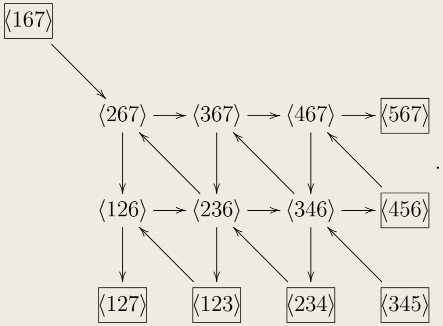
Landau Singularities and Symbology



Anastasia Volovich

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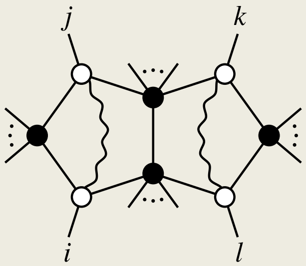


Nordita, June 2016

1512.07909

with Tristan Dennen and Marcus Spradlin

Introduction

- A general goal of modern S-matrix program is to compute amplitudes with minimal effort.
- This relies on understanding both the **physical** principles they satisfy and **mathematical** properties they have.
- In best case scenarios, we would like these conditions to determine amplitudes **uniquely**.

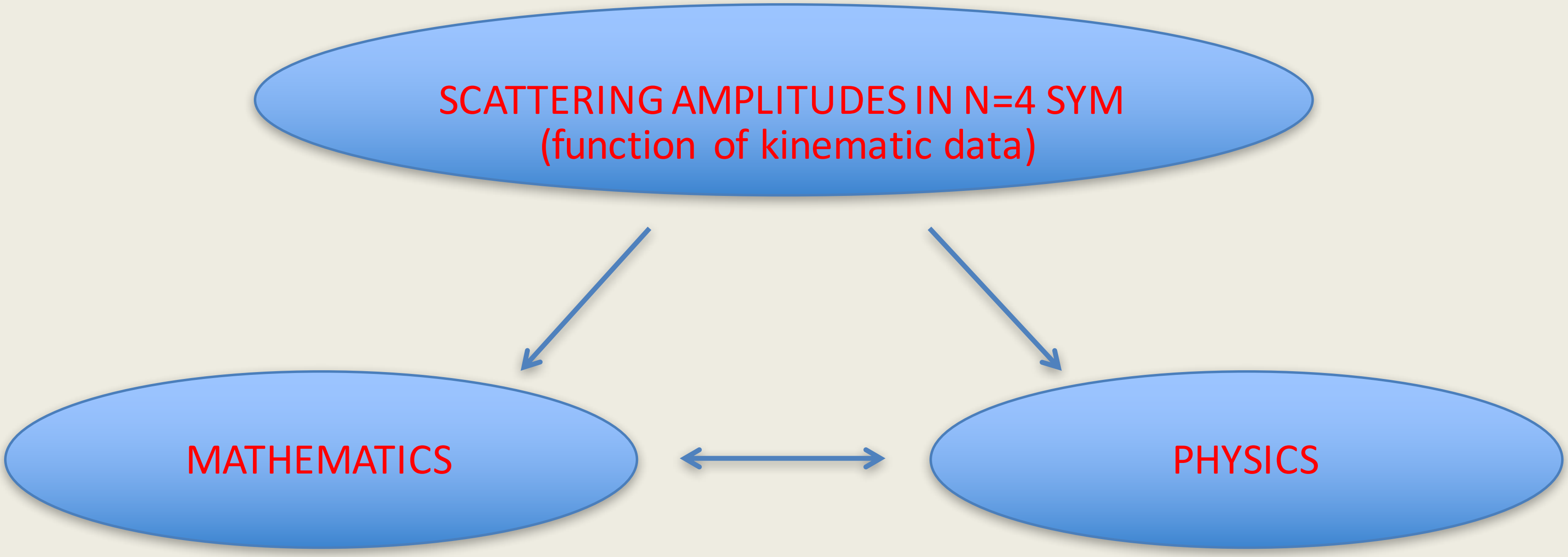
- Our previous work has revealed that the mathematical structure of N=4 Yang-Mills amplitudes is at least partially dictated by **cluster algebra** structure.

Goncharov, Spradlin, Vergu; Golden, Paulos; Parker, Scherlis, AV

- The goal of my talk is to explore the most basic physical principle --- locality, expressed through the Landau equations --- from which we will see cluster coordinates emerge.

Dennen, Spradlin, AV 1512.07909

Motivated by Maldacena, Simons-Duffin, Zhiboedov 1509.03612



Cluster Structure

Landau Singularities

THE GOAL OF OUR WORK IS TO EXPLORE THE CONNECTION

BETWEEN CLUSTER COORDINATES AND LANDAU SINGULARITIES

Plan

- Landau equations
- Solution: one and two-loop n-point MHV in N=4 SYM
- Connection with symbol alphabet of an amplitude
- Connection with cluster structure of an amplitude
- Conclusions and open questions

Landau Singularities

Landau equations for a given Feynman integral are a set of kinematic constraints that are necessary for the appearance of a pole or branch point in the integrated function

$$I = c \int \prod_{r=1}^{L} d^D \ell_r \int_{\alpha_i \geq 0} d^\nu \alpha \, \delta(1 - \sum_{i=1}^{\nu} \alpha_i) \frac{\mathcal{N}(\ell_r^\mu, p_i^\mu, \ldots)}{\mathcal{D}^\nu}$$

$$\mathcal{D} = \sum_{i=1}^{\nu} \alpha_i (q_i^2 - m_i^2) \,,$$

In this talk:
only focus on singularities describes by Landau equation

Landau Equations

$$\sum_{i \in \text{loop}} \alpha_i q_i^\mu = 0 \quad \forall \text{ loops},$$

$$\alpha_i (q_i^2 - m_i^2) = 0 \quad \forall i.$$

Landau 1959
Eden, Landshoff, Olive, Polkinghorne
"The Analytic S-Matrix"

