Gravity OS-Diagrams

Enrico Herrmann

Based on: arXiv:1604.03479

in collaboration with:

Jaroslav Trnka

see also: Heslop,Lipstein: arXiv:1604.03046

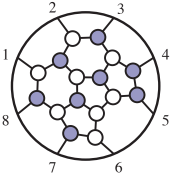
June 23, 2016

Caltech

Motivation

planar $\mathcal{N}=4$ sYM $\Rightarrow$ Hydrogen atom of the 21^{st} century!

- Dual Conformal Invariance [Drummond, Henn, Smirnov, Sokatchev, Korchemsky,...]
- Relation to Wilson loops and Correlation Functions [Mason, Skinner, Caron-Huot, Alday, Eden, Korchemsky, Maldacena, Sokatchev, ...]
- Yangian Invariance and Integrability [Drummond, Henn, Plefka, Beisert, Staudacher, Alday, Viera, Basso,...]



Mathematical structures beyond planar $\mathcal{N}=4$ sYM?

non-planar $\mathcal{N}=4$ sYM	$\mathcal{N}=8$ SUGRA

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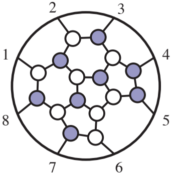

Mathematical structures beyond planar $\mathcal{N}=4$ sYM?

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Outline

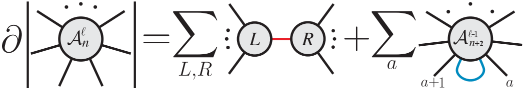
On-Shell methods, Cuts of loop amplitudes and Generalized Unitarity

[Britto, Cachazo, Feng, Witten; Bern, Dixon, Kosower; ...]

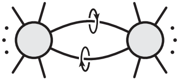
- core idea: on-shell amplitudes break up into products of simpler amplitudes on all factorization channels
- amplitudes are **fixed** from their singularities

locality: $\frac{1}{P^2}$ propagators

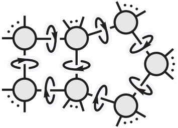
unitarity: factorization on poles



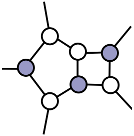
iterative cuts:



⇒



Maximal Cut



OS-Diagram

Three-Point Amplitudes

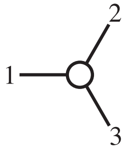
massless theories in $D=4$: 3pt-amplitude completely fixed by Poincaré invariance

$$p^\mu \to p^{\alpha\dot\alpha} \quad \Rightarrow \quad p^2 = \det p^{\alpha\dot\alpha} = 0 \quad \Rightarrow \quad p^{\alpha\dot\alpha} = \lambda^\alpha \widetilde{\lambda}^{\dot\alpha}$$

On-shell conditions have two solutions:

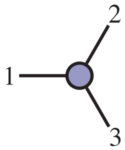
$\overline{\text{MHV}},\ k=1$

$\lambda_1 \sim \lambda_2 \sim \lambda_3$



$\text{MHV},\ k=2$

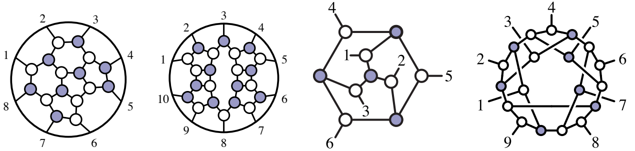
$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$



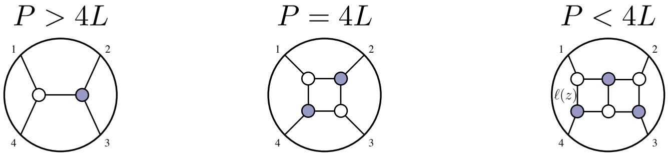
$$\mathcal{A}_3^{\overline{\text{MHV}}} = \frac{\delta^4(P)\delta^4([12]\widetilde{\eta}_3+[23]\widetilde{\eta}_1+[31]\widetilde{\eta}_2)}{[12][23][31]} \qquad \mathcal{A}_3^{\text{MHV}} = \frac{\delta^4(P)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

