$\mathcal{N}$=2 Theories:
S-duality, Instantons and All That

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This talk is mainly based on:

- M. Billò, M. Frau, F. Fucito, A.L. and J.F. Morales, *``S-duality and the prepotential in N=2*theories (I): the ADE algebras,''* JHEP **1511** (2015) 024, arXiv:1507.07709
- M. Billò, M. Frau, F. Fucito, A.L. and J.F. Morales, *``S-duality and the prepotential in N=2*theories (II): the non-simply laced algebras,''* JHEP **1511** (2015) 026, arXiv:1507.08027
- M. Billò, M. Frau, F. Fucito, A.L. and J.F. Morales, *``Resumming instantons in N=2* theories,''* XIV Marcel Grossmann Meeting, arXiv:1602.00273

and

- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, A.L. and M. Raman, *``Modular anomaly equations and S-duality in N=2 conformal SQCD,"* JHEP 1510 (2015) 091, arXiv:1507.07476
- S.K. Ashok, E. Dell'Aquila, A.L. and M. Raman, *``S-duality, triangle groups and modular anomalies in N=2 SQCD,''* JHEP 1604 (2016) 118, arXiv:1601.01827

but it builds on a very vast literature (relevant references will be given during the talk)

1. Introduction
2. $\mathcal{N}$=4 SYM
3. $\mathcal{N}$=2* SYM
4. $\mathcal{N}$=2 SQCD
5. Conclusions

Introduction

- Non-perturbative effects are important:
  - in gauge theories: confinement, chiral symmetry breaking, AGT, ...
  - in string theories: D-branes, duality, AdS/CFT, ...
- They are essential to complete the perturbative expansion and lead to results valid at all couplings
- In supersymmetric theories, tremendous progress has been possible thanks to the developement of localization techniques

  (Nekrasov ‘02, Nekrasov-Okounkov ’03, Pestun ‘07, …, Nekrasov-Pestun ‘13, ….)
- In superconformal theories these methods allowed us to compute exactly several quantities:
  - Sphere partition function and free energy
  - Wilson loops
  - Correlation functions, amplitudes
  - Cusp anomalous dimensions and bremsstrahlung function

- We will focus on SYM theories in *4d* with *N*=2 supersymmetry
    - They are less constrained than the *N*=4 theories
    - They are sufficiently constrained to be analyzed exactly
- Building on the Seiberg-Witten approach, there has been a quest for an exact quantum description of these theories and their duality pattern:
    - Insights from M-theory embedding and *6d* realizations (Gaiotto)
    - *4d/2d* relations (AGT)
    - Resurgence
    - Formulation on curved manifolds
    - Large N limit, holography

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- We will be interested in studying how S-duality on the quantum effective couplings costrains the prepotential of N=2 theories. (earlier work by Minahan et al. ’96, ‘97)
- We will make use of these constraints to obtain exact expressions valid at all couplings

$\mathcal{N}$=4 SYM

N=4 SYM

- Consider *N* =4 SYM in d=4
  - This theory is maximally supersymmetric (16 SUSY charges)
  - The field content is

A		1 vector
λ^a	$(a = 1, \cdots, 4)$	4 Weyl spinors
X^i	$(i = 1, \cdots, 6)$	6 real scalars

  - All fields are in the adjoint representation of the gauge group G .
  - The β−function vanishes to all orders in perturbation theory.
  - If $\langle X^i \rangle = 0$, the theory is superconformal (*i.e.* invariant under $\mathrm{SU}(2,2|4)$) also at the quantum level.

$\mathcal{N}$=4 SYM

- The relevant ingredients of $\mathcal{N}$ =4 SYM are:
  - The gauge group G (or the gauge algebra $\mathfrak{g}$)
  - The (complexified) coupling constant

$$\tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2} \quad \in \mathbb{H}_+$$

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- Many <u>exact</u> results have been obtained using:
  - Explicit expressions of scattering amplitudes
  - Integrability
  - AdS/CFT correspondence
  - Duality

N=4 SYM

- *N* =4 SYM is believed to possess an exact duality invariance which contains the electro-magnetic duality S

(Montonen-Olive '77, Vafa-Witten '94, Sen '94, ...)

- If the gauge algebra $\mathfrak{g}$ is simply laced (ADE)
  - S maps the theory to itself but with electric and magnetic states exchanged
  - It is a weak/strong duality, acting on the coupling by