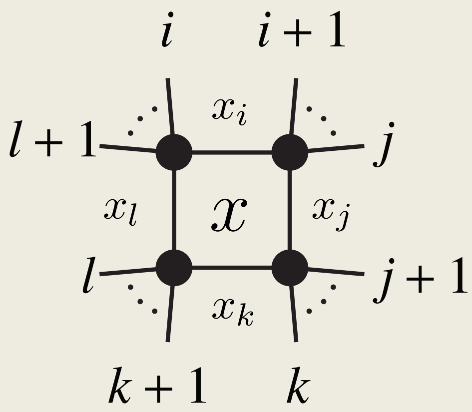
Landau Singularities
locus in external kinematic data
where Landau equations admit solutions

Leading LS	all $\alpha_i \neq 0$	LLS
Subleading LS	some $\alpha_i = 0$	$SLLS$, S^2LLS etc

One-Loop Box

Landau 1959
Eden, Landshoff, Olive, Polkinghorne
"The Analytic S-Matrix"

The Landau equations are easily solved for one-loop box integrals in four dimensions.
(and bubbles and triangles)



$$p_i^\mu = x_i^\mu - x_{i-1}^\mu$$

$$x_{ij}^2 \equiv (x_i - x_j)^2$$

$$(x_i - x_j)^2 = (p_{i+1} + p_{i+2} + \cdots + p_j)^2$$

The second Landau equation puts the propagators on-shell
(no constraints on external kinematics).

$$(x - x_i)^2 = 0, \quad (x - x_j)^2 = 0, \quad (x - x_k)^2 = 0, \quad (x - x_l)^2 = 0$$

The solvability of the first equation gives a determinant constraint.

$$\alpha_i(x - x_i) + \alpha_j(x - x_j) + \alpha_k(x - x_k) + \alpha_l(x - x_l) = 0$$

Leading Landau Singularities

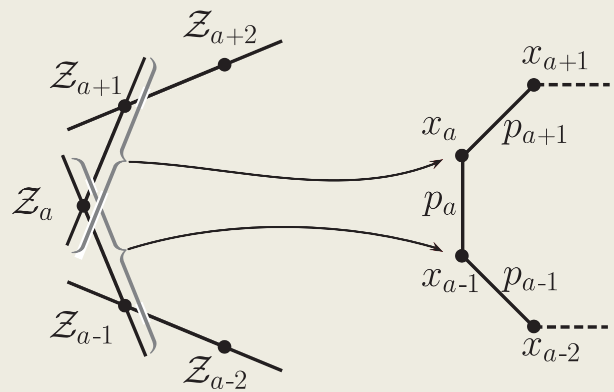
$$0 = (x_{ij}^2 x_{kl}^2 - x_{ik}^2 x_{jl}^2 + x_{il}^2 x_{jk}^2)^2 - 4 x_{ij}^2 x_{jk}^2 x_{kl}^2 x_{il}^2$$

For generic integrals it becomes a hard problem, so next we focus on specific N=4 SYM integrals.

Planar N=4 SYM and Momentum Twistors

Null momentum

$$p_a^\mu \mapsto (p_a)_{\underline{\alpha}\,\underline{\dot\alpha}} \equiv p_a^\mu (\sigma_\mu)_{\underline{\alpha\dot\alpha}} \equiv \lambda^{(a)}_{\underline{\alpha}} \widetilde{\lambda}^{(a)}_{\underline{\dot\alpha}}$$



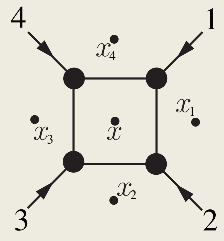
Momentum conservation

$$p_a \equiv x_a - x_{a-1}$$

$$Z = (\lambda, \mu) = (\lambda_\alpha, x_{\alpha\dot\alpha}\lambda^\alpha)$$

$$\langle A\,B\,C\,D\rangle \equiv \epsilon_{IJKL} Z_A^I Z_B^J Z_C^K Z_D^L$$

If x, y are points in Minkowski space associated to two lines $(A, B), (C, D)$ in $\mathbb{P}^3$ $\qquad (x-y)^2 = \frac{\langle A\,B\,C\,D\rangle}{\langle I\,A\,B\rangle\langle I\,C\,D\rangle}$



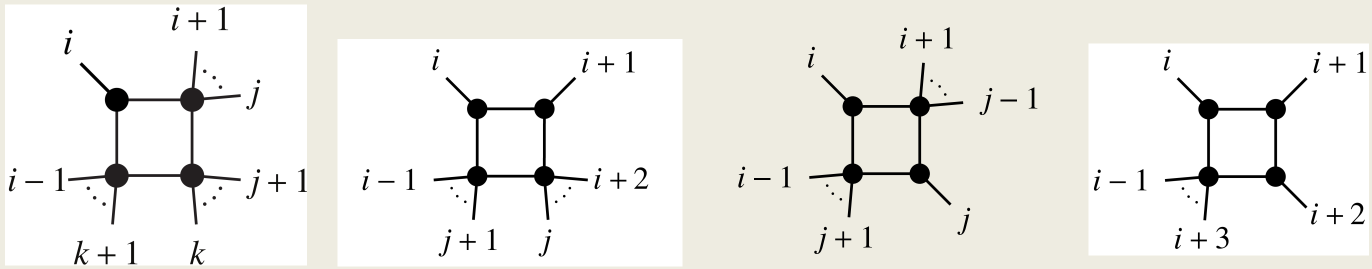
$$= \int d^4x \frac{N}{(x-x_1)^2(x-x_2)^2(x-x_3)^2(x-x_4)^2}$$

$$\int_{AB} \frac{\langle 1234\rangle^2}{\langle AB\,12\rangle\langle AB\,23\rangle\langle AB\,34\rangle\langle AB\,41\rangle}$$

Hodges
Arkani-Hamed, Bourjaily, Cachazo, Trnka

Momentum twistors simplify the problem of analyzing solutions to Landau equations.

