On the four-loop cusp anomalous dimension

Johannes M. Henn

Mainz University
PRISMA Cluster of Excellence

Nordita, Aspects of Amplitudes, June 30, 2016

Based on collaboration with Alexander Smirnov, Vladimir Smirnov and Matthias Steinhauser



J. Henn, A. Smirnov, V. Smirnov and M. Steinhauser, arXiv:1604.03126, JHEP 1605 (2016) 066

J. Henn, A. Smirnov, V. Smirnov and M. Steinhauser,
arXiv:1604.03126, JHEP 1605 (2016) 066

The first next-to-next-to-next-to-next-to-leading order (N^4LO) contribution to a three-point function within QCD.

J. Henn, A. Smirnov, V. Smirnov and M. Steinhauser,
arXiv:1604.03126, JHEP 1605 (2016) 066
The first next-to-next-to-next-to-next-to-leading order (N^4LO) contribution to a three-point function within QCD.

- Evaluating four-loop QCD form factors

J. Henn, A. Smirnov, V. Smirnov and M. Steinhauser,
arXiv:1604.03126, JHEP 1605 (2016) 066

The first next-to-next-to-next-to-next-to-leading order (N^4LO) contribution to a three-point function within QCD.

- Evaluating four-loop QCD form factors
- Massless planar four-loop vertex integrals

J. Henn, A. Smirnov, V. Smirnov and M. Steinhauser,
arXiv:1604.03126, JHEP 1605 (2016) 066

The first next-to-next-to-next-to-next-to-leading order (N^4LO) contribution to a three-point function within QCD.

- Evaluating four-loop QCD form factors
- Massless planar four-loop vertex integrals
- Perspectives

The photon-quark form factor, which is a building block for N^4LO cross sections.

The photon-quark form factor, which is a building block for N^4LO cross sections.
It is a gauge-invariant part of virtual forth-order corrections for the process $e^+e^- \to 2$ jets, or for Drell-Yan production at hadron colliders.

The photon-quark form factor, which is a building block for N^4LO cross sections.

It is a gauge-invariant part of virtual forth-order corrections for the process $e^+e^- \to 2$ jets, or for Drell-Yan production at hadron colliders.

Let Γ_q^μ be the photon-quark vertex function.

The scalar form factor is

$$F_q(q^2) = -\frac{1}{4(1-\epsilon)q^2} \mathrm{Tr}\left(\not{p}_2 \Gamma_q^\mu \not{p}_1 \gamma_\mu\right) ,$$

where $D = 4 - 2\epsilon$, $q = p_1 + p_2$ and p_1 (p_2) is the incoming (anti-)quark momentum.

The photon-quark form factor, which is a building block for N^4LO cross sections.
It is a gauge-invariant part of virtual forth-order corrections for the process $e^+e^- \to 2$ jets, or for Drell-Yan production at hadron colliders.
Let Γ^μ_q be the photon-quark vertex function.
The scalar form factor is

$$F_q(q^2) = -\frac{1}{4(1-\epsilon)q^2}\mathrm{Tr}\left(\not{p}_2\,\Gamma^\mu_q\not{p}_1\,\gamma_\mu\right) ,$$

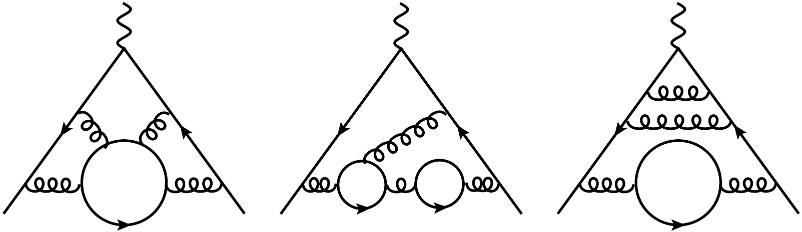
where $D = 4 - 2\epsilon$, $q = p_1 + p_2$ and p_1 (p_2) is the incoming (anti-)quark momentum.
The large-N_c asymptotics of $F_q(q^2) \to$ planar Feynman diagrams.

