Diagrammatic Hopf algebra of cut Feynman integrals: the one-loop case

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Landau conditions: contracting and cutting propagators

Cutkosky rules: relation between cuts and discontinuities

'First entry condition': physical thresholds or masses in first entry

Reverse unitarity: same relations for cut and uncut integrals

Differential equations: written in dlog-form

Can all these be unified in a single picture?

Good candidate is coproduct of Hopf algebra of Feynman integrals

[Goncharov, series of papers; Duhr, JHEP 1208 (2012) 043]

Some coproduct entries have graphical representation:

- entries related to discontinuities as cut integrals

[SA, Britto, Duhr, Gardi, JHEP 1410 (2014) 125; SA, Britto, Grönqvist, JHEP 1507 (2015) 111]

- evidence for first entries as uncut integrals

[M. Spradlin, A. Volovich; JHEP 1111 (2011) 084]

Is there a completely diagrammatic representation of the full coproduct of any one-loop Feynman integral?

In this presentation we show that yes:

- **the above properties are given a diagrammatic representation**
- **we find a recursive construction of the symbol of any (basis) one-loop integral**

Outline

Brief introduction to algebras, coalgebras and Hopf algebras

The Hopf algebra of MPLs and the Hopf algebra of Feynman graphs

One-loop Feynman integrals, cut and uncut

A map between the diagrammatic coproduct and the coproduct of MPLs

Conclusion and outlook

Brief introduction to algebras, coalgebras and Hopf algebras

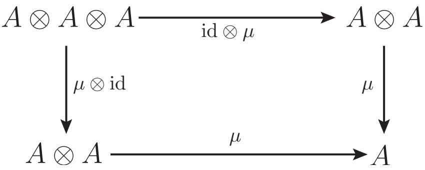
An algebra over a field K (like $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) is a K-vector space A together with a product μ (and a unit ε):

$$\mu : A \otimes A \longrightarrow A \qquad\qquad \epsilon : K \longrightarrow A$$

$$(a, b) \longmapsto \mu(a \otimes b) \equiv a \cdot b$$

Associativity of the product:

$$\mu(\mathrm{id} \otimes \mu) = \mu(\mu \otimes \mathrm{id})$$



$$\mu(\mathrm{id} \otimes \mu)(a \otimes b \otimes c) = \mu(a \otimes (b \cdot c)) = a \cdot (b \cdot c)$$

$$\mu(\mu \otimes \mathrm{id})(a \otimes b \otimes c) = \mu((a \cdot b) \otimes c) = (a \cdot b) \cdot c$$

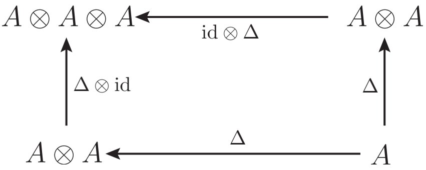
Coalgebra

A coalgebra is defined as the dual of an algebra, equipped with a coproduct Δ (dual to the product) and a counit η (dual to the unit). For simplicity, I assume the dual of A is A itself. Then:

$$\Delta : A \longrightarrow A \otimes A \qquad\qquad \eta : A \longrightarrow K$$

Cossociativity of the coproduct is dual to associativity of the product:

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$$



A bialgebra is an algebra that is at the same time a coalgebra (as our A), for which the product and the coproduct are compatible:

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$$

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (a_1 \cdot b_1) \otimes (a_2 \cdot b_2)$$

A Hopf algebra is a bialgebra equipped with an antipode $S : A \longrightarrow A$:

$$S(a \cdot b) = S(b) \cdot S(a)\,; \qquad \mu(\mathrm{id} \otimes S)\Delta = \mu(S \otimes \mathrm{id})\Delta = 0$$

for reviews, [Duhr, JHEP 1208 (2012) 043; Weinzierl, arXiv:1506.09119]

The Hopf algebra of MPLs and the Hopf algebra of Feynman graphs

Example 1: MPL and their coproduct

Multiple Polylogarithms:

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \qquad a_i, z \in \mathbb{C}$$

A large class of Feynman diagrams can be written in terms of MPL.

[Goncharov, series of papers]

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A large class of Feynman diagrams can be written in terms of MPL.

[Goncharov, series of papers]

$\mathbb{Q}$-vector space of MPL forms a Hopf algebra $\mathcal{H}$:

Graded by weight: $\mathcal{H}=\bigoplus_{n=0}^{\infty}\mathcal{H}_n$; [weight of $G(a_1,\ldots,a_n;z)$ is n]

Coproduct respects weight: $\mathcal{H}_n \xrightarrow{\Delta^{\text{MPL}}} \bigoplus_{k=0}^{n}\mathcal{H}_k\otimes\mathcal{H}_{n-k}$;

Action of Δ^{MPL} on $\mathcal{H}_n$: $\Delta^{\text{MPL}}=\sum_{p+q=n}\Delta^{\text{MPL}}_{p,q}$; [$\Delta^{\text{MPL}}_{p,q}$ takes values in $\mathcal{H}_p\otimes\mathcal{H}_q$]

