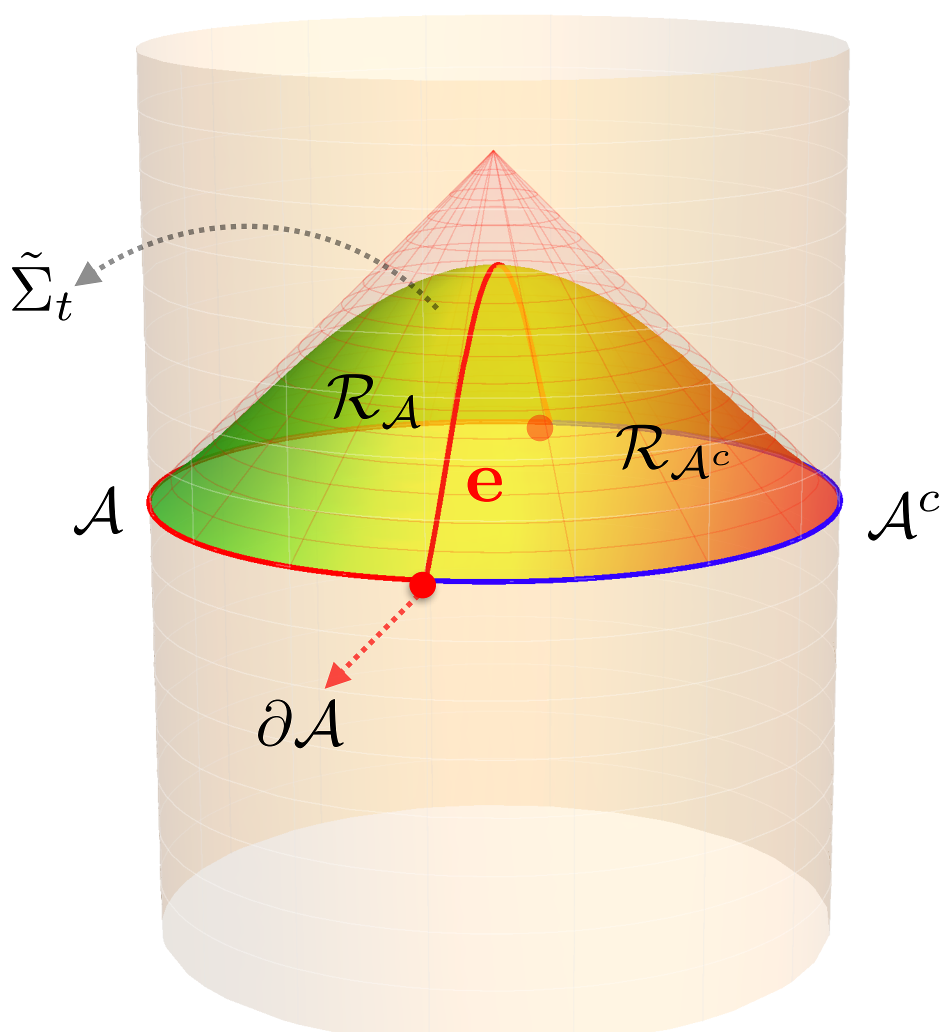
Real time replica: Boundary

- We cut open the path integral along the region $\mathcal{A}$ on the Cauchy slice Σ_t.
- Imposing suitable boundary conditions in the future/past segments leads to the matrix elements of the reduced density matrix $(\rho_{\mathcal{A}})_{\pm}$.