Three-loop results
[P. A. Baikov, K. G. Chetyrkin, A. V. Smirnov, V. A. Smirnov and M. Steinhauser'09,
T. Gehrmann, E. W. N. Glover, T. Huber, N. Ikizlerli, and C. Studerus'10]

Three-loop results
[P. A. Baikov, K. G. Chetyrkin, A. V. Smirnov, V. A. Smirnov and M. Steinhauser'09,
T. Gehrmann, E. W. N. Glover, T. Huber, N. Ikizlerli, and C. Studerus'10]
Analytic results for the three missing coefficients
[R. N. Lee, A. V. Smirnov and V. A. Smirnov'10]

Three-loop results
[P. A. Baikov, K. G. Chetyrkin, A. V. Smirnov, V. A. Smirnov and M. Steinhauser'09,
T. Gehrmann, E. W. N. Glover, T. Huber, N. Ikizlerli, and C. Studerus'10]
Analytic results for the three missing coefficients
[R. N. Lee, A. V. Smirnov and V. A. Smirnov'10]
Analytic results for the three-loop master integrals up to weight 8
[R. N. Lee and V. A. Smirnov'10]
motivated by a future four-loop calculation.

The fermionic corrections ($\sim n_f$) to F_q in the large-N_c limit, to the four-loop order.



Numerical four-loop calculations
[R. H. Boels, B. A. Kniehl & G. Yang'16]

Numerical four-loop calculations
[R. H. Boels, B. A. Kniehl & G. Yang'16]

Partial results for some individual integrals
[A. von Manteuffel, E. Panzer & R. M. Schabinger'15]

We apply

- `qgraf` for the generation of Feynman amplitudes;

We apply

- `qgraf` for the generation of Feynman amplitudes;
- `q2e` and `exp` for writing down form factors in terms of Feynman integrals

We apply

- `qgraf` for the generation of Feynman amplitudes;
- `q2e` and `exp` for writing down form factors in terms of Feynman integrals
- `FIRE` and `LiteRed` for the IBP reduction to master integrals.

We apply

- `qgraf` for the generation of Feynman amplitudes;
- `q2e` and `exp` for writing down form factors in terms of Feynman integrals
- `FIRE` and `LiteRed` for the IBP reduction to master integrals.

Calculations in generic ξ-gauge for checks.

We apply

- `qgraf` for the generation of Feynman amplitudes;
- `q2e` and `exp` for writing down form factors in terms of Feynman integrals
- `FIRE` and `LiteRed` for the IBP reduction to master integrals.

Calculations in generic ξ-gauge for checks.

$$F_q = 1 + \sum_{n \geq 1} \left(\frac{\alpha_s^0}{4\pi} \right)^n \left(\frac{\mu^2}{-q^2} \right)^{(n\epsilon)} F_q^{(n)} .$$

We apply

- `qgraf` for the generation of Feynman amplitudes;
- `q2e` and `exp` for writing down form factors in terms of Feynman integrals
- `FIRE` and `LiteRed` for the IBP reduction to master integrals.

Calculations in generic ξ-gauge for checks.

$$F_q = 1 + \sum_{n\geq 1} \left(\frac{\alpha_s^0}{4\pi}\right)^n \left(\frac{\mu^2}{-q^2}\right)^{(n\epsilon)} F_q^{(n)} \,.$$

Our result is the fermionic contribution to $F_q^{(4)}$ in the large-N_c limit.

$$F_q^{(4)}\Big|_{\text{large-}N_c} =$$

$$\frac{1}{\epsilon^7}\left[\frac{1}{12}N_c^3 n_f\right] + \frac{1}{\epsilon^6}\left[\frac{41}{648}N_c^2 n_f^2 - \frac{37}{648}N_c^3 n_f\right] + \frac{1}{\epsilon^5}\left[\frac{1}{54}N_c n_f^3 + \frac{277}{972}N_c^2 n_f^2\right.$$

$$\left. + \left(\frac{41\pi^2}{648} - \frac{6431}{3888}\right) N_c^3 n_f\right] + \frac{1}{\epsilon^4}\left[\left(\frac{215\zeta_3}{108} - \frac{72953}{7776} - \frac{227\pi^2}{972}\right) N_c^3 n_f\right.$$

$$\left. + \frac{11}{54}N_c n_f^3 + \left(\frac{5}{24} + \frac{127\pi^2}{1944}\right) N_c^2 n_f^2\right] + \frac{1}{\epsilon^3}\left[\left(\frac{229\zeta_3}{486} - \frac{630593}{69984} + \frac{293\pi^2}{2916}\right) N_c^2 n_f^2\right.$$