Example 1: MPL and their coproduct

Example: $\text{Li}_3(x) = -G(0,0,1;z)$, function of weight 3.

$$\Delta^{\text{MPL}}(\text{Li}_3(x)) = \underbrace{1 \otimes \text{Li}_3(x)}_{\Delta^{\text{MPL}}_{0,3}} + \underbrace{\text{Li}_3(x) \otimes 1}_{\Delta^{\text{MPL}}_{3,0}} + \underbrace{\text{Li}_2(x) \otimes \log(x)}_{\Delta^{\text{MPL}}_{2,1}} + \underbrace{\text{Li}_1(x) \otimes \frac{\log^2(x)}{2}}_{\Delta^{\text{MPL}}_{1,2}}$$

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Coassociativity of the coproduct of MPL: $\Delta^{\mathrm{MPL}}_{1,1}\left(\Delta^{\mathrm{MPL}}\mathrm{Li}_3(x)\right)$

$$\left(\Delta^{\mathrm{MPL}}_{1,1} \otimes \mathrm{id}\right)\left[\Delta^{\mathrm{MPL}}(\mathrm{Li}_3(x))\right] = \left(\Delta^{\mathrm{MPL}}_{1,1} \otimes \mathrm{id}\right)\left[\Delta^{\mathrm{MPL}}_{2,1}(\mathrm{Li}_3(x))\right] = \mathrm{Li}_1(x) \otimes \log(x) \otimes \log(x)$$

$$\left(\mathrm{id} \otimes \Delta^{\mathrm{MPL}}_{1,1}\right)\left[\Delta^{\mathrm{MPL}}(\mathrm{Li}_3(x))\right] = \left(\mathrm{id} \otimes \Delta^{\mathrm{MPL}}_{1,1}\right)\left[\Delta^{\mathrm{MPL}}_{1,2}(\mathrm{Li}_3(x))\right] = \underbrace{\mathrm{Li}_1(x) \otimes \log(x) \otimes \log(x)}_{=\Delta^{\mathrm{MPL}}_{1,1,1}(\mathrm{Li}_3(x))}$$

Symbol tensor and maximal iteration of coproduct: $\mathcal{S}(F) \sim \Delta^{\mathrm{MPL}}_{1,\ldots,1}(F)$

Example 1: MPL coproduct and discontinuities

Discontinuities act on the **first entry** of the coproduct

$$\Delta^{\text{MPL}}\text{Disc} = (\text{Disc} \otimes \text{id})\, \Delta^{\text{MPL}}$$

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$$\Delta^{\mathrm{MPL}}\mathrm{Disc} = (\mathrm{Disc} \otimes \mathrm{id})\, \Delta^{\mathrm{MPL}}$$

- Discontinuity lowers weight by one: $\mathrm{Disc}(F_n) = 2\pi i \tilde{F}_{n-1}$;
- Trivial to identify $\tilde{F}_{n-1}$ in the coproduct of F_n — fixed by $\Delta^{\mathrm{MPL}}_{1,n-1}$

$$\Delta^{\mathrm{MPL}}\left(\mathrm{Li}_3(x)\right) = 1 \otimes \mathrm{Li}_3(x) + \mathrm{Li}_3(x) \otimes 1 + \mathrm{Li}_2(x) \otimes \log(x) + \mathrm{Li}_1(x) \otimes \frac{\log^2(x)}{2}$$

$$\mathrm{Disc}_x[\mathrm{Li}_3(x)] = \mathrm{Disc}_x[\mathrm{Li}_1(x)]\frac{\log^2(x)}{2} \sim 2\pi i\frac{\log^2(x)}{2}\theta(x > 1)$$

Differential operators act on the **last entry** of the coproduct

$$\Delta^{\mathrm{MPL}} \frac{\partial}{\partial z} = \left(\mathrm{id} \otimes \frac{\partial}{\partial z} \right) \Delta^{\mathrm{MPL}}$$

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$$\Delta^{\mathrm{MPL}} \frac{\partial}{\partial z} = \left(\mathrm{id} \otimes \frac{\partial}{\partial z} \right) \Delta^{\mathrm{MPL}}$$

- Derivative lowers weight by one: $\frac{\partial}{\partial x} F_n(x) = r(x) \hat{F}_{n-1}(x)$;
- Trivial to identify $\hat{F}_{n-1}$ in the coproduct of F_n — fixed by $\Delta^{\mathrm{MPL}}_{n-1,1}$

$$\Delta^{\mathrm{MPL}} \left(\mathrm{Li}_3(x) \right) = 1 \otimes \mathrm{Li}_3(x) + \mathrm{Li}_3(x) \otimes 1 + \mathrm{Li}_2(x) \otimes \log(x) + \mathrm{Li}_1(x) \otimes \frac{\log^2(x)}{2}$$

$$\frac{\partial}{\partial x} [\mathrm{Li}_3(x)] = \mathrm{Li}_2(x) \frac{\partial \log x}{\partial x} = \frac{1}{x} \mathrm{Li}_2(x)$$

Example 2: A diagrammatic coproduct on Feynman graphs

Two natural operations on graph G with propagators E_G:

- Cutting propagators
- Contracting propagators

We can construct a family of coproducts acting on one-loop graphs with these two operations.