$$S(\tau) = -1/\tau$$

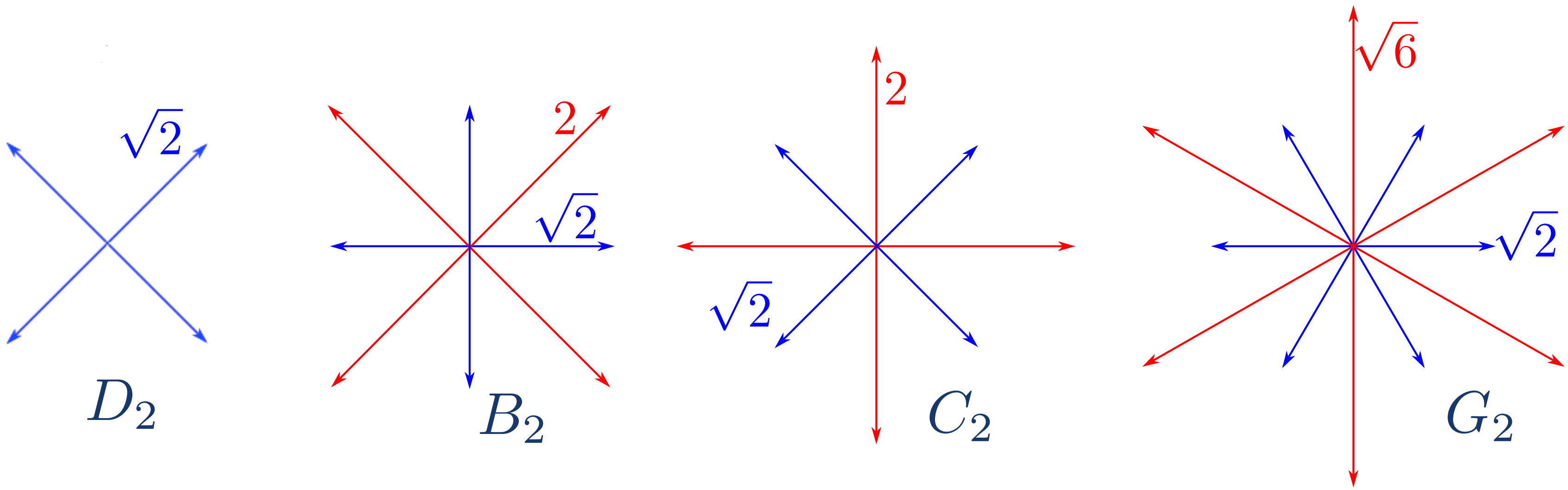
  - Together with $T(\tau) = \tau + 1$ ($\theta \to \theta + 2\pi$), it generates the modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ ; \quad S^2 = -1 \ , \quad (ST)^3 = -1$$

$\mathcal{N}$=4 SYM

- This can be extended to the non-simply laced algebras (BCFG) (Goddard et al '77, Dorey et al '96, Argyres et al. '06, Kapustin-Witten '07, ...)

- S-duality maps the algebra $\mathfrak{g}$ to its GNO dual $\mathfrak{g}^\vee$
    - in $\mathfrak{g}^\vee$ the long and short roots are exchanged



- $$A_n^\vee = A_n, \quad D_n^\vee = D_n, \quad E_n^\vee = E_n,$$
  $$B_n^\vee = C_n, \quad C_n^\vee = B_n, \quad F_4^\vee = F_4'^\vee, \quad G_2^\vee = G_2'^\vee$$

$\mathcal{N}$=4 SYM

- We can treat all algebras $\mathfrak{g} \in \{A_n, B_n, C_n, D_n, E_{6,7,8}, F_4, G_2\}$ at the same time, introducing

$$n_{\mathfrak{g}} = \frac{\alpha_L \cdot \alpha_L}{\alpha_S \cdot \alpha_S}$$

with α_L and α_S being the long and short roots of $\mathfrak{g}$

- One has

$$n_{\mathfrak{g}} = 1 \quad \text{for} \quad \mathfrak{g} = A_n, D_n, E_{6,7,8}$$
$$n_{\mathfrak{g}} = 2 \quad \text{for} \quad \mathfrak{g} = B_n, C_n, F_4$$
$$n_{\mathfrak{g}} = 3 \quad \text{for} \quad \mathfrak{g} = G_2$$

$$n_{\mathfrak{g}} = n_{\mathfrak{g}^\vee}$$

N=4 SYM

- For $\mathfrak{g} \in \{A_n, B_n, C_n, D_n, E_{6,7,8}, F_4, G_2\}$, the duality group is generated by

$$S(\tau) = -\frac{1}{n_{\mathfrak{g}}\tau} \quad , \qquad T(\tau) = \tau + 1$$

- They generate the so-called Hecke group $\mathrm{H}(p_{\mathfrak{g}}) \subset \mathrm{SL}(2, \mathbb{R})$

$$S = \begin{pmatrix} 0 & -1/\sqrt{n_{\mathfrak{g}}} \\ \sqrt{n_{\mathfrak{g}}} & 0 \end{pmatrix} , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} ;$$