One-loop boxes



(b) (c) (d) (e)

LLS

Box (b): $0 = \langle i(i{-}1, i{+}1)(j, j{+}1)(k, k{+}1)\rangle$

Box (c): $0 = \langle i{-}1\; i\; i{+}1\; i{+}2\rangle\langle i\; i{+}1\; j\; j{+}1\rangle$

Box (d): $0 = \langle i\, \bar{j}\rangle\langle \bar{i}\, j\rangle$

Box (e): $0 = \langle i{-}1\; i\; i{+}1\; i{+}2\rangle\langle i\; i{+}1\; i{+}2\; i{+}3\rangle$

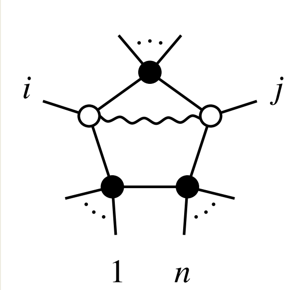
$\bar{a}$ is the plane $(a-1, a, a+1)$

$\langle C(A,B)(D,E)(G,H)\rangle \equiv \langle (A,B,C)\cap(D,E,C)\, G\, H\rangle$

$\langle (A,B,C)\cap(D,E,F)\, G\, H\rangle = \langle A\,B\,C\,G\rangle\langle D\,E\,F\,H\rangle - \langle A\,B\,C\,H\rangle\langle D\,E\,F\,G\rangle$

One-loop n-point MHV in N=4 SYM

$$\frac{\mathcal{A}^{1-\text{loop}}_{\text{MHV}}}{\mathcal{A}^{\text{tree}}_{\text{MHV}}} = \int_{AB} \sum_{1<i<j<n}$$



Bern, Dixon, Dunbar, Kosower

Arkani-Hamed, Bourjaily, Cachazo, Trnka

chiral pentagon

$$\frac{\langle AB\,\bar{i}\cap\bar{j}\rangle\langle i\,j\,n\,1\rangle}{\langle AB\,i{-}1\,i\rangle\langle AB\,i\,i{+}1\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,n\,1\rangle}$$

(LLS)

$$\langle i\,j\,n\,1\rangle\langle n\,1\,\bar{i}\cap\bar{j}\rangle = 0$$

$(A,B) = (i,j)$ or $(A,B) = \bar{i}\cap\bar{j}$

$\langle AB\,n\,1\rangle = 0$

(SLLS)

$$\begin{aligned}
&\langle j(j{-}1,j{+}1)(i,i{+}1)(n,1)\rangle = 0\,,\\
&\langle j(j{-}1,j{+}1)(i{-}1,i)(n,1)\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j,j{+}1)(n,1)\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j{-}1,j)(n,1)\rangle = 0\,,\\
&\langle \bar{i}j\rangle\langle i\bar{j}\rangle = 0\,.
\end{aligned}$$

Reduces to boxes

(S²LLS)

$$\begin{aligned}
&\langle i{-}1\;i\;j{-}1\;j\rangle\langle j{-}1\;j\;n\;1\rangle\langle n\;1\;i{-}1\;i\rangle = 0\,,\\
&\langle i\;i{+}1\;j{-}1\;j\rangle\langle j{-}1\;j\;n\;1\rangle\langle n\;1\;i\;i{+}1\rangle = 0\,,\\
&\langle i{-}1\;i\;j\;j{+}1\rangle\langle j\;j{+}1\;n\;1\rangle\langle n\;1\;i{-}1\;i\rangle = 0\,,\\
&\langle i\;i{+}1\;j\;j{+}1\rangle\langle j\;j{+}1\;n\;1\rangle\langle n\;1\;i\;i{+}1\rangle = 0\,.
\end{aligned}$$

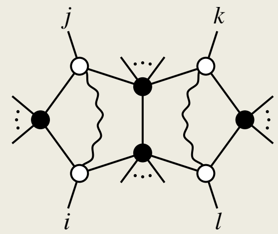
Reduces to triangles

Dennen, Spradlin, AV

Two-loop n-point MHV in N=4 SYM

Arkani-Hamed, Bourjaily, Cachazo, Trnka

$$\frac{\mathcal{A}^{2-\text{loop}}_{\text{MHV}}}{\mathcal{A}^{\text{tree}}_{\text{MHV}}} = \int_{AB}\int_{CD} \frac{1}{2} \sum_{i<j<k<l<i}$$



$$\frac{\langle i\, j\, k\, l\rangle}{\langle ABCD\rangle} \times \frac{\langle AB\, \bar{i}\cap\bar{j}\rangle}{\langle AB\, i{-}1\, i\rangle\langle AB\, i\, i{+}1\rangle\langle AB\, j{-}1\, j\rangle\langle AB\, j\, j{+}1\rangle} \times \frac{\langle CD\, \bar{k}\cap\bar{l}\rangle}{\langle CD\, k{-}1\, k\rangle\langle CD\, k\, k{+}1\rangle\langle CD\, l{-}1\, l\rangle\langle CD\, l\, l{+}1\rangle}$$

chiral double pentagon

(LLS)

$$\langle i\ j\ k\ l\rangle\langle i\ j\ \bar{k}\cap\bar{l}\rangle\langle\bar{i}\cap\bar{j}\ k\ l\rangle\langle\bar{i}\cap\bar{j}\ \bar{k}\cap\bar{l}\rangle = 0 \qquad (A,B) = (i,j) \text{ or } \bar{i}\cap\bar{j} \quad \text{and} \quad (C,D) = (k,l) \text{ or } \bar{k}\cap\bar{l}\,.$$