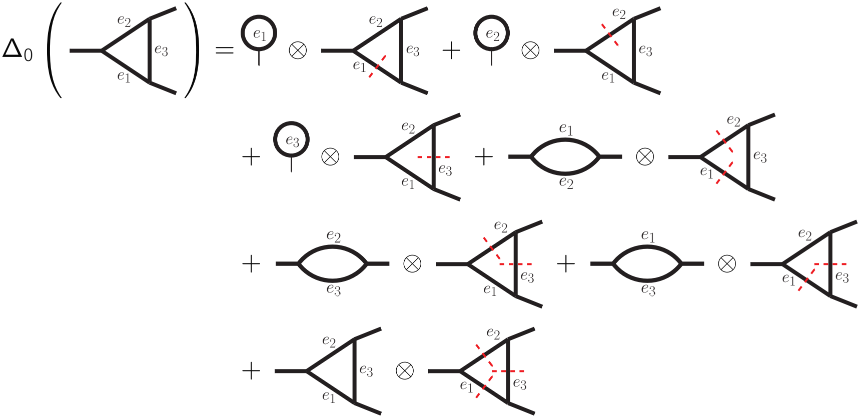
Different coproducts are labeled by a rational number a_X.

Δ_{a_X}: coproduct on one-loop graphs

Example 2: A diagrammatic coproduct on Feynman graphs

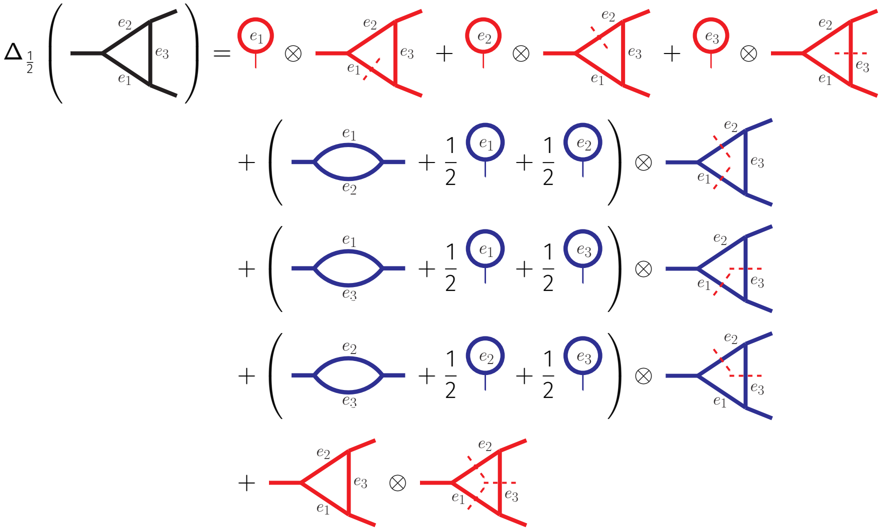
Incidence Hopf algebra: For graph G, with propagators E_G

- Last entry: cut subset of propagators $X \subseteq E_G$
- First entry: contract uncut propagators

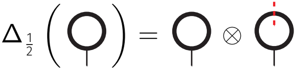


Example 2: $|E_G| = 3$, $C = \emptyset$

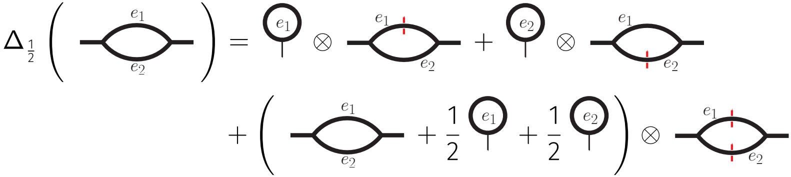
Less trivial construction: distinguish odd and even cuts.



One edge ($|E_G| = 1$, $C = \emptyset$) — **tadpole**:

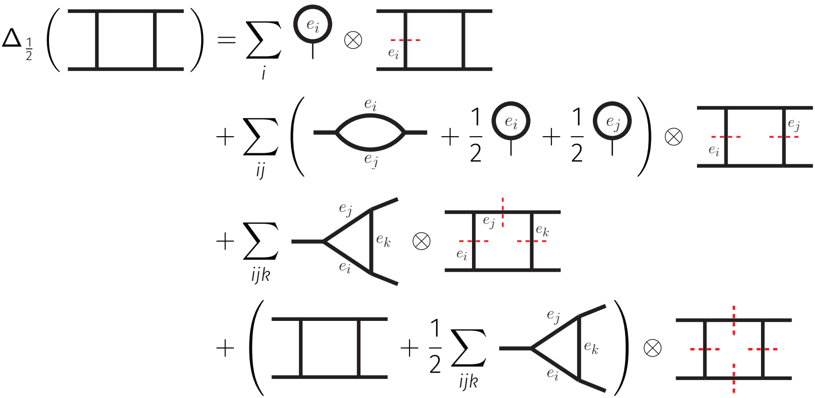


Two edges ($|E_G| = 2$, $C = \emptyset$) — **bubble**:

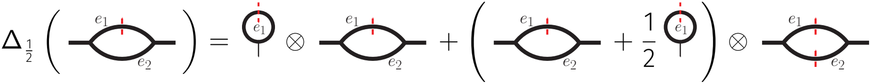


Example 2: Diagrammatic coproduct of uncut graphs ($C = \emptyset$)

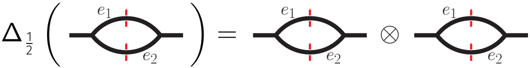
Four edges ($|E_G| = 4$, $C = \emptyset$) — **box**:



Two edges, one cut ($|E_G| = 2$, $|C| = 1$) — **single cut bubble**:



Two edges, two cuts ($|E_G| = 2$, $|C| = 2$) — **double cut bubble**:



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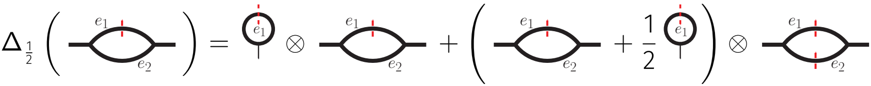

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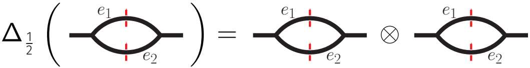
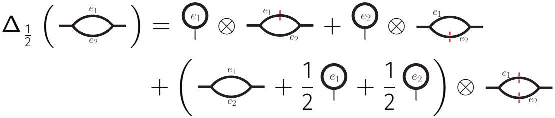


Compare with uncut bubble:



Example 2: A diagrammatic coproduct on Feynman graphs

$$\Delta_{a_X}(G,C) = \sum_{\substack{C\subseteq X\subseteq E_G,\\ X\neq\emptyset}} \left((G_X, C) + a_X \sum_{e\in X\setminus C} (G_{X\setminus e}, C) \right) \otimes (G,X)$$

[SA, Britto, Duhr, Gardi, to appear 16xx.xxxx]

(G, C): Feynman graph G, with E_G propagators, and the ones in $C \subseteq E_G$ cut.

G_X: Feynman graph with X edges, built from G by contracting all edges but those in X.

a_X: rational number. We will be particularly interested in the case where $a_X = 1/2$ if $|X|$ even and 0 otherwise.

Example 2: Coassociativity

Δ_{ax} is coassociative:

$$(\mathrm{id} \otimes \Delta_{ax})\, \Delta_{ax}(G, C) = (\Delta_{ax} \otimes \mathrm{id})\, \Delta_{ax}(G, C)$$

Can construct all other structures necessary to have a **Hopf algebra on graphs**. A bit technical, and not too relevant for the rest of the talk.

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Note: Coproduct VS coaction

Δ^{MPL} and Δ_{a_X} are really coactions: the first-entry of the tensor is special.

Δ^{MPL}: powers of π only appear in first entry.

Δ_{a_X}: first entry with same number of cuts as (G, C).

One-loop Feynman integrals, cut and uncut

Diagram with n external legs of momenta p_l, in dim. reg.,

$$\widetilde{J}_n = \frac{e^{\gamma_E \epsilon}}{\pi^{\frac{D}{2}}} \int d^D k \prod_{j=0}^{n-1} \frac{1}{(k-q_j)^2 - m_j^2 + i0}$$

$$q_j = \sum_l \beta_{jl} p_l, \quad \beta_{jl} \in \{-1, 0, 1\}$$

We choose $D = d - 2\epsilon$ with $d \in \mathbb{N}$, $d = 2\lceil n/2 \rceil$:

- tadpoles and bubbles: $D = 2 - 2\epsilon$;
- triangles and boxes: $D = 4 - 2\epsilon$;
- pentagons and hexagons: $D = 6 - 2\epsilon$;
- $\ldots$;

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- $\ldots$;

$\widetilde{J}_n$ evaluates to MPLs and is a pure function of weight $\frac{d}{2} = \lceil \frac{n}{2} \rceil$

(N.B.: we assume $w(\epsilon) = -1$, coefficient of ϵ^j has weight $\frac{d}{2} + j$)

A basis for one-loop Feynman integrals

Diagram with n external legs of momenta p_l, in dim. reg.,

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Relation to other integrals:

- Non-scalar integrals: tensor reduction ;

Passarino, Veltman, Nucl.Phys. B160 (1979)

- Propagators raised to a power: Integration-By-Parts (IBP) relations ;

Tkachov, Phys.Lett. B100 (1981); Chetyrkin, Tkachov, Nucl.Phys. B192 (1981); Laporta, Int.J.Mod.Phys. A15 (2000)

- Integrals in other dimensions: dimensional shift and IBP relations.