On-Shell Diagrams as Gluing of 3pt-amplitudes

use 3pt-amplitudes as building blocks for more complicated On-Shell diagrams



results are functions of external kinematics,



Extra δ-functions   Leading Singularity   Unfixed Parameters

Grassmannian Formulation for On-Shell Diagrams

[Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka]

deep connection: On-Shell Diagrams $\Leftrightarrow$ Grassmannian $G(k,n)$

$G(k,n)$: space of k-planes in n-dimensions

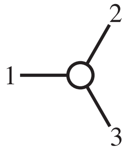
$\cong$ $(k \times n)$-matrices modulo $GL(k)$

planar OS-diags $\Rightarrow$ positive Grassmannian $\rightarrow$ connection to combinatorics $\&$ algebraic geometry [Lusztig,Postnikov,Speyer,Williams,Knutson,Lam,...]

Motivation from physics: linearize momentum conservation!

Grassmannian Formulation for On-Shell Diagrams

Remember 3pt-amplitude ($k=1$):



$$\mathcal{A}_3^{\overline{\text{MHV}}} = \frac{\delta^4(\sum_{i=1}^3 \lambda_i\widetilde{\lambda}_i)\delta^4([12]\widetilde{\eta}_3+[23]\widetilde{\eta}_1+[31]\widetilde{\eta}_2)}{[12][23][31]}$$

- mom. conservation is **quadratic** constraint on λ and $\widetilde{\lambda}$
- not manifest that $\lambda_1 \sim \lambda_2 \sim \lambda_3$

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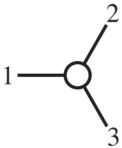

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- mom. conservation is **quadratic** constraint on λ and $\widetilde{\lambda}$
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cure: $\lambda_i^\alpha = \alpha_i\ \rho^\alpha,\ \rho:$ reference spinor

$$\delta(\sum \lambda_i\widetilde{\lambda}_i) \stackrel{\lambda_i=\alpha_i\rho}{\longrightarrow} \delta(\rho\sum \alpha_i\widetilde{\lambda}_i) \Rightarrow \alpha_1\widetilde{\lambda}_1+\alpha_2\widetilde{\lambda}_2+\alpha_3\widetilde{\lambda}_3=0$$

$$\alpha_1=[23],\quad \alpha_2=[31],\quad \alpha_3=[12]$$

suggests:

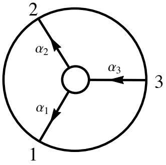
$$\mathcal{A}_3^{\overline{\text{MHV}}} \stackrel{?}{=} \int \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\int d^2\rho \left(\prod_{i=1}^3 \delta^2(\alpha_i\rho-\lambda_i)\right)\delta^2(\sum \alpha_i\widetilde{\lambda}_i)\delta^4(\sum \alpha_i\widetilde{\eta}_i)$$

Grassmannian Formulation for On-Shell Diagrams

$$\mathcal{A}_3^{\overline{\text{MHV}}} \stackrel{?}{=} \int \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3} \int d^2\rho \left(\prod_{i=1}^{3} \delta^2(\alpha_i \rho - \lambda_i)\right) \delta^2(\sum \alpha_i \widetilde{\lambda}_i)\delta^4(\sum \alpha_i \widetilde{\eta}_i)$$

GL(1)-redundancy: $\alpha_i \to t\alpha_i,\ \rho \to \frac{1}{t}\rho \Rightarrow$ gauge-fix!

Encode linear relations in terms of $(k \times n)$-matrix C mod $GL(k)$.



