Integration-by-parts reductions from unitarity cuts and algebraic geometry

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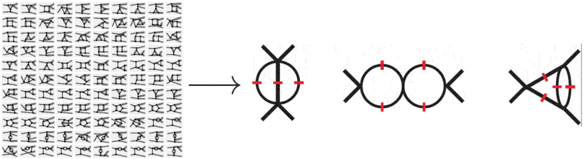


Aspects of Amplitudes
Nordita, June 27, 2016

(based on Phys. Rev. D **93** (2016) 041701 with Yang Zhang)

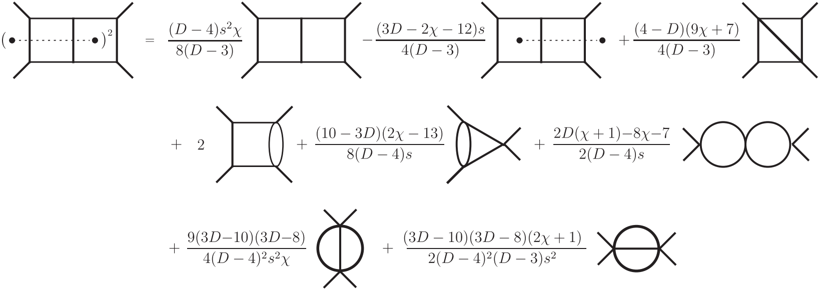
What are integration-by-parts identities?

Integration-by-parts identities arise from the vanishing integration of total derivatives, [Chertyrkin, Tchakov, Nucl. Phys. B 192, 159 (1981)]

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu \, P}{D_1^{a_1} \cdots D_k^{a_k}} \;=\; 0\,.$$

where P and v_j^μ are polynomials in ℓ_i, p_j, and $a_i \in \mathbb{N}$.

An example of an IBP relation ($\chi \equiv t/s$):



How are integration-by-parts identities useful?

Integration-by-parts (IBP) identities play a central role in loop-level QFT calculations.

- **Reduction.** IBP identities enable the reduction of any set of loop integrals to a typically much smaller set of master integrals.

  The reduction is quite dramatic:
  1) $gg \to gg$ at two loops: $\sim 400 \longrightarrow 13$ integrals
  2) $gg \to H$ at N^{3}LO in α_s: $5 \cdot 10^8 \longrightarrow \sim 1000$ integrals.

- **Computing master integrals.** Using IBP reduction, the master integrals $\mathcal{I}_j$ can be computed via differential equations:
  [T. Gehrmann and E. Remiddi, Nucl. Phys., B580, 485 (2000)]

$$\frac{\partial}{\partial x_m}\boldsymbol{\mathcal{I}}(\boldsymbol{x},\epsilon) = A_m(\boldsymbol{x},\epsilon)\boldsymbol{\mathcal{I}}(\boldsymbol{x},\epsilon)\,,$$

where x_m denotes a kinematical invariant.

IBP reductions on unitarity cuts

Key idea: study IBP reductions on generalized-unitarity cuts

$$\frac{1}{D_i} \longrightarrow \delta(D_i)\,, \qquad i \in S$$

where S can be an arbitrary subset of propagators. Also: [Ita, 1510.05626]

The cuts break the construction of IBPs into simpler pieces.

- Any integral missing any of the propagators in S is set to zero by the cut.
- By choosing appropriate sets $S_1, \ldots, S_c$ of cuts we can reconstruct all terms in the IBPs.

I will focus on the two-loop case. A generic integral takes the form

$$I^{(2)} = \int \frac{\mathrm{d}^D \ell_1}{\pi^{D/2}} \frac{\mathrm{d}^D \ell_2}{\pi^{D/2}} \frac{P(\ell_1, \ell_2)}{D_1 \cdots D_k} \, .$$

Decompose $\ell_i = \bar{\ell}_i + \ell_i^{\perp}$ where $\bar{\ell}_i \in \mathbb{R}^{1,3}$, and change to the hyperspherical coordinates $\mu_{ii} \equiv -(\ell_i^{\perp})^2 \geq 0$ and $\mu_{12} \equiv -\ell_1^{\perp} \cdot \ell_2^{\perp}$.

The integral then takes the form

$$I^{(2)} = \frac{2^{D-6}}{\pi^5 \Gamma(D-5)} \int_0^\infty \mathrm{d}\mu_{11} \int_0^\infty \mathrm{d}\mu_{22} \int_{-\sqrt{\mu_{11}\mu_{22}}}^{\sqrt{\mu_{11}\mu_{22}}} \mathrm{d}\mu_{12} \times \left(\mu_{11}\mu_{22} - \mu_{12}^2\right)^{\frac{D-7}{2}} \int \mathrm{d}^4 \bar{\ell}_1 \, \mathrm{d}^4 \bar{\ell}_2 \frac{P(\bar{\ell}_i, \mu_{ij})}{D_1 \cdots D_k} \, .$$

Useful variables

The use of cuts motivates the following choice of variables:

$$z_i \equiv \begin{cases} D_i & 1 \leq i \leq k \\ g_{i-k} & k+1 \leq i \leq m\,. \end{cases}$$

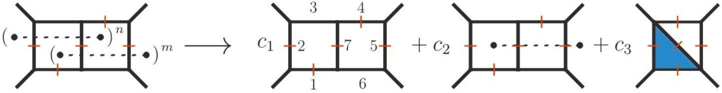
The g_j are irreducible numerator insertions. If the g_j are chosen as $\frac{1}{2}(\ell_i + K_j)^2$, the map $\{\overline{\ell}_i, \mu_{ij}\} \longrightarrow \{z_1, \ldots, z_m\}$ has a *polynomial* inverse.