(SLLS)

$$\begin{aligned}
&\langle j(j{-}1,j{+}1)(i{-}1,i)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\ \bar{k}\cap\bar{l}\rangle = 0\,,\\
&\langle j(j{-}1,j{+}1)(i,i{+}1)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\ \bar{k}\cap\bar{l}\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j{-}1,j)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\ \bar{k}\cap\bar{l}\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j,j{+}1)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\ \bar{k}\cap\bar{l}\rangle = 0\,.
\end{aligned}$$

Dennen, Spradlin, AV

$$\langle i\bar{j}\rangle\langle\bar{i}j\rangle = 0 \quad \text{and} \quad \langle k\bar{l}\rangle\langle\bar{k}l\rangle = 0$$

(S²LLS)

$$\begin{aligned}
&\langle i\ i{+}1\ j{-}1\ j\rangle\langle j{-}1\ j\ k\ l\rangle\langle k\ l\ i\ i{+}1\rangle\langle j{-}1\ j\ \bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\ i\ i{+}1\rangle = 0\,, &\langle\bar{i}\cap(i,j{-}1,j)\ \bar{l}\cap(k,k{+}1,l)\rangle = 0\,,\\
&\langle i{-}1\ i\ j{-}1\ j\rangle\langle j{-}1\ j\ k\ l\rangle\langle k\ l\ i{-}1\ i\rangle\langle j{-}1\ j\ \bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\ i{-}1\ i\rangle = 0\,, &\langle\bar{i}\cap(i,j,j{+}1)\ \bar{l}\cap(k,k{+}1,l)\rangle = 0\,,\\
&\langle i\ i{+}1\ j\ j{+}1\rangle\langle j\ j{+}1\ k\ l\rangle\langle k\ l\ i\ i{+}1\rangle\langle j{-}1\ j\ \bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\ i{-}1\ i\rangle = 0\,, &\langle\bar{i}\cap(i,j{-}1,j)\ \bar{l}\cap(k{-}1,k,l)\rangle = 0\,,\\
&\langle i{-}1\ i\ j\ j{+}1\rangle\langle j\ j{+}1\ k\ l\rangle\langle k\ l\ i{-}1\ i\rangle\langle j{-}1\ j\ \bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\ i{-}1\ i\rangle = 0\,, &\langle\bar{i}\cap(i,j,j{+}1)\ \bar{l}\cap(k{-}1,k,l)\rangle = 0\,,
\end{aligned}$$

(S³LLS)

It would have been very difficult to solve Landau equations without momentum twistors.

- We have produced a long list of Landau singularities for one and two-loop N=4 SYM integrals.

- For amplitudes of generalized polylogarithm form there should be a close connection between Landau singularities and symbol alphabet.

Maldacena, Simons-Duffin, Zhiboedov

Abreu, Britto, Duhr, Gardi, Gronqvist

Symbol and Singularities

Many of the simplest (and hence best understood) amplitudes can be expressed in terms of a class of generalized polylogs defined by iterated integrals

$$Li_k(z) = \int_0^z Li_{k-1}(t) d\log t \qquad Li_1(z) = -\log(1-z)$$

$$G(a_k, a_{k-1}, \ldots; z) = \int_0^z G(a_{k-1}, \ldots; t) \frac{dt}{t - a_k}, \qquad G(z) \equiv 1$$

Example: 2-loop 6-point MHV

GSVV

$$R_6^{(2)} = \sum_{\text{cyclic}} \text{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\text{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

$+$ products of $\text{Li}_k(-x)$ functions of lower weight

$$v_1 = \frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}, \qquad v_2 = \frac{\langle 1235\rangle\langle 2456\rangle}{\langle 1256\rangle\langle 2345\rangle}, \qquad v_3 = \frac{\langle 1356\rangle\langle 2346\rangle}{\langle 1236\rangle\langle 3456\rangle},$$

$$x_1^+ = \frac{\langle 1456\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3456\rangle}, \qquad x_2^+ = \frac{\langle 1346\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 3456\rangle}, \qquad x_3^+ = \frac{\langle 1236\rangle\langle 1245\rangle}{\langle 1234\rangle\langle 1256\rangle},$$

$$x_1^- = \frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}, \qquad x_2^- = \frac{\langle 1256\rangle\langle 1346\rangle}{\langle 1236\rangle\langle 1456\rangle}, \qquad x_3^- = \frac{\langle 1245\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2345\rangle},$$

$$x_a^{\pm} = \frac{u_a}{2u_1u_2u_3}\left(u_1+u_2+u_3-1 \pm \sqrt{(u_1+u_2+u_3-1)^2 - 4u_1u_2u_3}\right)$$

$$v_a = \frac{1}{u_a} - 1$$

$$u_a = \frac{(p_a + p_{a+1})^2 (p_{a+3} + p_{a+4})^2}{(p_a + p_{a+1} + p_{a+2})^2 (p_{a+2} + p_{a+3} + p_{a+4})^2}$$

Example: 2-loop 6-point MHV

- Function

$$R_6^{(2)} = \sum_{\text{cyclic}} \text{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\text{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$
$$+ \text{ products of } \text{Li}_k(-x) \text{ functions of lower weight}$$

GSVV

- Symbol : much of the information about the analytic structure of such function is captured in an object called symbol

$$\langle 1256\rangle \otimes \langle 1346\rangle \otimes \langle 1246\rangle \otimes \langle 1456\rangle + \cdots$$

7272 terms

Symbol of Transcendental Function

Goncharov, Spradlin, Vergu, AV

$$T_k \to S(T_k) = R_1 \otimes \cdots \otimes R_k$$

Symbol is an element of the k-fold tensor product of the multiplicative group of rational functions.

$$dT_k = \sum_i T^i_{k-1} d\log R_i \to S(T_k) = \sum_i S(T^i_{k-1}) \otimes R_i$$

$$log(R) \to R$$

$$Li_2(R) \to -(1-R) \otimes R$$

Symbol converts polylog functional equation into rational function identities.