Tarasov, Phys.Rev.D54 (1996); Lee, Nucl.Phys. B830 (2010)

Discontinuities on external channels, apply *Cutkosky prescription*:

[Landau ('59); Cutkosky ('60); t'Hooft & Veltman ('73); SA, Britto, Duhr, Gardi, JHEP 1410 (2014) 125]

- cut propagators identifying channel, replace by delta functions ;
- θ-function to fix energy flow ;
- complex-conjugate one side of cut diagram ;
- keep integration contour of uncut diagram, evaluate in specific kinematic region.

Discontinuities on internal mass:

[SA, Britto, Grönqvist, JHEP 1507 (2015) 111]

- replace propagator with the specific mass by delta function ;
- keep integration contour of uncut diagram, evaluate in specific kinematic region.

* Well defined integration contour ☺

* Vanish if pole of propagator outside integration region ☹

Can we generalise rules to capture all poles **and** keep a well defined countour?

For $g(x)$ behaving well enough around $x = a$, and $a_1 < a < a_2$

$$\int_{a_1}^{a_2} dx g(x)\delta(x-a) = \mathrm{Res}_{x=a}\frac{g(x)}{x-a} = g(a).$$

But: Res_a still non-zero for $a \notin [a_1, a_2]$!

New prescription: [SA, Britto, Duhr, Gardi, to appear 16yy.yyyy]

- change variables $k_j \to x_j$ such that propagator D_j is linear in x_j: $D_j \equiv B_j(x_j - x_{j,p})$;
- if propagator D_j is not cut, do nothing ;
- if propagator D_j is cut,

$$\int_{x_{j,min}}^{x_{j,max}} dx_j \frac{g(x_j)}{B_j(x_j - x_{j,p})} \to \mathrm{Res}_{x_j = x_{j,p}} \frac{g(x_j)}{B_j(x_j - x_{j,p})} = \frac{g(x_{j,p})}{B_j}$$

$$\mathcal{C}_C\widetilde{J}_n = \frac{(2\pi)^{\lfloor c/2 \rfloor} e^{\gamma_E \epsilon}}{2^c \pi^{D/2}} \frac{\sqrt{Y_C}^{D-c-1}}{\sqrt{\text{Gram}_C}^{D-c}} \int d\Omega_{D-c+1} \left[\prod_{j \notin C} \frac{1}{(k-q_j)^2 - m_j^2} \right]_C$$

- $c = |C|$: number of cut propagators
- Y_C: modified Cayley determinant

$$Y_C = \left| \det \left(\frac{1}{2} \left(m_i^2 + m_j^2 - (q_i - q_j)^2 \right) \right)_{i,j \in C} \right|$$

- Gram_C: Gram determinant (e arbitrary element of C)

$$G_C = \left| \det \left((q_i - q_e) \cdot (q_j - q_e) \right)_{i,j \in C \setminus e} \right|$$

- keep contour for uncut propagators, evaluate under cut conditions

Maximal cuts and Leading Singularity:

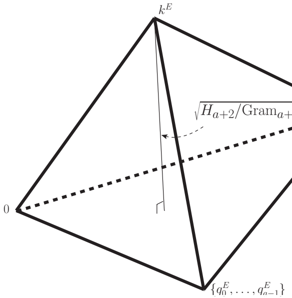
$$\mathcal{C}_G \widetilde{J}_G = 2^{1-2\epsilon-n/2} \frac{e^{\gamma_E \epsilon}\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{Y_G^{-1/2-\epsilon}}{\text{Gram}_G^{-\epsilon}} = \frac{2^{1-n/2}}{\sqrt{Y_G}} + \mathcal{O}(\epsilon)\,, \qquad n \text{ even.}$$

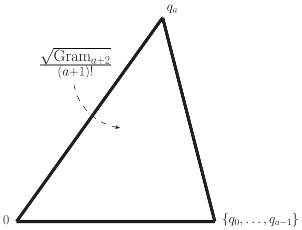
$$\mathcal{C}_G \widetilde{J}_G = 2^{-(1+n)/2} \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \frac{Y_G^{-\epsilon}}{\text{Gram}_G^{1/2-\epsilon}} = \frac{2^{-(1+n)/2}}{\sqrt{\text{Gram}_G}} + \mathcal{O}(\epsilon)\,, \qquad n \text{ odd.}$$

Integrals normalised to leading singularity:

$$J_n = \widetilde{J}_n / \text{LS}\left[\widetilde{J}_n\right]$$

Next-to-maximal cuts in closed formula for both even and odd.