$$\left. + \left(\frac{2411\zeta_3}{243} - \frac{1074359}{69984} - \frac{2125\pi^2}{1296} + \frac{413\pi^4}{3888}\right) N_c^3 n_f + \left(\frac{127}{81} + \frac{5\pi^2}{162}\right) N_c n_f^3\right]$$

$$+ \mathcal{O}\left(\frac{1}{\epsilon^2}\right)$$

Exponentiation of infrared (soft and collinear) divergences

Exponentiation of infrared (soft and collinear) divergences

$$\log(F_q)|_{\text{pole part}}\Big|_{\left(\frac{\alpha_s}{4\pi}\right)^4} =$$

$$\Bigg\{ \frac{1}{\epsilon^5}\left[\frac{25}{96}\beta_0^3 C_F \gamma^0_{\text{cusp}}\right] + \frac{1}{\epsilon^4}\left[C_F\left(-\frac{13}{96}\beta_0^2\gamma^1_{\text{cusp}} - \frac{5}{12}\beta_1\beta_0\gamma^0_{\text{cusp}}\right) - \frac{1}{4}\beta_0^3\gamma_q^0\right]$$

$$+ \frac{1}{\epsilon^3}\left[C_F\left(\frac{5}{32}\beta_2\gamma^0_{\text{cusp}} + \frac{3}{32}\beta_1\gamma^1_{\text{cusp}} + \frac{7}{96}\beta_0\gamma^2_{\text{cusp}}\right) + \frac{1}{4}\beta_0^2\gamma_q^1 + \frac{1}{2}\beta_1\beta_0\gamma_q^0\right]$$

$$+ \frac{1}{\epsilon^2}\left[-\frac{1}{4}\beta_2\gamma_q^0 - \frac{1}{4}\beta_1\gamma_q^1 - \frac{1}{4}\beta_0\gamma_q^2 - \frac{1}{32}C_F\gamma^3_{\text{cusp}}\right] + \frac{1}{\epsilon}\left[\frac{\gamma_q^3}{4}\right]\Bigg\},$$

The coefficients of the cusp and collinear anomalous dimensions

$$\gamma_x = \sum_{n \geq 0} \left(\frac{\alpha_s(\mu^2)}{4\pi} \right)^n \gamma_x^n ,$$

The coefficients of the cusp and collinear anomalous dimensions

$$\gamma_x = \sum_{n\geq 0} \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^n \gamma_x^n ,$$

with $x \in \{\text{cusp}, q\}$.

$$\begin{aligned}
\gamma^0_{\rm cusp} &= 4\,, \\
\gamma^1_{\rm cusp} &= \left(-\frac{4\pi^2}{3}+\frac{268}{9}\right) N_c - \frac{40 n_f}{9}\,, \\
\gamma^2_{\rm cusp} &= \left(\frac{44\pi^4}{45}+\frac{88\zeta_3}{3}-\frac{536\pi^2}{27}+\frac{490}{3}\right) N_c^2 \\
&\quad + \left(-\frac{64\zeta_3}{3}+\frac{80\pi^2}{27}-\frac{1331}{27}\right) N_c n_f - \frac{16 n_f^2}{27}\,, \\
\gamma^3_{\rm cusp} &= \left(-\frac{32\pi^4}{135}+\frac{1280\zeta_3}{27}-\frac{304\pi^2}{243}+\frac{3463}{81}\right) N_c n_f^2 + \left(\frac{128\pi^2\zeta_3}{9}+224\zeta_5\right. \\
&\quad \left. -\frac{44\pi^4}{27}-\frac{16252\zeta_3}{27}+\frac{13346\pi^2}{243}-\frac{60391}{81}\right) N_c^2 n_f + \left(\frac{64\zeta_3}{27}-\frac{32}{81}\right) n_f^3 + \ldots
\end{aligned}$$