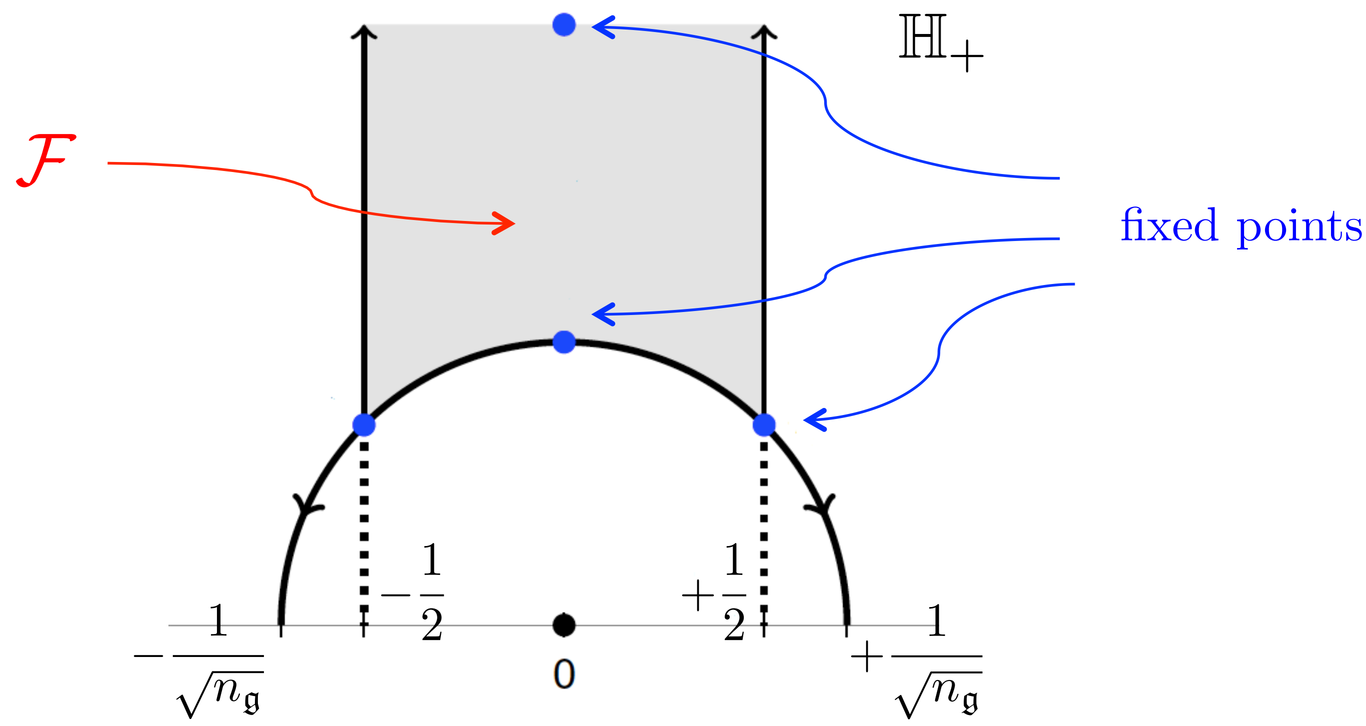
$$S^2 = -1 \, , \quad (ST)^{p_{\mathfrak{g}}} = -1$$

where $n_{\mathfrak{g}} = 4\cos^2\left(\frac{\pi}{p_{\mathfrak{g}}}\right)$.

$n_{\mathfrak{g}}$	1	2	3
$p_{\mathfrak{g}}$	3	4	6

$\mathcal{N}$=4 SYM

- The fundamental domain $\mathcal{F}$ of the Hecke group $\mathrm{H}(p_{\mathfrak{g}})$



- (STS) and T generate a subgroup $\Gamma_0(n_{\mathfrak{g}}) \subset \mathrm{SL}(2,\mathbb{Z})$

N=4 SYM as a *N*=2 theory

Let us decompose the *N*=4 multiplet into

- one *N*=2 vector multiplet

  1 vector → A , 2 Weyl fermions → ψ^I , 1 complex scalar → ϕ

$$A \, , \quad \psi^I \, , \quad \phi$$

- one *N*=2 hypermultiplet

  2 complex scalars → q , $\tilde{q}$; 2 Weyl fermions → χ , $\tilde{\chi}$

$$q \, , \quad \tilde{q} \, , \quad \chi \, , \quad \tilde{\chi}$$

By introducing the v.e.v.

$$\langle \phi \rangle = a = \mathrm{diag}(a_1, ..., a_n)$$

- we break the gauge group $G \rightarrow U(1)^n$
- we spontaneously break conformal invariance
- we can describe the dynamics in terms of a holomorphic prepotential $F(a)$, as in *N*=2 theories.