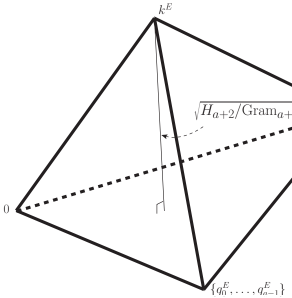


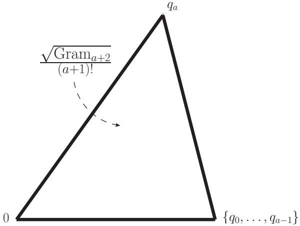


$$\mathrm{Gram}_{a+2} = \left|\det\left((q_0^E, \ldots, q_a^E)^T (q_0^E, \ldots, q_a^E)\right)\right|$$

$$H_{a+2} = \left|\det\left((k^E, q_0^E, \ldots, q_a^E)^T (k^E, q_0^E, \ldots, q_a^E)\right)\right| \xrightarrow[\text{conditions}]{\text{cut}} Y_{a+2}$$

Cuts of one-loop Feynman integrals—polytope geometry





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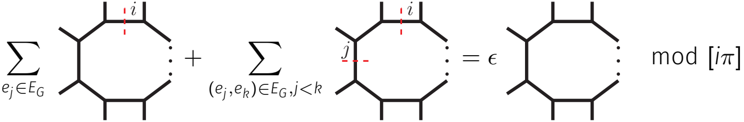
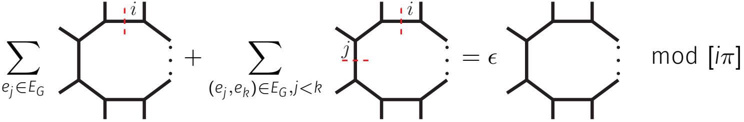
One- and two-propagator cuts of any J_G



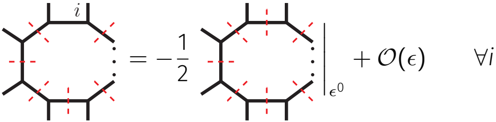
The sum of all one- and two-propagator cuts at order ϵ^n equals the uncut function at order ϵ^{n-1} up to analytic continuation.

One- and two-propagator cuts of any J_G



The sum of all one- and two-propagator cuts at order ϵ^n equals the uncut function at order ϵ^{n-1} up to analytic continuation.

Maximal and next-to-maximal cuts of J_G with $|E_G|$ even propagators



Unless they vanish, the $|E_G|$ different next-to-maximal cuts of a diagram with n even propagators are equal to $-1/2$.

A map between the diagrammatic coproduct and the coproduct of MPLs

Quick summary of what we have done so far

Coproduct of one-loop Feynman integrals — Δ^{MPL}

Coproduct of one-loop cut and uncut Feynman graphs — $\Delta_{\frac{1}{2}}$

Basis for one-loop Feynman integrals — J_G

Well defined cutting rules — $\mathcal{C}_C$

We combine these elements in a map between the diagrammatic coproduct and the coproduct of MPLs

Use the coproduct of MPLs to check diagrammatic coproduct

Make the following identifications:

$$\text{Feynman diagram } G \longleftrightarrow \text{integral } J_G$$

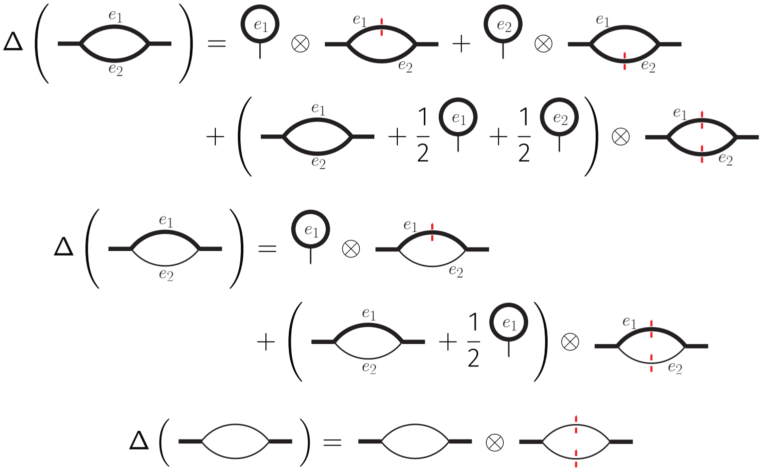
$$\text{Cut Feynman diagram } (G, C) \longleftrightarrow \text{cut integral } \mathcal{C}_C J_G$$

$$\Delta_{\frac{1}{2}} \longleftrightarrow \Delta^{\text{MPL}}$$

- Graphs (G, C) understood in dimensional regularisation: can take massless limit, this is why we only considered graphs with generic masses;
- Δ^{MPL} acts order by order in $\epsilon \Rightarrow$ new relations to check at each order ;
- As order in ϵ increases, more and more $\Delta^{\text{MPL}}_{p,q}$ terms to check ;
- Relation for some coproduct entries can be proven to all orders in ϵ.