$$
\begin{aligned}
\gamma_q^0 &= -\frac{3N_c}{2}\,, \quad \gamma_q^1 = \left(\frac{\pi^2}{6}+\frac{65}{54}\right) N_c n_f + \left(7\zeta_3 - \frac{5\pi^2}{12} - \frac{2003}{216}\right) N_c^2\,, \\
\gamma_q^2 &= \left(-\frac{\pi^4}{135} - \frac{290\zeta_3}{27} + \frac{2243\pi^2}{972} + \frac{45095}{5832}\right) N_c^2 n_f + \left(-\frac{4\zeta_3}{27} - \frac{5\pi^2}{27} + \frac{2417}{1458}\right) \\
&\quad + N_c^3 \left(-68\zeta_5 - \frac{22\pi^2\zeta_3}{9} - \frac{11\pi^4}{54} + \frac{2107\zeta_3}{18} - \frac{3985\pi^2}{1944} - \frac{204955}{5832}\right)\,, \\
\gamma_q^3 &= N_c^3 \left[\left(-\frac{680\zeta_3^2}{9} - \frac{1567\pi^6}{20412} + \frac{83\pi^2\zeta_3}{9} + \frac{557\zeta_5}{9} + \frac{3557\pi^4}{19440} - \frac{94807\zeta_3}{972}\right.\right. \\
&\quad \left.\left. + \frac{354343\pi^2}{17496} + \frac{145651}{1728}\right) n_f\right] + \left(-\frac{8\pi^4}{1215} - \frac{356\zeta_3}{243} - \frac{2\pi^2}{81} + \frac{18691}{13122}\right) N_c n_f^3 \\
&\quad + \left(-\frac{2}{3}\pi^2\zeta_3 + \frac{166\zeta_5}{9} + \frac{331\pi^4}{2430} - \frac{2131\zeta_3}{243} - \frac{68201\pi^2}{17496} - \frac{82181}{69984}\right) N_c^2 n_f^2 + \ldots
\end{aligned}
$$

We reproduce results up to three loops
[A. Vogt'01; C.F. Berger'02; S. Moch, J.A.M. Vermaseren & A. Vogt'04,05; P.A. Baikov, K.G. Chetyrkin, A.V. Smirnov, V.A. Smirnov & M. Steinhauser'09; T. Becher & M. Neubert'09; T. Gehrmann, E.W.N. Glover, T. Huber, N. Ikizlerli & C. Studerus'10]

We reproduce results up to three loops
[A. Vogt'01; C.F. Berger'02; S. Moch, J.A.M. Vermaseren & A. Vogt'04,05; P.A. Baikov, K.G. Chetyrkin, A.V. Smirnov, V.A. Smirnov & M. Steinhauser'09; T. Becher & M. Neubert'09; T. Gehrmann, E.W.N. Glover, T. Huber, N. Ikizlerli & C. Studerus'10]

The $N_c^3 n_f^3$ term of $\gamma^3_{\rm cusp}$ agrees with
[M. Beneke & V.M. Braun'94]

We reproduce results up to three loops
[A. Vogt'01; C.F. Berger'02; S. Moch, J.A.M. Vermaseren & A. Vogt'04,05; P.A. Baikov, K.G. Chetyrkin, A.V. Smirnov, V.A. Smirnov & M. Steinhauser'09; T. Becher & M. Neubert'09; T. Gehrmann, E.W.N. Glover, T. Huber, N. Ikizlerli & C. Studerus'10]

The $N_c^3 n_f^3$ term of γ^3_{cusp} agrees with
[M. Beneke & V.M. Braun'94]

Agreement of the n_f^2 term with
[Davies, B. Ruijl, T.Ueda, J. Vermaseren & A. Vogt]
talk at Loops and Legs 2016 by A. Vogt

We reproduce results up to three loops
[A. Vogt'01; C.F. Berger'02; S. Moch, J.A.M. Vermaseren & A. Vogt'04,05; P.A. Baikov, K.G. Chetyrkin, A.V. Smirnov, V.A. Smirnov & M. Steinhauser'09; T. Becher & M. Neubert'09; T. Gehrmann, E.W.N. Glover, T. Huber, N. Ikizlerli & C. Studerus'10]

The $N_c^3 n_f^3$ term of γ^3_{cusp} agrees with
[M. Beneke & V.M. Braun'94]

Agreement of the n_f^2 term with
[Davies, B. Ruijl, T.Ueda, J. Vermaseren & A. Vogt]
talk at Loops and Legs 2016 by A. Vogt

All the other four-loop terms in γ^3_{cusp} and γ^3_q, and for the finite part, are new.