$\mathbb{Z}_q$ symmetric gluing conditions

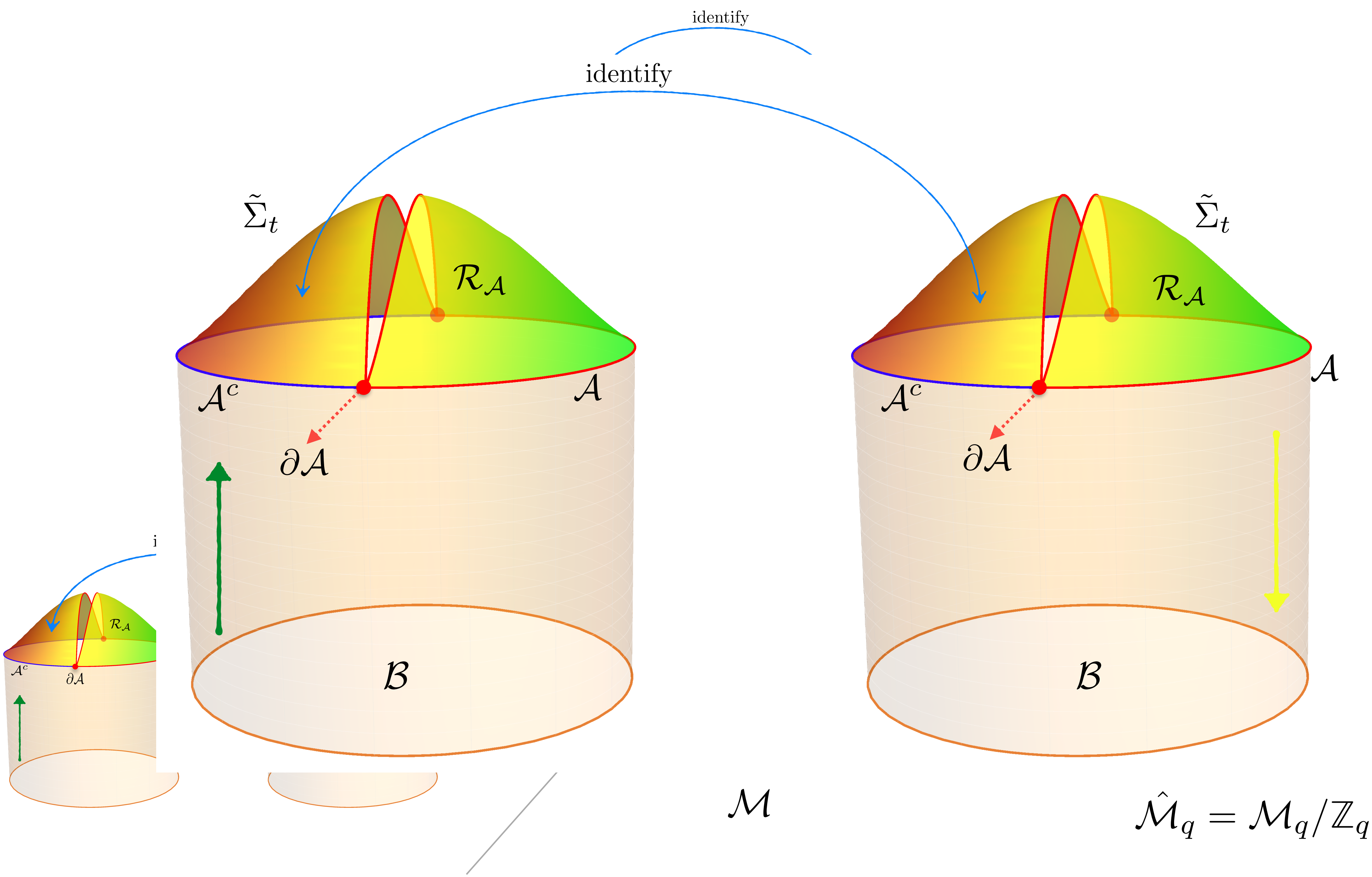
The bulk ansatz



- Prescription: Pick some bulk Cauchy slice $\tilde{\Sigma}_t$ within the FRW wedge.
- We will glue copies of the geometry past of $\tilde{\Sigma}_t$ to obtain the dual of the SK contour.
- The choice of $\tilde{\Sigma}_t$ is irrelevant for computing time-ordered correlation functions.
- For entanglement entropy we will find that $\tilde{\Sigma}_t$ is forced to contain the extremal surface.