Very useful for practical computations.

Symbol and Singularities

- Much of the information about the analytic structure of such function is captured in an object called symbol.
- We expect that the symbol entries appearing in any amplitude should be such that their zeros specify values of the external momenta where solutions of the Landau equations exist.

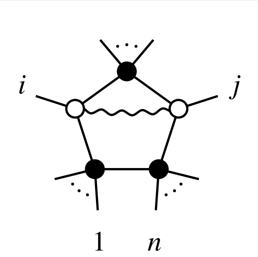
Maldacena, Simons-Duffin, Zhiboedov

Abreu, Britto, Duhr, Gardi, Gronqvist

One-loop n-point MHV in N=4 SYM

Bern, Dixon, Dunbar, Kosower

Arkani-Hamed, Bourjaily, Cachazo, Trnka

$\int_{AB}$ 

$$= \mathrm{Li}_2\left(1-u_{n,i-1,i,j}\right)-\mathrm{Li}_2\left(1-u_{j,n,i,j-1}\right)-\mathrm{Li}_2\left(1-u_{i,j-1,n,i-1}\right)$$
$$-\mathrm{Li}_2\left(1-u_{i,j-1,n,i-1}\right)+\mathrm{Li}_2\left(1-u_{i,j-1,j,i-1}\right)$$
$$+\log\left(u_{j,n,i-1,j-1}\right)\log\left(u_{n,i-1,i,j}\right)$$

$$u_{i,j,k,l}=\frac{\langle i\,i{+}1\,j\,j{+}1\rangle\langle k\,k{+}1\,l\,l{+}1\rangle}{\langle l\,l{+}1\,j\,j{+}1\rangle\langle k\,k{+}1 i\,i{+}1\rangle}=\frac{x_{ij}^2x_{kl}^2}{x_{lj}^2x_{ki}^2}.$$

Symbol:

First Entry

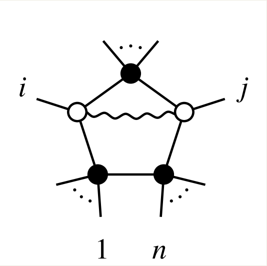
$$\langle i{-}1\;i\;j{-}1\;j\rangle\,,\quad \langle i{-}1\;i\;j\;j{+}1\rangle\,,\quad \langle i{-}1\;i\;n\;1\rangle\,,\quad \langle i\;i{+}1\;j{-}1\;j\rangle$$
$$\langle i\;i{+}1\;j\;j{+}1\rangle\,,\quad \langle i\;i{+}1\;n\;1\rangle\,,\quad \langle j{-}1\;j\;n\;1\rangle\,,\quad \langle j\;j{+}1\;n\;1\rangle$$

Second Entry

$\langle i{-}1\,i\,n\,1\rangle$, $\langle i\,i{+}1\,n\,1\rangle$, $\langle j{-}1\,j\,n\,1\rangle$ and $\langle j\,j{+}1\,n\,1\rangle$

$$\langle\bar{i}j\rangle\,,\quad \langle i(i{-}1,i{+}1)(j,j{+}1)(n,1)\rangle\,,\quad \langle i(i{-}1,i{+}1)(j{-}1,j)(n,1)\rangle$$
$$\langle\bar{j}i\rangle\,,\quad \langle j(j{-}1,j{+}1)(i,i{+}1)(n,1)\rangle\,,\quad \langle j(j{-}1,j{+}1)(i{-}1,i)(n,1)\rangle$$

One-loop n-point MHV in N=4 SYM



$$= \frac{\langle AB\,\bar{i} \cap \bar{j}\rangle \langle i\,j\,n\,1\rangle}{\langle AB\,i{-}1\,i\rangle\langle AB\,i\,i{+}1\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,n\,1\rangle}$$

(LLS) $\langle i\,j\,n\,1\rangle\langle n\,1\,\bar{i}\cap\bar{j}\rangle = 0$ Prefactor

(SLLS)

$$\begin{aligned}
&\langle j(j{-}1,j{+}1)(i,i{+}1)(n,1)\rangle = 0\,,\\
&\langle j(j{-}1,j{+}1)(i{-}1,i)(n,1)\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j,j{+}1)(n,1)\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j{-}1,j)(n,1)\rangle = 0\,,\\
&\langle \bar{i}j\rangle\langle i\bar{j}\rangle = 0\,.
\end{aligned}$$

Second entries of the symbol

(S²LLS)

$$\begin{aligned}
&\langle i{-}1\ i\ j{-}1\ j\rangle\langle j{-}1\ j\ n\ 1\rangle\langle n\ 1\ i{-}1\ i\rangle = 0\,,\\
&\langle i\ i{+}1\ j{-}1\ j\rangle\langle j{-}1\ j\ n\ 1\rangle\langle n\ 1\ i\ i{+}1\rangle = 0\,,\\
&\langle i{-}1\ i\ j\ j{+}1\rangle\langle j\ j{+}1\ n\ 1\rangle\langle n\ 1\ i{-}1\ i\rangle = 0\,,\\
&\langle i\ i{+}1\ j\ j{+}1\rangle\langle j\ j{+}1\ n\ 1\rangle\langle n\ 1\ i\ i{+}1\rangle = 0\,.
\end{aligned}$$

First/Second
entries of the symbol

Dennen, Spradlin, AV

Two-loop n-point MHV in N=4 SYM

- Explicit analytic results for the chiral double pentagon have only been obtained in n=6.