$\Updownarrow$

$$C = \begin{pmatrix} \alpha_1\alpha_3 & \alpha_2\alpha_3 & 1 \end{pmatrix}$$

$$\delta(P) \equiv \delta(\lambda \cdot \widetilde{\lambda}) \to \delta(C \cdot \widetilde{\lambda})\delta(C^\perp \cdot \lambda)$$

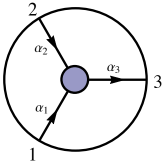
$C^\perp$: orthogonal matrix to C

arrows $\leftrightarrow$ gauge-fixing

$$\mathcal{A}_3^{\overline{\text{MHV}}} = \frac{1}{\text{vol}(GL(1))} \int \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3} \delta(C \cdot \widetilde{\lambda})\delta(C^\perp \cdot \lambda)\delta(C \cdot \widetilde{\eta})$$

Grassmannian Formulation for On-Shell Diagrams

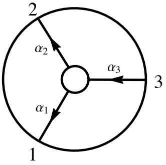
MHV$(k = 2)$



$\Updownarrow$

$$C = \begin{pmatrix} 1 & 0 & \alpha_1\alpha_3 \\ 0 & 1 & \alpha_2\alpha_3 \end{pmatrix}$$

$\overline{\text{MHV}}(k = 1)$



$\Updownarrow$

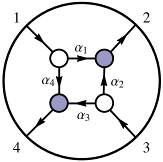
$$C = \begin{pmatrix} \alpha_1\alpha_3 & \alpha_2\alpha_3 & 1 \end{pmatrix} .$$

Note: for MHV-amplitude we partially gauge-fixed $GL(2)$!

Convenient to keep $GL(1)_v$ redundance in each vertex v

Grassmannian Formulation for On-Shell Diagrams

- glue 3pt-amplitudes together $\to$ build bigger on-shell diagrams
- encode momentum conservation by $(k \times n)$-matrix C
- n: number of external legs,
  $k = 2n_B + n_W - n_I$: MHV-degree



$$C = \begin{pmatrix} 1 & \alpha_1 & 0 & \alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix}$$

Grassmannian Formulation for On-Shell Diagrams

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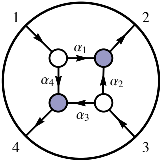

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math

α_i definite sign $\Rightarrow$ C has pos. minors $\Rightarrow$ pos. Grassmannian

physics

$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\delta(C\cdot Z)$$

$$\delta(C\cdot Z) \equiv \delta(C\cdot\widetilde{\lambda})\delta(C^{\perp}\cdot\lambda)\delta(C\cdot\widetilde{\eta})$$

equal to product of 3pt amplitudes!

Note: $d\log$-measure! $\rightarrow$ special for Yang-Mills

relation to amplitudes: hidden symmetries, geometric formulation,...

[see Trnka talk at Amplitudes]

A first look at gravity on-shell diagrams

- 3pt-amplitudes: squaring relation

$$A_3 = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \to M_3 = \left(\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\right)^2$$

- general on-shell diagram (product of 3pt amplitudes)

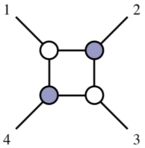
$$(\text{YM})^2 = (\text{GR}) \times (\phi^3)$$

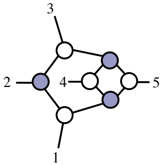
  "don't square the propagators"

- (ϕ^3) factor changes expressions drastically

A first look at gravity on-shell diagrams

study some data (MHV leading singularities)





$$\frac{[13][24]}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle}$$

$$\frac{[12][23][45]^2}{\langle 12\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 34\rangle\langle 35\rangle}$$

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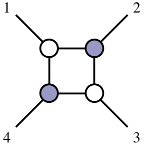

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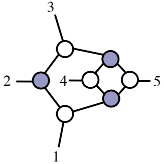

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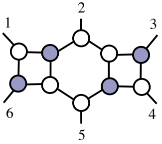
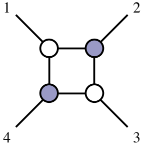


$$\frac{\langle 5|Q_{16}|2]\langle 2|Q_{34}|5][16]^2[34]^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle\langle 25\rangle^2}$$