Skenderis, Van Rees '08

Bulk density matrix elements



Gravitational dynamics

- Once we have the ansatz and the replica boundary conditions, all that remains is to solve the bulk equations of motion. Work in local coordinates adapted to the normal bundle of the singular locus:

$$ds^2 = \left(q^2 dr^2 - r^2\, d\tau^2\right) + \left(\gamma_{ij} + 2\, K^x_{ij}\, r^q \cosh\tau + 2\, K^t_{ij}\, r^q \sinh\tau\right) dy^i\, dy^j \\ + \left[r^{f_q\,(q-1)} - 1\right] \delta g_{\mu\nu}\, dx^\mu\, dx^\nu + \cdots$$

- Bulk equations of motion then fix the geometry of the singular locus. To leading order in q-1 we fix the geometry. In Einstein-Hilbert theory this gives the extremal surface condition

$$\mathrm{EOM}^a \propto \frac{q-1}{r}\, K^a + \mathrm{regular}^a$$

$$K^a = 0 \quad\Longrightarrow\quad \theta^\pm = \frac{1}{\sqrt{2}}\left(K^0 \pm K^1\right) = 0\,,$$

$$\Longrightarrow\quad \lim_{q\to 1} \mathbf{e}_q = \mathcal{E}_\mathcal{A}\,, \qquad \mathcal{E}_\mathcal{A} \in \mathcal{M} \text{ is extremal.}$$

The on-shell gravitational action

- In addition to ascertaining the saddle point solution, we have to also compute the on-shell action atop it.

- It is useful to compute directly the *modular entropy* which localizes:

$$\tilde{S}^{(q)}_{\mathcal{A}} = -q^2\, \partial_q \left[\frac{1}{q} \log \mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}})^q\right]$$

Dong ‘16

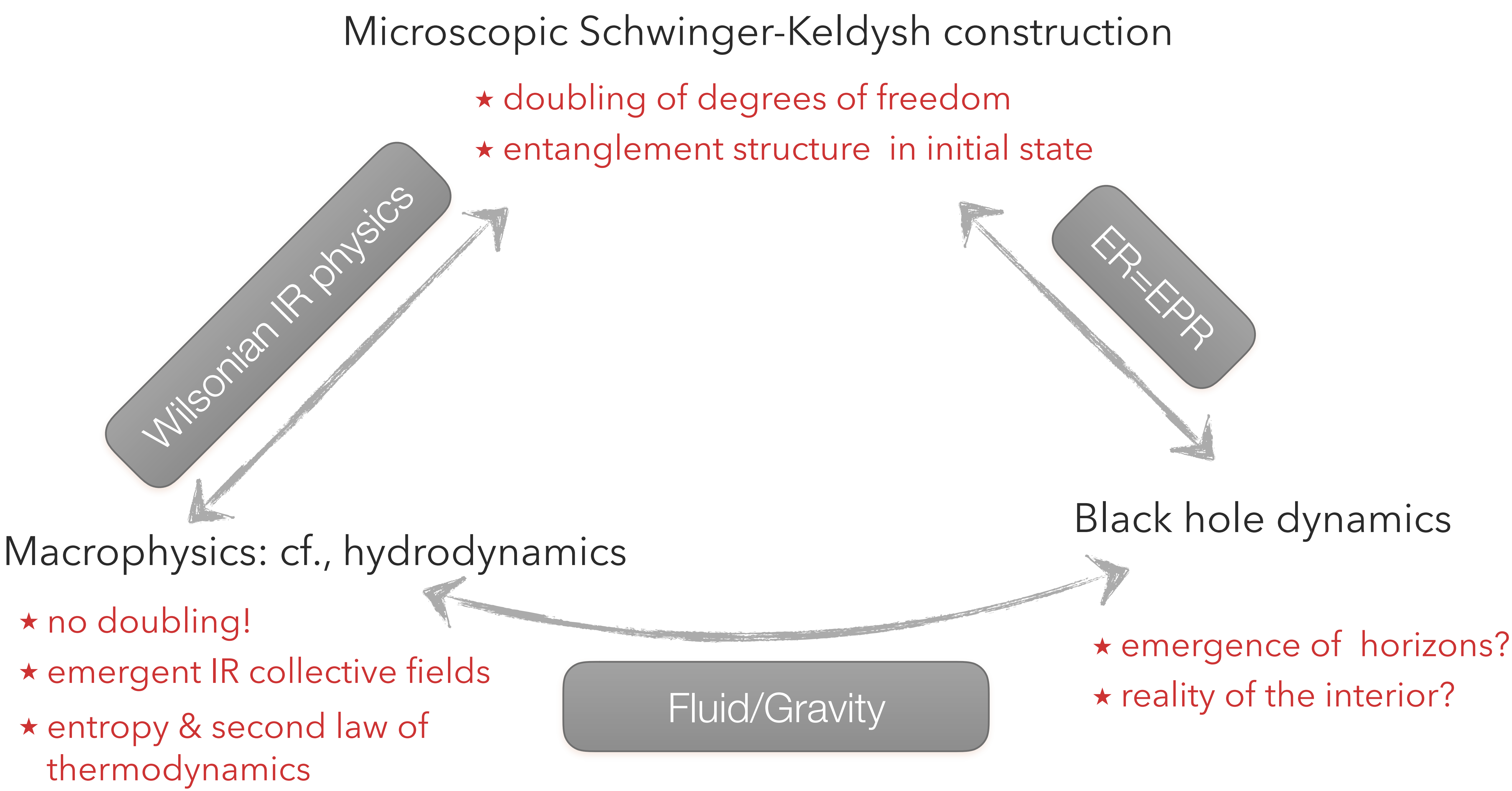
- While there are some slight subtleties, at the end of the day the result for the on-shell action computed directly in the Lorentzian replica geometry leads to the expected answer:

$$\partial_q I[\hat{\mathcal{M}}_q] = i\,\frac{\mathrm{Area}(\mathbf{e}_q)}{4\, q^2 G_N} \qquad\qquad \Longrightarrow \quad S_{\mathcal{A}} = \frac{\mathrm{Area}(\mathcal{E}_{\mathcal{A}})}{4\, G_N}$$

Looking ahead…

- An understanding of the microscopic symmetries in the SK construction, and the symmetry breaking pattern, results in an universal low energy dynamical theory of fluids.
- The macrophysics derived is consistent with phenomenological expectations (yet to make contact with the eightfold classification).
- Perhaps the most important lesson here is that microscopic unitary which enforces fluctuation-dissipation etc., is upheld thanks to the ghost couplings. Lessons for gravity?
- General principles for gravitational implementation of Schwinger-Keldysh?

A roadmap for the framework



Holographic fluids

- Known second order transport of holographic fluids follows from:

$$\mathcal{L}_{\mathrm{wv}} = c_{\mathrm{eff}} \int d\theta\, d\bar{\theta}\, \frac{\sqrt{-\mathring{\mathsf{g}}}}{1+\beta^e\, \mathring{\mathcal{A}}_e} \Bigg\{ \left(\frac{4\pi\, \mathring{\mathsf{T}}}{d}\right)^d \left(1 - \frac{i\, d}{8\pi}\, \mathring{\mathsf{P}}^{c\langle a} \mathring{\mathsf{P}}^{b\rangle d}\, \mathring{\mathcal{D}}_\theta \mathring{\mathsf{g}}_{ab}\, \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{\mathsf{g}}_{cd}\right) - \left(\frac{4\pi\, \mathring{\mathsf{T}}}{d}\right)^{d-2} \left[\frac{{}^{\mathcal{W}}\mathring{\mathsf{R}}}{d-2} + \frac{1}{d}\,\mathrm{Harmonic}\left(\frac{2}{d}-1\right)\mathring{\sigma}^2 + \frac{1}{2}\mathring{\omega}^2\right]\Bigg\}$$

- How does the bulk gravity theory realize this effective action?
- Recent attempts get the ideal fluid part correct, but no clear story beyond…

Nickel, Son ‘10

Crossley, Glorioso, Liu, Wang ’15; deBoer, Heller, Pinzani-Fokeeva ‘15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