Drummond, Henn

- Symbol of two-loop n-point MHV amplitude

$$\langle a\, a{+}1\, b\, b{+}1\rangle \quad \otimes \quad \langle a \bar{b}\rangle \quad \otimes \quad \langle a\, a{+}1\, b\, c\rangle \quad \otimes \quad \langle a\, \bar{b}\rangle$$

$$\langle a\, a{+}1\, \bar{b} \cap \bar{c}\rangle$$

$$a, b \in \{i{-}1, i, j{-}1, j, k{-}1, k, l{-}1, l\} \qquad \langle a(a{-}1, a{+}1)(c, c{+}1)(d, d{+}1)\rangle$$

$$a, b \in \{i, j, k, l\} \text{ and } c, d \in \{i{-}1, i, j{-}1, j, k{-}1, k, l{-}1, l\}$$

Caron-Huot

- It can be that individual chiral double pentagon integrals have an even larger symbol alphabet, with nontrivial cancelation in the sum which gives the amplitude.

- All of symbol entries are on the list of Landau singularities.

Dennen, Spradlin, AV

Landau Singularities, Symbology and Cluster Structure

- All symbol entries are Landau singularities.
- Can we make a stronger statement? Why various other Landau singlularities don't appear in the symbol?
- SSLLS involve more complicated four-brackets than those which appear in amplitudes, but they are similar to cluster A-coordinates for the Grassmannian cluster algebra that it relevant to planar SYM.

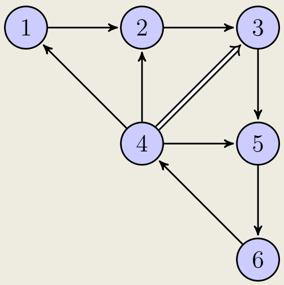
$$\langle \bar{i} \cap (i, j{-}1, j)\ \bar{l} \cap (k, k{+}1, l) \rangle = 0$$
$$\langle \bar{i} \cap (i, j, j{+}1)\ \bar{l} \cap (k, k{+}1, l) \rangle = 0$$
$$\langle \bar{i} \cap (i, j{-}1, j)\ \bar{l} \cap (k{-}1, k, l) \rangle = 0$$
$$\langle \bar{i} \cap (i, j, j{+}1)\ \bar{l} \cap (k{-}1, k, l) \rangle = 0$$

All evidence to date says that for the simplest amplitudes in planar N=4 Yang-Mills symbol entries are cluster coordinates on Gr(4,n).

Goncharov, Spradlin, Vergu; Golden, Paulos, Parker, Scherlis, AV

Quivers and Cluster Algebra

Encode a quiver by a skew-symmetric matrix



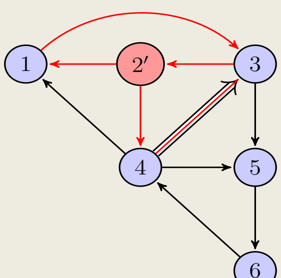
$$b_{ij} = (\#\text{arrows } i \to j) - (\#\text{arrows } j \to i).$$

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

Cluster algebra is defined by

"a set of all cluster coordinates produced via mutations"

- Associate variable a_i (cluster coordinate) to each vertex i
- Define mutation relation at vertex k



$$a_k a_k' = \prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}},$$

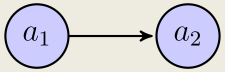
$$a_2' = \frac{1}{a_2}(a_1 a_4 + a_3)$$

Fomin, Zelevinsky 2002

Cluster Algebra Portal

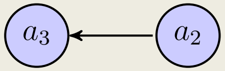
http://www.math.lsa.umich.edu/~fomin/cluster.html

A_2 Cluster Algebra



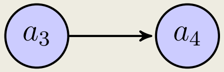
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$a_3 := a_1' = \frac{1}{a_1}\left[\prod_{i\to 1} a_i + \prod_{1\to j} a_j\right] = \frac{1}{a_1}\left[a_1^0 a_2^0 + a_1^0 a_2^1\right] = \frac{1+a_2}{a_1}$$



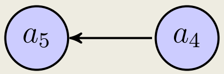
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$a_4 := a_2' = \frac{1}{a_2}\left[\prod_{i\to 2} a_i + \prod_{2\to j} a_2\right] = \frac{1}{a_2}(1+a_3) = \frac{1+a_1+a_2}{a_1 a_2}$$



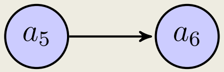
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$a_5 := a_3' = \frac{1}{a_1}\left[\prod_{i\to 3} x^i + \prod_{3\to j} x^j\right] = \frac{1}{a_3}(1+a_4) = \frac{a_1 a_2 + 1 + a_1 + a_2}{a_1 a_2}\frac{a_1}{1+a_2} = \frac{1+a_1}{a_2}$$

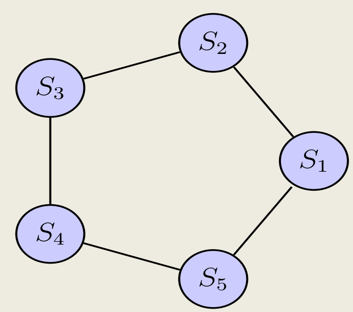


$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$a_6 := a_4' = \frac{1}{a_2}\left[\prod_{i\to 4} a_i + \prod_{4\to j} a_2\right] = \frac{1}{a_4}(a_5+1) = \frac{1+a_1+a_2}{a_2}\frac{a_1 a_2}{1+a_1+a_2} = a_1$$



$$a_1,\ a_2,\ a_3 = \frac{1+a_2}{a_1},\ a_4 = \frac{1+a_1+a_2}{a_1 a_2},\ a_5 = \frac{1+a_1}{a_2}$$



$\{a_1, a_2\}$ $\qquad$ $a_{n+1} = \dfrac{1+a_n}{a_{n-1}}$

What does this have to do with amplitudes?

Amplitudes are functions on

$$\mathrm{Gr}(4,n)/(\mathbb{C}^*)^{n-1} \simeq \mathrm{Conf}_n(\mathbb{P}^3)$$

Drummond, Henn, Korchemsky, Sokatchev

Hodges

Let us look at Grassmannian cluster algebras.