A generic two-loop integral now reads [Baikov, Phys.Lett. B385 (1996) 404-410]

$$I^{(2)}_{n\geq 5} = \frac{2^{D-6}}{\pi^5 \Gamma(D-5) J} \int \prod_{i=1}^{11} \mathrm{d}z_i \, F(\boldsymbol{z})^{\frac{D-7}{2}} \frac{P(\boldsymbol{z})}{z_1 \cdots z_k}\,.$$

Caveat for multiplicity $n \leq 4$. $\exists \omega : p_i \cdot \omega = 0$. The component of $\overline{\ell}_i$ along ω integrates out, replacing $D - D_c \rightarrow D - (D_c - 1)$ above, and leaving 9 z_j.

Let us find the IBP reductions of the double-box integral. We start by allowing *only integrals which contain all Zurich-flag propagators:*



Define $S_{\text{cut}} = \{1,2,4,5,7\}$. We use the $\widetilde{z}_i$-variables

$$\widetilde{z}_i = D_i\,, \quad i = 1,\ldots,7, \qquad \widetilde{z}_8 = \frac{1}{2}(\ell_1 + p_4)^2\,, \qquad \widetilde{z}_9 = \frac{1}{2}(\ell_2 + p_1)^2\,.$$

After cutting $\frac{1}{\widetilde{z}_i} \to \delta(\widetilde{z}_i)$, $i \in S_{\text{cut}}$, the double-box integral takes the form

$$I^{\text{DB}}_{\text{cut}}[P] \;=\; \int \prod_{i=1}^{9} \mathrm{d}\widetilde{z}_i \frac{F(\widetilde{\boldsymbol{z}})^{\frac{D-6}{2}}}{\widetilde{z}_3\,\widetilde{z}_6} \prod_{j\in S_{\text{cut}}} \delta(\widetilde{z}_j) P(\widetilde{\boldsymbol{z}})\Big|_{\widetilde{z}_{S_{\text{cut}}}=0}\,.$$

As the cut sets $\widetilde{z}_{\{1,2,4,5,7\}}$ to zero, we set $z_{\{1,2,3,4\}} = \widetilde{z}_{\{3,6,8,9\}}$ in the following.

Generic total derivative

After integrating out the delta functions and relabeling we have

$$I_{\text{cut}}^{\text{DB}}[P] \;=\; \int \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}{z_1 z_2} F(\boldsymbol{z})^{\frac{D-6}{2}} P(\boldsymbol{z})\,.$$

An IBP relation corresponds to a total derivative or, equivalently, an exact diff. form. The generic exact diff. form of the form $I_{\text{cut}}^{\text{DB}}$ is

$$\begin{aligned}
0 &= \int \mathrm{d}\left[\sum_{i=1}^{4} \frac{(-1)^{i+1} a_i(\boldsymbol{z}) F(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_i} \wedge \cdots \wedge \mathrm{d}z_4\right] \\
&= \int \left[\sum_{i=1}^{4} \frac{\partial}{\partial z_i}\left(\frac{a_i(\boldsymbol{z}) F(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2}\right)\right] \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4 \\
&= \int \left[\sum_{i=1}^{4}\left(\frac{\partial a_i}{\partial z_i} + \frac{D-6}{2F} a_i \frac{\partial F}{\partial z_i}\right) - \sum_{j=1,2} \frac{a_j}{z_j}\right] \frac{F(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4\,.
\end{aligned}$$

The red term corresponds to an integral in $(D-2)$ dimensions, and the purple term in general produces squared propagators.

IBPs from syzygy equations

To get the generic exact form

$$0 = \int \left[\sum_{i=1}^{4} \left(\frac{\partial a_i}{\partial z_i} + \frac{D-6}{2F} a_i \frac{\partial F}{\partial z_i} \right) - \sum_{j=1,2} \frac{a_j}{z_j} \right] \frac{F(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4$$

to correspond to an IBP relation in D dimensions with only single-power propagators, we demand that each term is *polynomial*,

$$\sum_{i=1}^{4} \frac{D-6}{2F} a_i \frac{\partial F}{\partial z_i} = \widetilde{b} \quad \Longrightarrow \quad \sum_{i=1}^{4} a_i \frac{\partial F}{\partial z_i} + bF = 0 \;\; \left(\text{with} \;\; b = \tfrac{2}{6-D} \widetilde{b}\right)$$

$$a_j = \widetilde{b}_j z_j \quad \Longrightarrow \quad a_j + b_j z_j = 0 \;\; \left(\text{with} \quad b_j = -\widetilde{b}_j\right),$$

with a_i, b_i, b polynomials in z. Such equations, with polynomial solutions, are known in algebraic geometry as *syzygy equations*.