All planar four-loop on-shell form-factor integrals with $p_1^2 = p_2^2 = 0$, with $q^2 \equiv p_3^2 = (p_1 + p_2)^2$

$$
\begin{aligned}
F a_1, \ldots, a_{18} \quad = \quad & \int \ldots \int \frac{\mathrm{d}^D k_1 \ldots \mathrm{d}^D k_4}{(-(k_1+p_1)^2)^{a_1}(-(k_2+p_1)^2)^{a_2}(-(k_3+p_1)^2)^{a_3}} \\
& \times \frac{1}{(-(k_4+p_1)^2)^{a_4}(-(k_1-p_2)^2)^{a_5}(-(k_2-p_2)^2)^{a_6}(-(k_3-p_2)^2)^{a_7}} \\
& \times \frac{1}{(-(k_4-p_2)^2)^{a_8}(-k_1^2)^{a_9}(-k_2^2)^{a_{10}}(-k_3^2)^{a_{11}}(-k_4^2)^{a_{12}}} \\
& \times \frac{1}{(-(k_1-k_2)^2)^{-a_{13}}(-(k_1-k_3)^2)^{-a_{14}}(-(k_1-k_4)^2)^{-a_{15}}} \\
& \times \frac{1}{(-(k_2-k_3)^2)^{-a_{16}}(-(k_2-k_4)^2)^{-a_{17}}(-(k_3-k_4)^2)^{-a_{18}}} \, .
\end{aligned}
$$

All planar four-loop on-shell form-factor integrals with $p_1^2 = p_2^2 = 0$, with $q^2 \equiv p_3^2 = (p_1 + p_2)^2$

$$
\begin{aligned}
F a_1, \dots, a_{18} \quad = \quad & \int \dots \int \frac{\mathrm{d}^D k_1 \dots \mathrm{d}^D k_4}{(-(k_1+p_1)^2)^{a_1}(-(k_2+p_1)^2)^{a_2}(-(k_3+p_1)^2)^{a_3}} \\
& \times \frac{1}{(-(k_4+p_1)^2)^{a_4}(-(k_1-p_2)^2)^{a_5}(-(k_2-p_2)^2)^{a_6}(-(k_3-p_2)^2)^{a_7}} \\
& \times \frac{1}{(-(k_4-p_2)^2)^{a_8}(-k_1^2)^{a_9}(-k_2^2)^{a_{10}}(-k_3^2)^{a_{11}}(-k_4^2)^{a_{12}}} \\
& \times \frac{1}{(-(k_1-k_2)^2)^{-a_{13}}(-(k_1-k_3)^2)^{-a_{14}}(-(k_1-k_4)^2)^{-a_{15}}} \\
& \times \frac{1}{(-(k_2-k_3)^2)^{-a_{16}}(-(k_2-k_4)^2)^{-a_{17}}(-(k_3-k_4)^2)^{-a_{18}}} \,.
\end{aligned}
$$

At most 12 indices can be positive.

FIRE $\rightarrow$ 99 master integrals.

FIRE $\rightarrow$ 99 master integrals.

[J. Henn, A.&V. Smirnov'13]: introduce an additional scale.
$p_2^2 \neq 0$

FIRE $\rightarrow$ 99 master integrals.

[J. Henn, A.&V. Smirnov'13]: introduce an additional scale.
$p_2^2 \neq 0$

504 master integrals

FIRE $\rightarrow$ 99 master integrals.

[J. Henn, A.&V. Smirnov'13]: introduce an additional scale.
$p_2^2 \neq 0$

504 master integrals

Use differential equations
[A.V. Kotikov'91, Bern, Dixon & Kosower'94, E. Remiddi'97, T. Gehrmann & E. Remiddi'00, J. Henn'13,...]

FIRE $\rightarrow$ 99 master integrals.

[J. Henn, A.&V. Smirnov'13]: introduce an additional scale.
$p_2^2 \neq 0$

504 master integrals

Use differential equations
[A.V. Kotikov'91, Bern, Dixon & Kosower'94, E. Remiddi'97, T. Gehrmann & E. Remiddi'00, J. Henn'13,...]

FIRE $\rightarrow$ 99 master integrals.

[J. Henn, A.&V. Smirnov'13]: introduce an additional scale.
$p_2^2 \neq 0$

504 master integrals

Use differential equations
[A.V. Kotikov'91, Bern, Dixon & Kosower'94, E. Remiddi'97, T. Gehrmann & E. Remiddi'00, J. Henn'13,...]