N=4 SYM as a *N*=2 theory

- The prepotential of the *N*=4 theory is simply

$$F^{\mathfrak{g}} = n_{\mathfrak{g}}\, i\, \pi\, \tau\, a^2$$

- S-duality acts as $\tau \to -\dfrac{1}{n_{\mathfrak{g}}\tau}$, $\mathfrak{g} \to \mathfrak{g}^\vee$

- S-duality also relates the electric variable a of the $\mathfrak{g}$ theory to the magnetic variable a_D of the $\mathfrak{g}^\vee$ theory:

$$S\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} 0 & -1/\sqrt{n_{\mathfrak{g}}} \\ \sqrt{n_{\mathfrak{g}}} & 0 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} -a/\sqrt{n_{\mathfrak{g}}} \\ \sqrt{n_{\mathfrak{g}}}\, a_D \end{pmatrix}$$

- The dual variables are defined as

$$a_D \equiv \frac{1}{2\pi i\, n_{\mathfrak{g}}} \frac{\partial F^{\mathfrak{g}^\vee}}{\partial a} = \tau\, a$$

$\mathcal{N}$=4 SYM as a $\mathcal{N}$=2 theory

- Let's find the S-dual prepotential:

$$S(F^{\mathfrak{g}}) = n_{\mathfrak{g}}\, i\, \pi \Big(-\frac{1}{n_{\mathfrak{g}}\tau}\Big) \left(\sqrt{n_{\mathfrak{g}}}\, a_D\right)^2 = -n_{\mathfrak{g}}\, i\, \pi\, \frac{1}{\tau}\, a_D^2$$

- S-duality exchanges the description based on a with its Legendre-transform, based on a_D :

$$\mathcal{L}(F^{\mathfrak{g}^\vee}) = F^{\mathfrak{g}^\vee} - a\,\frac{\partial F^{\mathfrak{g}^\vee}}{\partial a} = n_{\mathfrak{g}}\, i\, \pi\, \tau\, a^2 - 2\pi\, i\, n_{\mathfrak{g}}\, a\, a_D$$

$$= -n_{\mathfrak{g}}\, i\, \pi\, \frac{1}{\tau}\, a_D^2$$

$\mathcal{N}$=4 SYM as a $\mathcal{N}$=2 theory

- Let's find the S-dual prepotential:

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$$= -n_{\mathfrak{g}}\, i\,\pi\,\frac{1}{\tau}\,a_D^2$$

- Thus

$$S(F^{\mathfrak{g}}) = \mathcal{L}(F^{\mathfrak{g}^\vee})$$

N=2 theories

- We want to show that this structure is present also in *N*=2 theories and investigate its consequences on their strong coupling dynamics.

- We consider two cases:

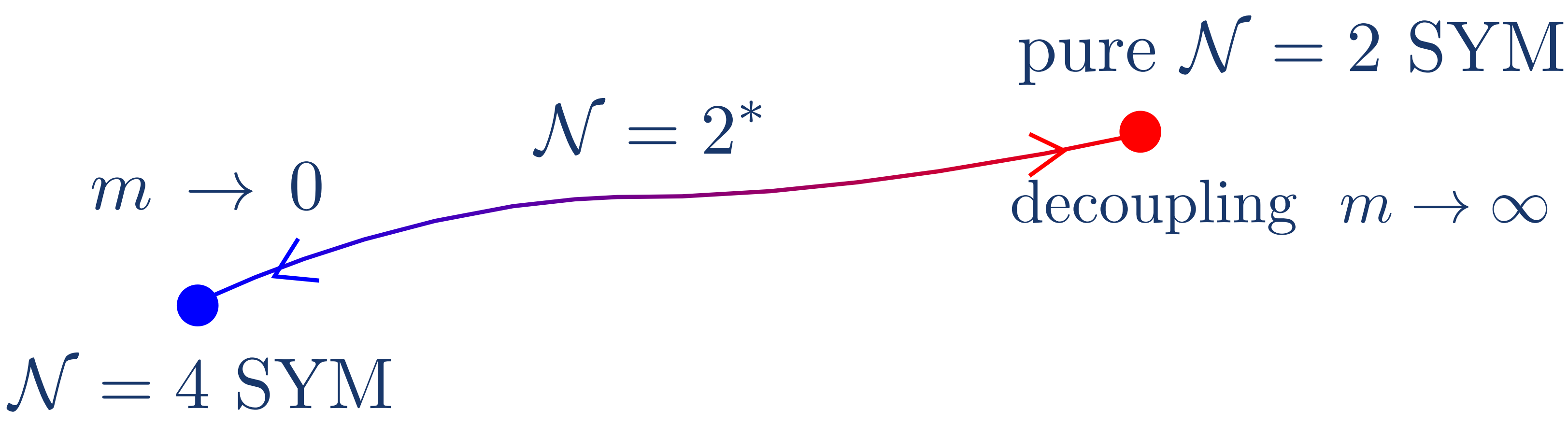
  1. *N*=2* theories

  2. *N*=2 SQCD theories with $N_f = 2N_c$

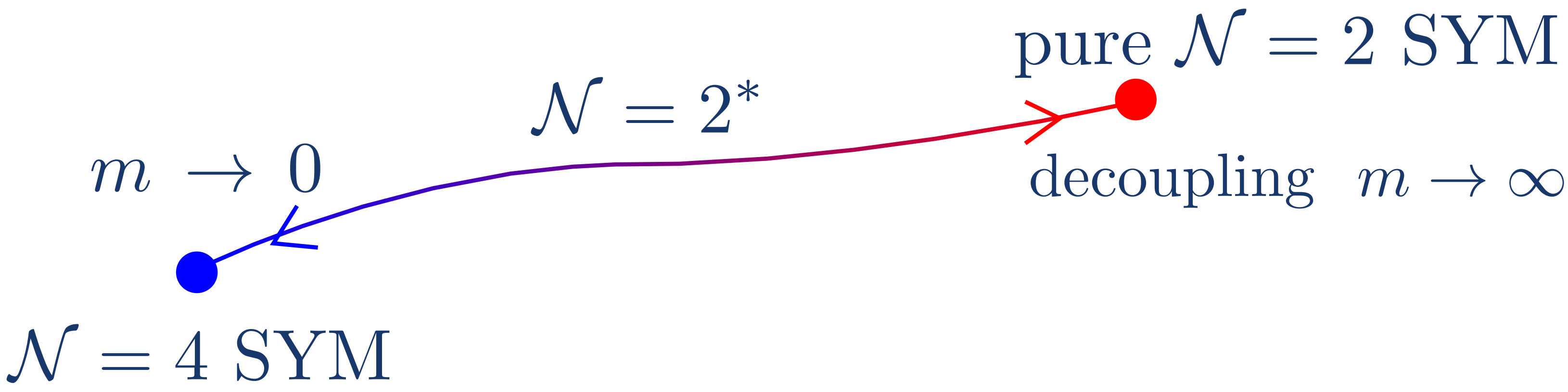
$\mathcal{N}$=2* SYM

The *N*=2* set-up

- Field content:
  - one *N*=2 vector multiplet for the algebra $\mathfrak{g}$
  - one *N*=2 hypermultiplet in the adjoint rep. of $\mathfrak{g}$ with **mass *m***
- Half of the supercharges are broken, and we have *N*=2 SUSY
- The β-function still vanishes, but the superconformal invariance is explicitly broken by the mass m



The *N*=2* set-up



- The *N*=2* theory is a mass deformation of the *N*=4 SYM
- By decoupling the massive hypermultiplet with

$$m \to \infty \quad \text{and} \quad \Lambda^{2h^\vee} \equiv q\, m^{2h^\vee} \quad \text{fixed}$$

one recovers the pure *N*=2 SYM theory where

- $h^\vee$ is the dual Coxeter number for $\mathfrak{g}$
- $q = e^{2\pi i \tau}$ is the instanton counting parameter
- $2h^\vee$ is the β-function coefficient of the pure N=2 SYM

Structure of the *N*=2* prepotential

- The *N*=2* prepotential contains classical, 1-loop and non-perturbative terms

$$F^{\mathfrak{g}} = n_{\mathfrak{g}}\, i\,\pi\,\tau\, a^2 + f^{\mathfrak{g}} \quad \text{with} \quad f^{\mathfrak{g}} = f^{\mathfrak{g}}_{1-loop} + f^{\mathfrak{g}}_{non-pert}$$

- The 1-loop term reads

$$\frac{1}{4}\sum_{\alpha\in\Psi_{\mathfrak{g}}}\left[-(\alpha\cdot a)^2\log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2+(\alpha\cdot a+m)^2\log\left(\frac{\alpha\cdot a+m}{\Lambda}\right)^2\right]$$

  - $\Psi_{\mathfrak{g}}$ is the set of the roots α of the algebra $\mathfrak{g}$
  - $\alpha\cdot a$ is the mass of the W-boson associated to the root α

- The non-perturbative contributions come from all instanton sectors and are proportional to q^k and can be explicitly computed using localization for all classical algebras

(Nekrasov '02, Nekrasov-Okounkov '03, …, Billò et al 15, ...)

S-duality and the prepotential

- Take $n_{\mathfrak{g}} = 1$ for simplicity (*i.e.* ADE algebras $\mathfrak{g} = \mathfrak{g}^{\vee}$)

- The dual variables are defined as

$$a_D \equiv \frac{1}{2\pi i}\frac{\partial F^{\mathfrak{g}}}{\partial a} = \tau\left(a + \frac{1}{2\pi i\tau}\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)$$

- Applying S-duality we get

$$S(F^{\mathfrak{g}}) = i\,\pi\Big(-\frac{1}{\tau}\Big)a_D^2 + f^{\mathfrak{g}}\Big(-\frac{1}{\tau}, a_D\Big)$$

- Computing the Legendre transform we get

$$\begin{aligned}\mathcal{L}(F^{\mathfrak{g}}) &= F^{\mathfrak{g}} - 2i\pi\, a\cdot a_D \\ &= i\,\pi\Big(-\frac{1}{\tau}\Big)a_D^2 + f^{\mathfrak{g}}(\tau, a) + \frac{1}{4i\pi\tau}\left(\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)^2\end{aligned}$$

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S-duality and the prepotential

- Requiring

$$S(F^{\mathfrak{g}}) = \mathcal{L}(F^{\mathfrak{g}})$$

implies

$$f^{\mathfrak{g}}\left(-\frac{1}{\tau}, a_D\right) = f^{\mathfrak{g}}(\tau, a) + \frac{1}{4i\pi\tau}\left(\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)^2$$

- We now exploit this very powerful constraint and show its implications.