- nontrivial numerators!
- higher power poles possible! (unlike YM)

A first look at gravity on-shell diagrams

study some data (MHV leading singularities)



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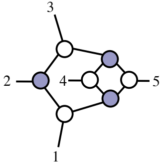

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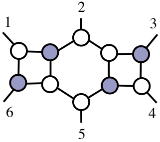

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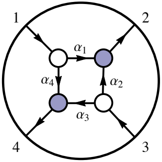
Detailed analysis

- nontrivial numerators! $\rightarrow$ collinearity condition in the vertices
- higher power poles possible! (unlike YM) $\rightarrow$ infinite momenta

Grassmannian Formula for Gravity



$$C = \begin{pmatrix} 1 & \alpha_1 & 0 & \alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix}$$

Inspired by the explicit data, can "discover" the gravity formula

- Yang–Mills: $\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\delta(C\cdot Z)$
- Gravity: $\Omega = \frac{d\alpha_1}{\alpha_1^3}\frac{d\alpha_2}{\alpha_2^3}\frac{d\alpha_3}{\alpha_3^3}\frac{d\alpha_4}{\alpha_4^3}\left(\prod_v \Delta_v\right)\delta(C\cdot Z)$

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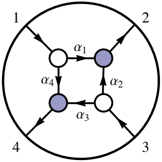

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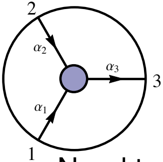
- special numerator Δ_v for each vertex
- α_i^3 poles

Can motivate this formula by looking at 3pt amplitudes

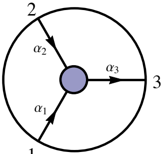
Grassmannian Formula for Gravity



$$C = \begin{pmatrix} 1 & 0 & \alpha_1\alpha_3 \\ 0 & 1 & \alpha_2\alpha_3 \end{pmatrix}, \quad C^\perp = \begin{pmatrix} -\alpha_1\alpha_3 & -\alpha_2\alpha_3 & 1 \end{pmatrix}$$

- Need to modify measure by some dimensionful, permutation invariant object Δ
- $\delta(C^\perp \cdot \lambda) \Rightarrow -\overbrace{\alpha_1\lambda_1}^{\lambda_A} - \overbrace{\alpha_2\lambda_2}^{\lambda_B} + \overbrace{\frac{1}{\alpha_3}\lambda_3}^{\lambda_E} = 0 \Rightarrow \Delta = \langle AB\rangle = \langle BE\rangle = \langle EA\rangle$

Grassmannian Formula for Gravity



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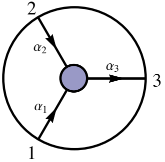
Ansatz:

$$\Omega_3^{\text{MHV}} = \kappa \frac{\Delta^\rho d\alpha_1 d\alpha_2 d\alpha_3}{\alpha_1^{\sigma_1}\alpha_2^{\sigma_2}\alpha_3^{\sigma_3}} \delta^{2\times 2}(C\cdot\widetilde{\lambda})\delta^{1\times 2}(C^\perp\cdot\lambda)\delta^{2\times\mathcal{N}}(C\cdot\widetilde{\eta})$$

Impose: a) maximal SUSY $\mathcal{N} = 4s$ b) permutation invariance
c) independence of α_3 $(\Omega \sim \frac{d\alpha_3}{\alpha_3})$

$$\rho = s - 1\,, \qquad \sigma_1 = \sigma_2 = \sigma_3 = 2s - 1$$

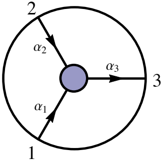
Grassmannian Formula for Gravity



$$\Omega_3^{\text{MHV},s} = \kappa \frac{\Delta^{s-1} d\alpha_1 d\alpha_2 d\alpha_3}{\alpha_1^{2s-1}\alpha_2^{2s-1}\alpha_3^{2s-1}} \overbrace{\delta^{2\times 2}(C\cdot\widetilde{\lambda})\delta^{1\times 2}(C^\perp\cdot\lambda)\delta^{2\times 4s}(C\cdot\widetilde{\eta})}^{=\delta(C\cdot Z)}$$

What does this mean for $s > 2$?