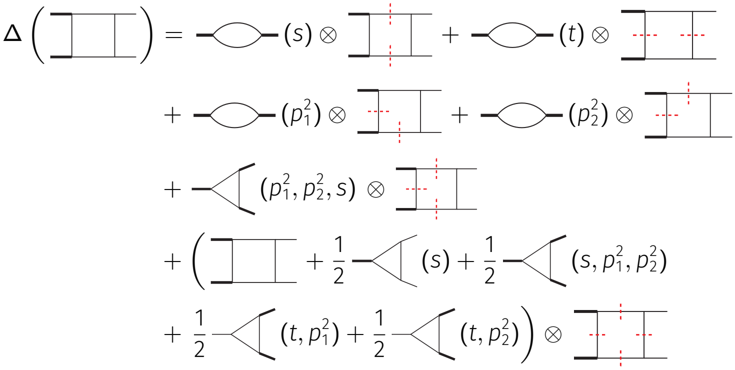
Massless limit: example, $|E_G| = 2$, $C = \emptyset$

Diagrammatic coproduct of all bubbles with massive external legs:



First take massless limit of $\Delta_{\frac{1}{2}}$ **then** check Δ_{MPL} order by order in ϵ.

Example: two-mass-hard box $B(s, t; p_1^2, p_2^2)$



Checked up to weight 4, i.e. $\mathcal{O}(\epsilon^2)$.

Relation between 3- and 4-propagator cuts explains why function is simple up to $\mathcal{O}(\epsilon)$: three-mass triangle starts contributing.

Checks

Explicitly checked for several orders in ϵ for:

tadpole: trivial ;

bubbles: $\mathrm{Bub}(p^2)$, $\mathrm{Bub}(p^2; m^2)$ and $\mathrm{Bub}(p^2; m_1^2, m_2^2)$;

triangles: several combinations of internal and external masses ;

box: $B(s,t)$, $B(s,t,p_1^2)$, $B(s,t,p_1^2,p_3^2)$, $B(s,t,p_1^2,p_2^2)$, $B(s,t;m_{12}^2)$ and $B(s,t;m_{12}^2,m_{23}^2)$.

Consistency checks for:

box: $B(s,t,p_1^2,p_2^2,p_3^2)$ and $B(s,t,p_1^2,p_2^2,p_3^2,p_4^2)$;

pentagon: zero mass ;

hexagon: zero mass. [M. Spradlin, A. Volovich; JHEP 1111 (2011) 084]

Discontinuity operators act on first entry of the coproduct

$$\Delta \text{Disc} = (\text{Disc} \otimes \text{id})\, \Delta$$

First entries of coproduct of graph have the same cut edges as graph

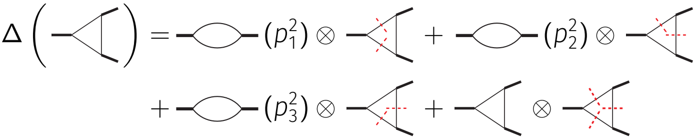
$\Rightarrow$They have the same discontinuity structure (Landau equations).

> The graphical coproduct is consistent with the action of discontinuity operators

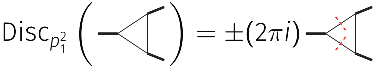
First entry condition: [Gaiotto, Maldacena, Sever, Vieira, JHEP 1112 (2011) 011]

Satisfied by construction by the diagrammatic coproduct of a Feynman diagram: first entry is always a Feynman diagram.

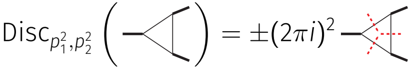
Diagrammatic coproduct encodes know relations between cuts and discontinuities



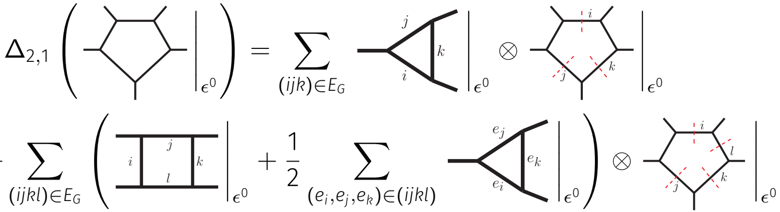
Single discontinuity:



Iterated discontinuities:

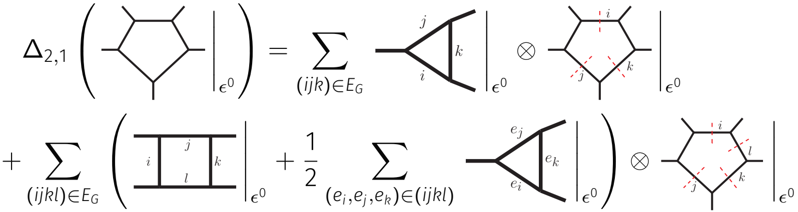


Very few contributions to last entry of weight one:

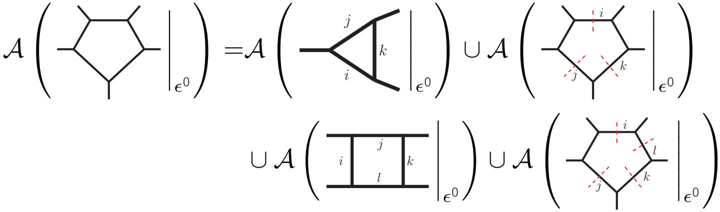


Weight one last entry in diagrammatic coproduct, $|E_G| = n$ odd

Very few contributions to last entry of weight one:

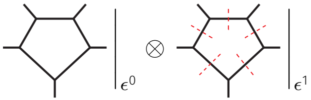


Alphabet $\mathcal{A}$ to $\mathcal{O}(\epsilon^0)$ (set of entries in symbol tensor):

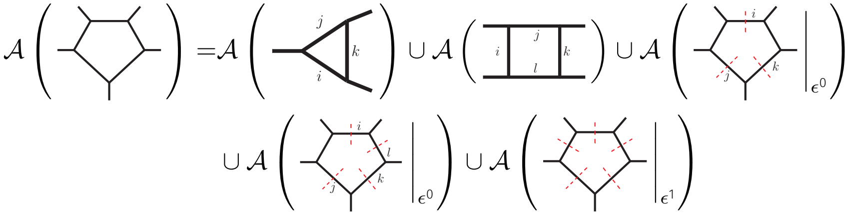


Symbol alphabet $\mathcal{A}$, $|E_G| = n$ odd

At order ϵ, one extra contribution:



Alphabet to all orders in ϵ:



Last entry of diagrammatic coproduct and symbol alphabet

To all orders in ϵ last entry of symbol of J_G is ($|E_G|$ odd):

- next-to-maximal (NMax) cut at order ϵ^0
- NNMax cut at order ϵ^0
- Max cut at order ϵ^1 contributes from order ϵ^1 of J_G)

These are the only new symbol letters introduced by this topology.

Similar results for $|E_G|$ even, one extra contribution:

- NNNMax and NNMax at order ϵ^0 cuts contribute from order ϵ^0
- NMax and Max cuts at order ϵ^1 contribute from order ϵ^1

Symbol alphabet $\mathcal{A}$—Some comments

From maximal cut $\mathcal{A}\,(\text{MaxCut}) = \{Y_G, \text{Gram}_G\}$:

Y_G: Leading Landau Singularity [T. Dennen, M. Spradlin, A. Volovich, JHEP 1603 (2016) 069]

(see also A. Volovich talk on Wednesday)

Gram_G: beyond Landau singularities

Ingredient for two-loop calculations.

We can be more precise and determine at which order in ϵ each new symbol letter appears.

Recursive construction of symbol of J_G:

For any J_G, symbol built iteratively by appending letters to the symbol of weight one integrals, the tadpole and bubble (new perspective on first entry condition)

Differential operators act on last entry of the coproduct

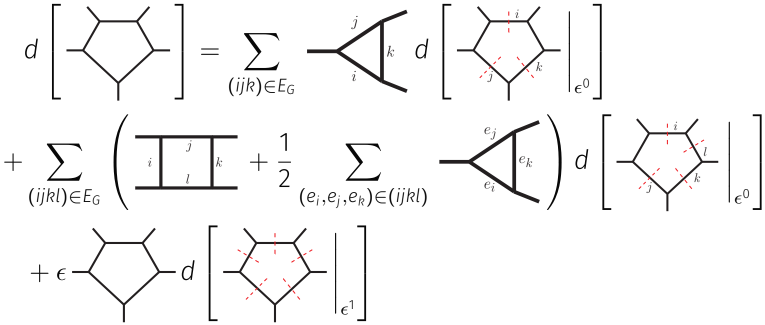
$$\Delta \frac{\partial}{\partial z} = \left(\mathrm{id} \otimes \frac{\partial}{\partial z}\right) \Delta$$

Last entries of coproduct of graph have same number of edges as graph

$\Rightarrow$They obey the same differential equations.

The graphical coproduct is consistent with the action of differential operators

Sufficient to consider last entries we just discussed, which are of weight one—**d log-forms**



Diagrammatic coproduct determines **differential equations**:

Coefficient of differential equations are derivatives of the weight one term in the ϵ-expansion of cuts

$|E_G|$ odd:

- NNMax and NMax cuts contribute from order ϵ^0
- Max cut contributes from order ϵ^1

$|E_G|$ even:

- NNNMax and NNMax cuts contribute from order ϵ^0
- NMax and Max cuts contribute from order ϵ^1

Conclusion and outlook

We conjecture and give evidence that:

The coproduct of all one-loop Feynman diagrams has a completely diagrammatic representation

We construct the diagrammatic coproduct of any one-loop diagram.

New definition of cut of Feynman integrals.

New relation between cut and uncut integrals.

Explicitly checked for several non-trivial examples.

Diagrammatic coproduct consistent with relation between cuts and discontinuities.

Construct iteratively the alphabet and even symbol of one-loop integrals.

Diagrammatic representation of differential equations.

Outlook

Can our construction be generalised to two and more loops?

What is a good basis of pure Feynman integrals beyond one-loop?

Which combinations of diagrams appear in the first entry?

Can our construction be generalised to diagrams that do not evaluate to MPLs?

Elliptic functions appear beyond one loop.

Possible connection with recent work by Francis Brown, independent of integrals evaluating to polylogarithms [*Notes on motivic periods*, arXiv:1512.06410].

Thank you!