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- We now exploit this very powerful constraint and show its implications.
- For a generic algebra $\mathfrak{g}$ we have:

$$f^{\mathfrak{g}}\left(-\frac{1}{n_{\mathfrak{g}}\tau}, \sqrt{n_{\mathfrak{g}}}a_D\right) = f^{\mathfrak{g}^\vee}(\tau, a) + \frac{1}{4i\pi n_{\mathfrak{g}}\tau}\left(\frac{\partial f^{\mathfrak{g}^\vee}}{\partial a}\right)^2$$

S-duality and the prepotential

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- We now exploit this very powerful constraint and show its implications.
- For SU(2) this is related to a recursion relation and a modular anomaly equation (also in the Omega-background)

(Minahan et al ‘98, Grimm et al ‘07, Huang et al 09, Mironov-Morozov ’09,…, Billò et al ’13, … Nemkov ’13, Billò et al ’15)

S-duality and the prepotential

- Requiring

$$S(F^{\mathfrak{g}}) = \mathcal{L}(F^{\mathfrak{g}})$$

implies

$$f^{\mathfrak{g}}\left(-\frac{1}{\tau}, a_D\right) = f^{\mathfrak{g}}(\tau, a) + \frac{1}{4i\pi\tau}\left(\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)^2$$

- We now exploit this very powerful constraint and show its implications.
- The modular anomaly equation is related to the holomorphic anomaly equation of the local CY topological string description of the low-energy effective theory

(BCOV '93, Witten '93, … Aganagic et al '06, Gunaydin et al '06, Huang et al 09, Huang '13, …)

Solving the modular anomaly eq.

- We organize the quantum prepotential $f^{\mathfrak{g}}$ in a mass expansion

$$f^{\mathfrak{g}}(\tau, a) = \sum_{n=1} f_n^{\mathfrak{g}}(\tau, a) \quad \text{with} \;\; f_n^{\mathfrak{g}} \propto m^{2n}$$

- From explicit calculations, one sees that:
  - $f_1^{\mathfrak{g}}$ is only 1-loop and thus τ-independent

$$f_1^{\mathfrak{g}}(a) = \frac{m^2}{4} \sum_{\alpha \in \Psi_{\mathfrak{g}}} \log\left(\frac{\alpha \cdot a}{\Lambda}\right)^2$$

  - $f_n^{\mathfrak{g}}$ $(n \geq 2)$ are both 1-loop and non-perturbative. They are homogeneous functions

$$f_n^{\mathfrak{g}}(\tau, \lambda\, a) = \lambda^{2-2n}\, f_n^{\mathfrak{g}}(\tau, a)$$

(This is because the prepotential has mass dimension 2)

S-duality and the prepotential

- The modular anomaly equation

$$f^{\mathfrak{g}}\Big(-\frac{1}{\tau}, a_D\Big) = f^{\mathfrak{g}}(\tau, a) + \frac{\delta}{24}\left(\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)^2 \quad , \qquad \delta = \frac{6}{i\pi\tau}$$

implies

$$f^{\mathfrak{g}}_n\Big(-\frac{1}{\tau}, a_D\Big) = f^{\mathfrak{g}}_n(\tau, a) + \cdots$$

S-duality and the prepotential

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implies

$$f_n^{\mathfrak{g}}\Big(-\frac{1}{\tau}, a_D\Big) = f_n^{\mathfrak{g}}(\tau, a) + \cdots$$

- n = 1
  - Using $f_1^{\mathfrak{g}}(a) = \frac{m^2}{4}\sum_{\alpha\in\Psi_{\mathfrak{g}}} \log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2$ and

    requiring that under S-duality $\Lambda \to \tau\Lambda$, we have

$$f_1^{\mathfrak{g}}\,(a_D) = f_1^{\mathfrak{g}}\,(\tau a + \cdots) = f_1^{\mathfrak{g}}\,(a) + \cdots \quad \checkmark$$

S-duality and the prepotential

- n = 2
  - Using the definition of the dual variable and the homogeneity property, we have

$$f_2^{\mathfrak{g}}\Big(-\frac{1}{\tau}, a_D\Big) = f_2^{\mathfrak{g}}\Big(-\frac{1}{\tau}, \tau(a+\cdots)\Big) = \tau^{-2}\, f_2^{\mathfrak{g}}\Big(-\frac{1}{\tau}, a+\cdots\Big)$$

  - In order to solve the equation, we must require that

$$f_2^{\mathfrak{g}}\Big(-\frac{1}{\tau}, a+\cdots\Big) = \tau^2\, f_2^{\mathfrak{g}}\Big(\tau, a+\cdots\Big) = \tau^2\, f_2^{\mathfrak{g}}\Big(\tau, a\Big) + \cdots$$

  - i.e. $f_2^{\mathfrak{g}}(\tau, a)$ should have modular weight 2 under S-duality !