Grassmannian Formula for Gravity



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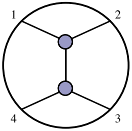
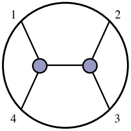
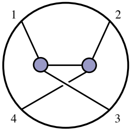
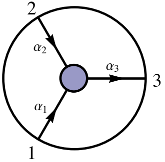
What does this mean for $s > 2$?

OS-diags in $\mathcal{N}=4$ sYM satisfy equivalence moves!

$0 =$  $+$  $-$ 

$$\Gamma_s(\langle 12\rangle\langle 34\rangle)^{s-1} + \Gamma_t(\langle 14\rangle\langle 23\rangle)^{s-1} = \Gamma_u(\langle 13\rangle\langle 24\rangle)^{s-1}$$

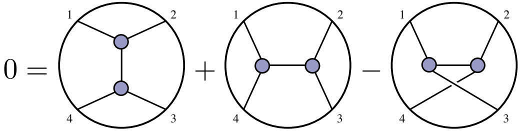
Grassmannian Formula for Gravity



$$\Omega_3^{\text{MHV},s} = \kappa \frac{\Delta^{s-1} d\alpha_1 d\alpha_2 d\alpha_3}{\alpha_1^{2s-1}\alpha_2^{2s-1}\alpha_3^{2s-1}} \overbrace{\delta^{2\times 2}(C\cdot\widetilde{\lambda})\delta^{1\times 2}(C^{\perp}\cdot\lambda)\delta^{2\times 4s}(C\cdot\widetilde{\eta})}^{=\delta(C\cdot Z)}$$

What does this mean for $s > 2$?

OS-diags in $\mathcal{N} = 4$ sYM satisfy equivalence moves!

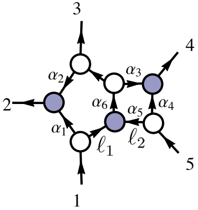


$$\Gamma_s(\langle 12\rangle\langle 34\rangle)^{s-1} + \Gamma_t(\langle 14\rangle\langle 23\rangle)^{s-1} = \Gamma_u(\langle 13\rangle\langle 24\rangle)^{s-1}$$

Two solutions:

1. $s = 1$: $\Gamma_s + \Gamma_t = \Gamma_u \Rightarrow$ Jacobi identity
   $\Gamma_s = f^{12a}f^{34a}$, $\Gamma_t = f^{14a}f^{23a}$, $\Gamma_u = f^{13a}f^{24a}$
2. $s = 2$: $\Gamma_s = \Gamma_t = \Gamma_u \Rightarrow$ Shouten identity
   universality of gravitational coupling

Grassmannian Formula for Gravity



$$\Omega_{\text{Gr}} = \frac{d\alpha_1}{\alpha_1^3} \cdots \frac{d\alpha_m}{\alpha_m^3} \left(\prod_v \Delta_v\right) \delta(C \cdot Z)$$

Features of the formula:

- higher poles present
- reduces to single pole if edge is erasable
- diagram vanishes if momenta in a given vertex become collinear

Can read off the λ's and $\widetilde{\lambda}$'s required for Δ similar to C-matrix!

Comments-Conclusion

- Extended Grassmannian formulation for On-Shell Diagrams to gravity
- Formulas for $\mathcal{N} < 8$ SUGRA only slightly modified
- presence of poles at infinity
- collinear vanishing of on-shell diagrams

Interesting other directions (not in this talk) [see Trnka talk at Amplitudes]

- relation to properties of amplitudes
- recursion relations
- UV-structure of gravity