[J. Henn'13]: use basis of integrals with constant leading singularities

Let $f = (f_1, \ldots, f_N)$ be *primary* master integrals (MI) for a given family of dimensionally regularized (with $D = 4 - 2\epsilon$) Feynman integrals.

Let $f = (f_1, \ldots, f_N)$ be *primary* master integrals (MI) for a given family of dimensionally regularized (with $D = 4 - 2\epsilon$) Feynman integrals.

Let $x = (x_1, \ldots, x_n)$ be kinematical variables and/or masses, or some new variables introduced to 'get rid of square roots'.

Let $f = (f_1, \ldots, f_N)$ be *primary* master integrals (MI) for a given family of dimensionally regularized (with $D = 4 - 2\epsilon$) Feynman integrals.

Let $x = (x_1, \ldots, x_n)$ be kinematical variables and/or masses, or some new variables introduced to 'get rid of square roots'.

DE:

$$\partial_i f(\epsilon, x) = A_i(\epsilon, x) f(\epsilon, x)\,,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and each A_i is an $N \times N$ matrix.

Let $f = (f_1, \ldots, f_N)$ be *primary* master integrals (MI) for a given family of dimensionally regularized (with $D = 4 - 2\epsilon$) Feynman integrals.

Let $x = (x_1, \ldots, x_n)$ be kinematical variables and/or masses, or some new variables introduced to 'get rid of square roots'.

DE:

$$\partial_i f(\epsilon, x) = A_i(\epsilon, x) f(\epsilon, x) \,,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and each A_i is an $N \times N$ matrix.

[J. Henn'13]: turn to a new basis where DE take the form

$$\partial_i f(\epsilon, x) = \epsilon \, A_i(x) f(\epsilon, x) \,.$$

In the case of two scales, i.e. with one variable in the DE, i.e. $n = 1$.

$$f'(\epsilon, x) = \epsilon \sum_k \frac{a_k}{x - x^{(k)}} f(\epsilon, x) \, .$$

where $x^{(k)}$ is the set of singular points of the DE and $N \times N$ matrices a_k are independent of x and ϵ.

In the case of two scales, i.e. with one variable in the DE, i.e. $n = 1$.

$$f'(\epsilon, x) = \epsilon \sum_k \frac{a_k}{x - x^{(k)}} f(\epsilon, x) \,.$$

where $x^{(k)}$ is the set of singular points of the DE and $N \times N$ matrices a_k are independent of x and ϵ.

For example, if $x_k = 0, -1, 1$ then results are expressed in terms of HPLs [E. Remiddi & J.A.M. Vermaseren]

$$H(a_1, a_2, \ldots, a_n; x) = \int_0^x f(a_1; t) H(a_2, \ldots, a_n; t) \mathrm{d}t \,,$$

where $f(\pm 1; t) = 1/(1 \mp t), \;\; f(0; t) = 1/t$

How to turn to a UT basis?

How to turn to a UT basis?

- [J. Henn'13] Select basis integrals that have constant leading singularities [F. Cachazo'08] which are multidimensional residues of the integrand. (Replace propagators by delta functions).
  Based on experience in super Yang-Mills and a conjecture by [N. Arkani-Hamed et al.'12]
- Transformations of the system of differential equations based on its singularities
  [Moser'59],[J. Henn'14],[R.N. Lee'14]

How to turn to a UT basis?

- [J. Henn'13] Select basis integrals that have constant leading singularities [F. Cachazo'08] which are multidimensional residues of the integrand. (Replace propagators by delta functions).
  Based on experience in super Yang-Mills and a conjecture by [N. Arkani-Hamed et al.'12]
- Transformations of the system of differential equations based on its singularities
  [Moser'59],[J. Henn'14],[R.N. Lee'14]

How to turn to a UT basis?

- [J. Henn'13] Select basis integrals that have constant leading singularities [F. Cachazo'08] which are multidimensional residues of the integrand. (Replace propagators by delta functions).
  Based on experience in super Yang-Mills and a conjecture by [N. Arkani-Hamed et al.'12]
- Transformations of the system of differential equations based on its singularities
  [Moser'59],[J. Henn'14],[R.N. Lee'14]
- If you have almost reached the ε-form, make a small final rotation of the current basis.
  See, e.g., [S. Caron-Huot, J. Henn,'14], [T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs'14]

We obtain differential equations with respect to $x = p_2^2/p_3^2$

$$\partial_x f(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1-x} \right] f(x,\epsilon)$$

where a and b are x- and ϵ-independent 504×504 matrices.