- The only quantity with this property is the second Eisenstein series E_2 (quasi-modular)

S-duality and the prepotential

- Generic n
  - The previous analysis can be easily generalized to arbitrary *n.*
  - In order to be able to solve the equation, we must have

$$f_n^{\mathfrak{g}}\left(-\frac{1}{\tau}, a+\cdots\right) = \tau^{2n-2}\, f_n^{\mathfrak{g}}\Big(\tau, a\Big) + \cdots$$

  - Thus we must require that $f_n^{\mathfrak{g}}$ depends on τ through "modular" functions with weight $2n-2$, *i.e.*

$$f_n^{\mathfrak{g}}(\tau, a) = f_n^{\mathfrak{g}}\Big(E_2(\tau), E_4(\tau), E_6(\tau), a\Big)$$

where $E_2(\tau), E_4(\tau), E_6(\tau)$ are the Eisenstein series

Eisenstein series

- The Eisenstein series are "modular" forms with a well-known Fourier expansion in $q = e^{2i\pi\tau}$:

$$E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 - 168q^4 + \cdots$$
$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \cdots$$
$$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \cdots$$

- E_4 and E_6 are truly modular forms of weight 4 and 6

$$E_4\Big(-\frac{1}{\tau}\Big) = \tau^4\, E_4(\tau) \quad , \quad E_6\Big(-\frac{1}{\tau}\Big) = \tau^6\, E_6(\tau)$$

- E_2 is quasi-modular of weight 2

$$E_2\Big(-\frac{1}{\tau}\Big) = \tau^2\, \big[E_2(\tau) + \delta\big] \quad , \quad \delta = \frac{6}{i\pi\tau}$$

- Thus a modular form of weight w is mapped under S into a form of weight w times τ^w, up to shifts induced by E_2

S-duality and the prepotential

- S-duality

$$f^{\mathfrak{g}}\Big(-\frac{1}{\tau}, a_D\Big) = f^{\mathfrak{g}}\left(E_2(-\frac{1}{\tau}), E_4(-\frac{1}{\tau}), E_6(-\frac{1}{\tau}), \tau(a + \frac{\delta}{12}\frac{\partial f^{\mathfrak{g}}}{\partial a})\right)$$

$$= f^{\mathfrak{g}}\left(E_2 + \delta, E_4, E_6, (a + \frac{\delta}{12}\frac{\partial f^{\mathfrak{g}}}{\partial a})\right)$$

$$= f^{\mathfrak{g}}(\tau, a) + \delta\left[\frac{\partial f^{\mathfrak{g}}}{\partial E_2} + \frac{1}{12}\left(\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)^2\right] + \mathcal{O}(\delta^2)$$

- Modular anomaly equation

$$f^{\mathfrak{g}}\Big(-\frac{1}{\tau}, a_D\Big) = f^{\mathfrak{g}}(\tau, a) + \delta\left[\frac{1}{24}\left(\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)^2\right]$$

S-duality and the prepotential

- S-duality

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$$= f^{\mathfrak{g}}\left(E_2 + \delta, E_4, E_6, \left(a + \frac{\delta}{12}\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)\right)$$

$$= f^{\mathfrak{g}}(\tau, a) + \delta\left[\frac{\partial f^{\mathfrak{g}}}{\partial E_2} + \frac{1}{12}\left(\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)^2\right] + \mathcal{O}(\delta^2)$$

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S-duality and the prepotential

- We thus obtain

$$\frac{\partial f^{\mathfrak{g}}}{\partial E_2} + \frac{1}{24}\left(\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)^2 = 0$$

which implies the following recursion relation

(Minahan et al '97)

$$\frac{\partial f_n^{\mathfrak{g}}}{\partial E_2} = -\frac{1}{24}\sum_{\ell=1}^{n-1} \frac{\partial f_\ell^{\mathfrak{g}}}{\partial a}\,\frac{\partial f_{n-\ell}^{\mathfrak{g}}}{\partial a}$$

  - This allows us to determine $f_n^{\mathfrak{g}}$ from the lower coefficients up to E_2-independent terms. These are fixed by comparison with the perturbative expressions (or the first instanton corrections).
  - The modular anomaly equation is a symmetry requirement; it does not eliminate the need of a dynamical input

- Once this is done, the result is valid to all instanton orders.