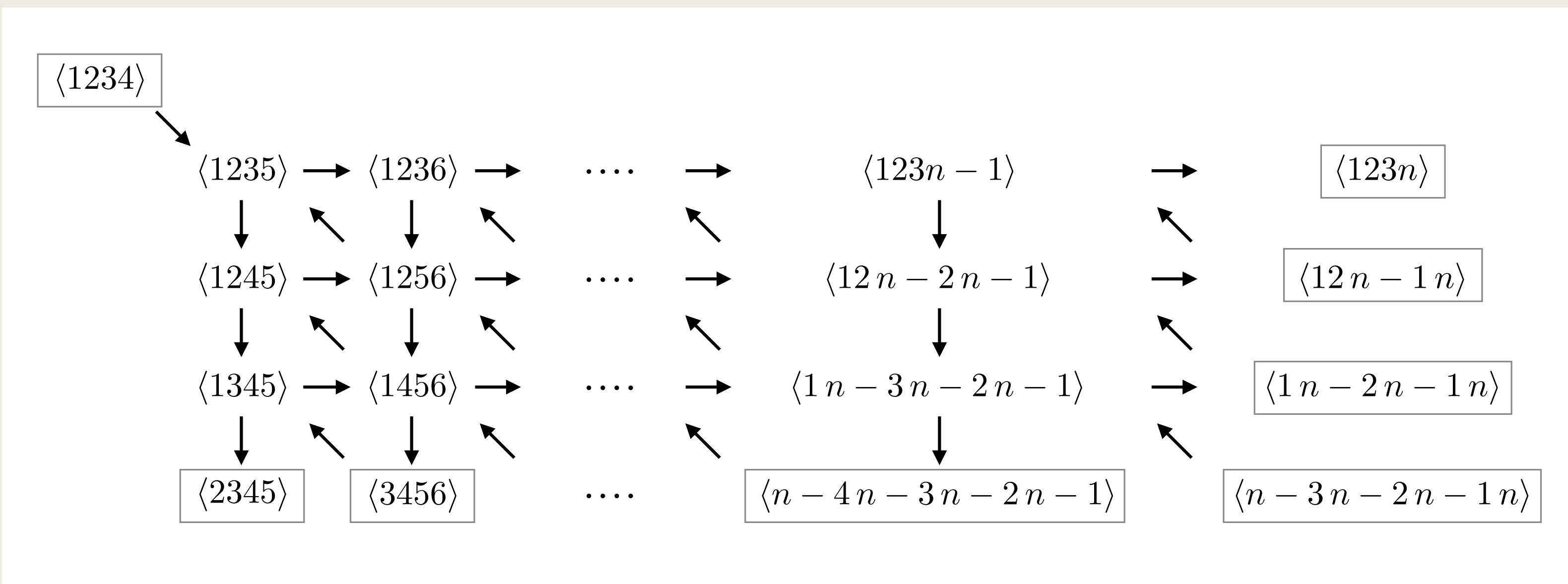
Grassmannian cluster algebras

Gekhtman, Shapiro, Vainshtein

$$Gr(4, n)$$

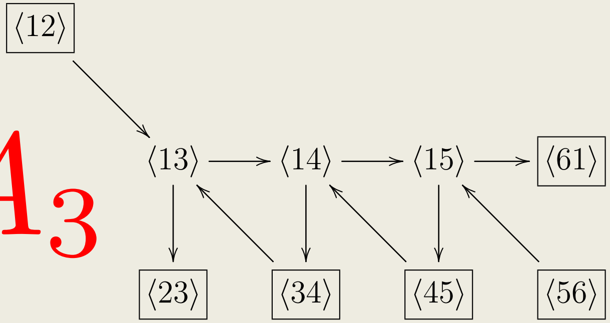
Fomin, Zelevinsky, Scott

3 x (n-5) initial quiver with initial cluster variables which we then mutate to obtain all cluster coordinates



Examples: n=6 & n=7

A_3



$\langle 13\rangle, \langle 14\rangle, \langle 15\rangle,$ $\langle 14\rangle, \langle 15\rangle, \langle 24\rangle,$ $\langle 13\rangle, \langle 15\rangle, \langle 35\rangle,$ $\langle 13\rangle, \langle 14\rangle, \langle 46\rangle,$

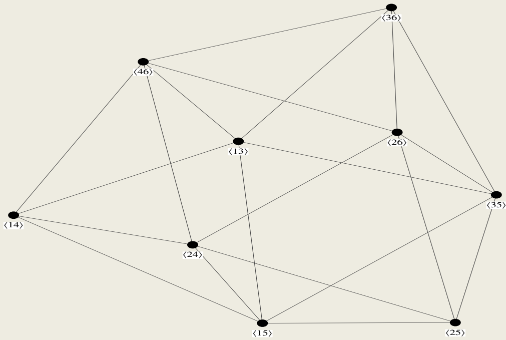
$\langle 15\rangle, \langle 24\rangle, \langle 25\rangle,$ $\langle 14\rangle, \langle 24\rangle, \langle 46\rangle,$ $\langle 15\rangle, \langle 25\rangle, \langle 35\rangle,$ $\langle 13\rangle, \langle 35\rangle, \langle 36\rangle,$

$\langle 13\rangle, \langle 36\rangle, \langle 46\rangle,$ $\langle 24\rangle, \langle 25\rangle, \langle 26\rangle,$ $\langle 24\rangle, \langle 26\rangle, \langle 46\rangle,$ $\langle 25\rangle, \langle 26\rangle, \langle 35\rangle,$