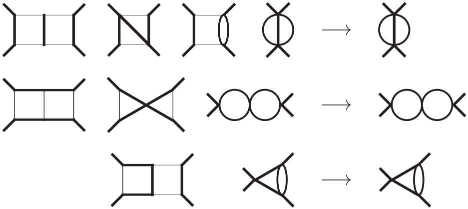
[Gluza, Kajda, Kosower, PRD 83 (2011) 045012], [Schabinger, JHEP 1201 (2012) 077], [Ita, 1510.05626]

$\longrightarrow$ [Harald Ita's talk]

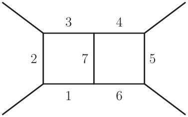
Obtain IBPs by plugging (a_i, b_i, b) into the top equation.
Note: (qa_i, qb_i, qb) is also a solution, for polynomial q.

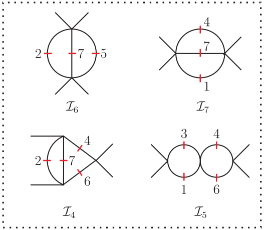
Complete set of cuts for IBPs

To find the complete IBP reduction, we must consider the cuts associated with "uncollapsible" masters:



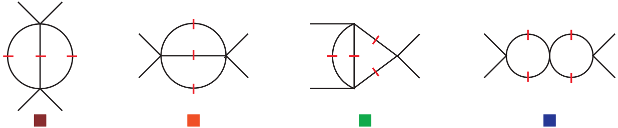
A bit more explicitly, the cuts we need to consider are





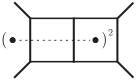
Example of IBP reduction obtained by our approach

By solving the syzygy equations on the following cuts

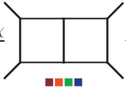


we can reconstruct the complete IBP reduction by merging the partial results.

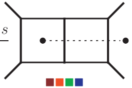
An example of an IBP relation produced by our method ($\chi \equiv t/s$):



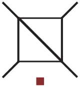
$= \frac{(D-4)s^2\chi}{8(D-3)}$



$-\frac{(3D-2\chi-12)s}{4(D-3)}$



$+\frac{(4-D)(9\chi+7)}{4(D-3)}$



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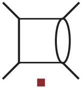
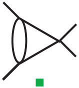
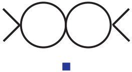


$+\frac{(10-3D)(2\chi-13)}{8(D-4)s}$



$+\frac{2D(\chi+1)-8\chi-7}{2(D-4)s}$



$+\frac{9(3D-10)(3D-8)}{4(D-4)^2s^2\chi}$

$+\frac{(3D-10)(3D-8)(2\chi+1)}{2(D-4)^2(D-3)s^2}$

Overview of our algorithm for generating IBPs

1. **Find a set of masters.** Solve syzygy equations without cuts for numerical external kinematics, then row-reduce linear equations and decide on a set of masters.
2. **Find the subset of uncollapsible masters.** Find the subset of masters with the property that their graphs cannot be obtained by adding propagators to another master.
3. **Solve syzygy equations on cuts.** For each uncollapsible master, solve the syzygy eqs. on the cut S_{cut} where all its propagators are on shell. Multiply $q = \prod_{i \notin S_{\text{cut}}} D_i^{a_i}$ onto the syzygy solutions and feed back into ansatz to find the IBP identities.
4. **Solve IBP identities linearly to get reductions.**

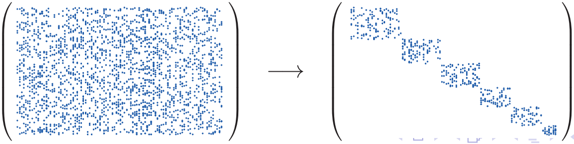
Timing compared to other IBP solvers

CPU time for reduction of $I_{\text{DB}}[(\ell_1 \cdot k_4)^{n_1}(\ell_2 \cdot k_1)^{n_2}]$ with $0 \leq n_i \leq 4$, $0 \leq n_1 + n_2 \leq 6$:

# of external masses	FIRE 5 (in C++ mode)	syzygy approach
zero	350 s	39 s
one	560 s	162 s

The origins of this are presumably

- The variables z_i simplify the syzygy equations.
- The syzygy equations are solved *on cuts*. As a result, fewer variables are involved in the polynomial eqs. This greatly speeds up the step of solving.
- The cuts *block-diagonalize* the linear system to be inverted



Conclusions and Outlook

- We have developed a new general method for generating integration-by-parts reductions.
- The method is based on reconstructing the IBP reductions on a set of generalized cuts, through solving polynomial equations, and merging the partial results.
- In the cases tested so far, the method in its current implementation is roughly a factor of 5 faster than publically available IBP solvers.
- Ongoing work:

  1) optimization of syzygy and linear solving

  2) generalization to squared propagators;
     linear (eikonal) propagators; μ-integrals.