We obtain differential equations with respect to $x = p_2^2/p_3^2$

$$\partial_x f(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1-x}\right] f(x,\epsilon)$$

where a and b are x- and ϵ-independent 504×504 matrices.

Solving these equations in terms of HPL with letters 0 and 1.

Asymptotic behaviour at the points $x = 0$ and $x = 1$

$$f(x,\epsilon) \overset{x\to 0}{=} \left[1 + \sum_{k\geq 1} p_k(\epsilon) x^k\right] x^{\epsilon a} f_0(\epsilon)\,,$$

$$f(x,\epsilon) \overset{x\to 1}{=} \left[1 + \sum_{k\geq 1} q_k(\epsilon) (1-x)^k\right] (1-x)^{-\epsilon b} f_1(\epsilon)\,,$$

Asymptotic behaviour at the points $x = 0$ and $x = 1$

$$f(x,\epsilon) \overset{x\to 0}{=} \left[1 + \sum_{k\geq 1} p_k(\epsilon) x^k\right] x^{\epsilon a} f_0(\epsilon)\,,$$

$$f(x,\epsilon) \overset{x\to 1}{=} \left[1 + \sum_{k\geq 1} q_k(\epsilon)(1-x)^k\right] (1-x)^{-\epsilon b} f_1(\epsilon)\,,$$

Natural boundary conditions at the point $x = 1$:
there is no singularity and it corresponds to propagator-type integrals which are known
[P. Baikov and K. Chetyrkin'10; R. Lee and V. Smirnov'11]

Asymptotic behaviour at the points $x = 0$ and $x = 1$

$$f(x,\epsilon) \overset{x\to 0}{=} \left[1 + \sum_{k\geq 1} p_k(\epsilon) x^k\right] x^{\epsilon a} f_0(\epsilon)\,,$$

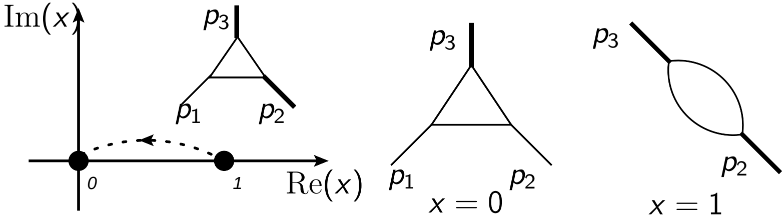
$$f(x,\epsilon) \overset{x\to 1}{=} \left[1 + \sum_{k\geq 1} q_k(\epsilon)(1-x)^k\right] (1-x)^{-\epsilon b} f_1(\epsilon)\,,$$

Natural boundary conditions at the point $x = 1$:
there is no singularity and it corresponds to propagator-type integrals which are known
[P. Baikov and K. Chetyrkin'10; R. Lee and V. Smirnov'11]

To obtain analytical results for the 99 one-scale MI, we perform (with the help of the HPL package [D. Maître'06]) matching at the point $x = 0$.

Transporting boundary conditions at $x = 1$ to the point $x = 0$.

Transporting boundary conditions at $x = 1$ to the point $x = 0$.



In mathematics, the object transporting the boundary information is called the Drinfeld associator.

In mathematics, the object transporting the boundary information is called the Drinfeld associator.
We construct the Drinfeld associator perturbatively in ϵ.

In mathematics, the object transporting the boundary information is called the Drinfeld associator.
We construct the Drinfeld associator perturbatively in ϵ.

$$x^{\epsilon a} = \sum x^{j\varepsilon} a_j$$

Results for the 99 one-scale MI can be obtained from the 'naive' part of the asymptotic expansion at $x = 0$, i.e. from the part corresponding to the zero eigenvalue of the matrix a, i.e. to $j = 0$.

In mathematics, the object transporting the boundary information is called the Drinfeld associator.
We construct the Drinfeld associator perturbatively in ϵ.

$$x^{\epsilon a} = \sum x^{j\varepsilon} a_j$$

Results for the 99 one-scale MI can be obtained from the 'naive' part of the asymptotic expansion at $x = 0$, i.e. from the part corresponding to the zero eigenvalue of the matrix a, i.e. to $j = 0$.