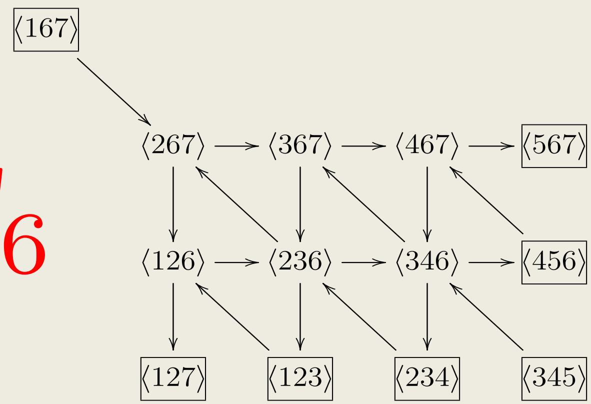
$\langle 26\rangle, \langle 35\rangle, \langle 36\rangle,$ $\langle 26\rangle, \langle 36\rangle, \langle 46\rangle.$

14 quivers
15 coordinates

$\langle i\, i+1\rangle \quad \langle ij\rangle$



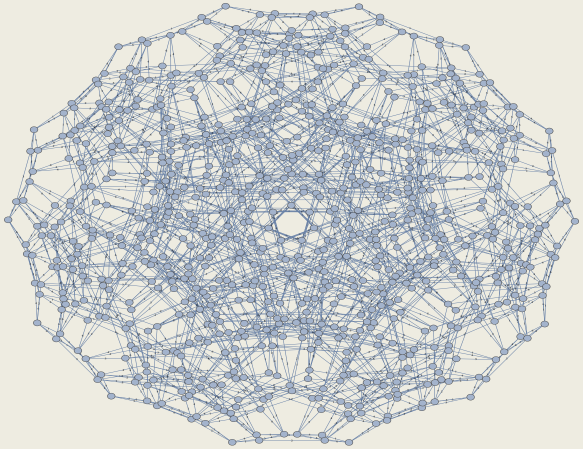
E_6

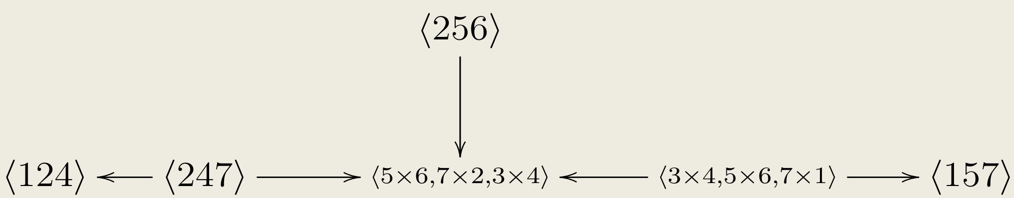


833 quivers
49 coordinates

$$\langle ijk\rangle$$

$\langle 1\times 2, 3\times 4, 5\times 6\rangle,$ $\langle 1\times 2, 3\times 4, 5\times 7\rangle$





$$\langle 1\times 2, 3\times 4, 5\times 6\rangle = \langle 512\rangle\langle 634\rangle - \langle 534\rangle\langle 612\rangle$$

Picture by D. Parker

What do cluster algebras have to do with amplitudes?

Goncharov, Spradlin, Vergu; Golden, Paulos, Parker, Scherlis, AV

- Symbols: all n-point amplitudes in SYM theory have symbol alphabet with subset of cluster A-coordinates on Gr(4,n)
- Coproduct: for two-loop MHV amplitudes, only cluster X-coordinates appear (with particular Poisson brackets) $\frac{\langle 1235\rangle\langle 1278\rangle\langle 2456\rangle\langle 5678\rangle}{\langle 1256\rangle\langle 2578\rangle\langle 78(123)\cap(456)\rangle}$ and $-\frac{\langle 2(13)(56)(78)\rangle\langle 5(12)(46)(78)\rangle}{\langle 1256\rangle\langle 2578\rangle\langle 78(123)\cap(456)\rangle}$
- Functions: there is a particular class of natural functions which exhibit these properties (cluster functions, A_n-functions)

Why is cluster structure useful?

- We can use cluster structure for advancing computations of multi-loop N=4 Yang-Mills amplitudes.

much more in Spradlin's talk

- Examples: 3-loop 7-point symbol,

  2-loop n-point function.

Goncharov, Spradlin, Vergu; Golden, Paulos, Parker, Scherlis, AV

Caron-Huot, He; Dixon, Drummond, Duhr, Henn, Pennington, Von Hippel

Landau Singularities and Cluster Structure

- SSLLS involve more complicated four-brackets then those which appear in MHV amplitudes.
- These brackets resemble cluster A-coordinates in Gr(4,n) cluster algebra.

$$\langle 1246\rangle\langle 1256\rangle\langle 1378\rangle\langle 3457\rangle - \langle 1246\rangle\langle 1257\rangle\langle 1378\rangle\langle 3456\rangle - \langle 1246\rangle\langle 1278\rangle\langle 1356\rangle\langle 3457\rangle + \langle 1278\rangle\langle 1257\rangle\langle 1346\rangle\langle 3456\rangle + \langle 1236\rangle\langle 1278\rangle\langle 1457\rangle\langle 3456\rangle$$

- It will be very interesting to understand the connection in detail.

in progress

Conclusion

- We initiated a study of Landau singularities of Feynman integrals relevant to one- and two-loop MHV amplitudes in N=4 SYM.
- A quantity appears in symbol of some Feynman integral/amplitude only it is Landau singularity.
- At one loop: SSLS/SLS correspond to first/second entries of the symbol.
- At two loops: all symbol entries are Landau singularities. We also found additional solutions which don’t seem to have direct connection with symbol alphabet.

Open questions

- We have only taken first steps in exploring connection between symbology, cluster algebras and Landau singularities.

- Many questions remain:

  - Role of numerator factors in SYM vs Landau singularity analysis
  - Generalization to other cases/non-DCI theories, etc
  - Symbol of integral vs symbol of amplitude
  - Connection between cluster structure and Landau singularities
  - Landau singularities from Amplituhedron

Dennen