Two alternative descriptions of the asymptotic expansion: with DE and with with expansion by regions
[M. Beneke and V. Smirnov'98]

In mathematics, the object transporting the boundary information is called the Drinfeld associator.
We construct the Drinfeld associator perturbatively in ϵ.

$$x^{\epsilon a} = \sum x^{j\varepsilon} a_j$$

Results for the 99 one-scale MI can be obtained from the 'naive' part of the asymptotic expansion at $x = 0$, i.e. from the part corresponding to the zero eigenvalue of the matrix a, i.e. to $j = 0$.

Two alternative descriptions of the asymptotic expansion: with DE and with with expansion by regions
[M. Beneke and V. Smirnov'98]

$j = 0$ corresponds to the hard-. . . -hard region while terms with $j < 0$ to other regions (soft, collinear, . . .).

In mathematics, the object transporting the boundary information is called the Drinfeld associator.
We construct the Drinfeld associator perturbatively in ϵ.

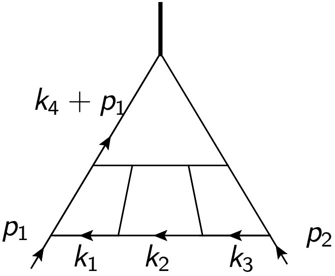
$$x^{\epsilon a} = \sum x^{j\varepsilon} a_j$$

Results for the 99 one-scale MI can be obtained from the 'naive' part of the asymptotic expansion at $x = 0$, i.e. from the part corresponding to the zero eigenvalue of the matrix a, i.e. to $j = 0$.

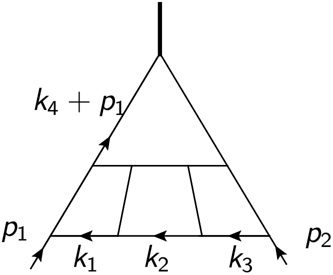
Two alternative descriptions of the asymptotic expansion: with DE and with with expansion by regions [M. Beneke and V. Smirnov'98]

$j = 0$ corresponds to the hard-…-hard region while terms with $j < 0$ to other regions (soft, collinear, …). No positive j.

An example of our result



An example of our result



$$I_{12} = \int \dots \int \prod_{j=1}^{4} \mathrm{d}^D k_j \, \frac{(k_4^2)^2}{k_1^2 k_2^2 k_3^2 (k_1 - k_2)^2 (k_2 - k_3)^2 (k_1 - k_4)^2}$$

$$\times \frac{1}{(k_2 - k_4)^2 (k_3 - k_4)^2 (k_1 + p_1)^2 (k_4 + p_1)^2 (k_4 - p_2)^2 (k_3 - p_2)^2}$$

$$= \frac{1}{576\varepsilon^8} + \frac{1}{216}\pi^2\frac{1}{\epsilon^6} + \frac{151}{864}\zeta_3\frac{1}{\epsilon^5} + \frac{173}{10368}\pi^4\frac{1}{\epsilon^4} + \left[\frac{505}{1296}\pi^2\zeta_3 + \frac{5503}{1440}\zeta_5\right]\frac{1}{\epsilon^3} +$$
$$+ \left[\frac{6317}{155520}\pi^6 + \frac{9895}{2592}\zeta_3^2\right]\frac{1}{\epsilon^2} + \left[\frac{89593}{77760}\pi^4\zeta_3 + \frac{3419}{270}\pi^2\zeta_5 - \frac{169789}{4032}\zeta_7\right]\frac{1}{\epsilon}$$
$$+ \left[\frac{407}{15}s_{8a} + \frac{41820167}{653184000}\pi^8 + \frac{41719}{972}\pi^2\zeta_3^2 - \frac{263897}{2160}\zeta_3\zeta_5\right] + \mathcal{O}(\epsilon)\,,$$

- General N_c. Non-planar diagrams

- General N_c. Non-planar diagrams
- Other form-factors. More complicated integrals.

- General N_c. Non-planar diagrams
- Other form-factors. More complicated integrals.
- No conceptual problems. More powerful algorithms for IBP reduction. More powerful machines.