Exploiting the recursion: first step

- Start from

$$f_1^{\mathfrak{g}} = \frac{m^2}{4} \sum_{\alpha \in \Psi_{\mathfrak{g}}} \log \left(\frac{\alpha \cdot a}{\Lambda} \right)^2$$

and get

$$\frac{\partial f_2^{\mathfrak{g}}}{\partial E_2} = -\frac{1}{24} \left(\frac{\partial f_1^{\mathfrak{g}}}{\partial a} \right)^2 = -\frac{m^4}{96} \sum_{\alpha, \beta \in \Psi_{\mathfrak{g}}} \frac{\alpha \cdot \beta}{(\alpha \cdot a)(\beta \cdot a)} \equiv -\frac{m^4}{24} C_2^{\mathfrak{g}}$$

- Here we introduced the root lattice sums

$$C_{n;\, m_1 \cdots\, m_\ell}^{\mathfrak{g}} = \sum_{\alpha \in \Psi_{\mathfrak{g}}} \sum_{\beta_1 \neq \cdots \beta_\ell \in \Psi_{\mathfrak{g}}(\alpha)} \frac{1}{(\alpha \cdot a)^n (\beta_1 \cdot a)^{m_1} \cdots (\beta_\ell \cdot a)^{m_\ell}}$$

with $\Psi_{\mathfrak{g}}(\alpha) = \{\beta \in \Psi_{\mathfrak{g}} \,:\, \alpha^\vee \cdot \beta = 1\}$

- Thus

$$f_2^{\mathfrak{g}} = -\frac{m^4}{24} E_2\, C_2^{\mathfrak{g}}$$

Exploiting the recursion: first step

- For example

$$C_2^{\mathrm{U}(2)} = \frac{1}{(a_1 - a_2)^2}$$

$$C_2^{\mathrm{U}(3)} = \frac{1}{(a_1 - a_2)^2} + \frac{1}{(a_1 - a_2)^2} + \frac{1}{(a_2 - a_3)^2}$$

and thus

$$f_2^{\mathrm{U}(2)} = -\frac{m^4}{24} E_2(\tau)\, C_2^{\mathrm{U}(2)}$$

$$f_2^{\mathrm{U}(3)} = -\frac{m^4}{24} E_2(\tau)\, C_2^{\mathrm{U}(3)}$$

- From the Fourier expansion of E_2 we get the perturbative and all non-perturbative contributions to the prepotential at order m^4 !
- There are no free parameters !

Exploiting the recursion: first step

- More explicitly for U(2)

$$f_2^{\mathrm{U}(2)} = -\frac{m^4}{24} E_2(\tau)\,\frac{1}{(a_1-a_2)^2}$$

$$= -\frac{m^4}{24}\left(1 - 24q - 72q^2 - 96q^3 + \cdots\right)\frac{1}{(a_1-a_2)^2}$$

$$= -\frac{m^4}{24(a_1-a_2)^2} + q\,\frac{m^4}{(a_1-a_2)^2} + q^2\,\frac{3m^4}{(a_1-a_2)^2} + q^3\,\frac{4m^4}{(a_1-a_2)^2}\cdots$$

1-loop 1-instanton 2-instanton 3-instanton

- One can check that these expressions exactly agree with the perturbative 1-loop calculations and the multi-instanton results from localization

(Nekrasov '02, … (Billò et al '13)^2, … (Billò et al '15)^2)

Exploiting the recursion: second step

- Knowing $f_1^{\mathfrak{g}}$ and $f_2^{\mathfrak{g}}$, from the recursion relation we find

$$\frac{\partial f_3^{\mathfrak{g}}}{\partial E_2} = -\frac{1}{12}\frac{\partial f_1^{\mathfrak{g}}}{\partial a}\cdot\frac{\partial f_2^{\mathfrak{g}}}{\partial a} = -\frac{m^6}{288}\,E_2\sum_{\alpha,\beta\in\Psi_{\mathfrak{g}}}\frac{\alpha\cdot\beta}{(\alpha\cdot a)^3(\beta\cdot a)}$$

$$\equiv -\frac{m^6}{72}\,E_2\Big(C_2^{\mathfrak{g}}+\frac{1}{4}C_{2;1,1}^{\mathfrak{g}}\Big)$$

- Hence

$$f_3^{\mathfrak{g}} = -\frac{m^6}{144}\,E_2^2\Big(C_2^{\mathfrak{g}}+\frac{1}{4}C_{2;1,1}^{\mathfrak{g}}\Big)+x\,E_4$$

- The integration constant is fixed by comparing the m^6 term with the perturbative 1-loop result, leading to

$$f_3^{\mathfrak{g}} = -\frac{m^6}{720}\,(5E_2^2+E_4)C_2^{\mathfrak{g}} - \frac{m^6}{576}\,(E_2^2-E_4)C_{2;1,1}^{\mathfrak{g}}$$

Exploiting the recursion

- The perturbative expression + the modular anomaly equation uniquely determine the exact result to all instantons !!

- This method can be generalized to all algebras, even the non-simply laced ones ($n_{\mathfrak{g}} = 2, 3$) (Billò et al '15)

- In this case a few technical issues have to be addressed:
  - the S-duality is

$$\tau \to -\frac{1}{n_{\mathfrak{g}}\tau}$$

  - there is a modular form of weight 2

$$H_2(\tau) = \left[\left(\frac{\eta^{n_{\mathfrak{g}}}(\tau)}{\eta(n_{\mathfrak{g}}\tau)} \right)^{\lambda_{\mathfrak{g}}} + \lambda_{\mathfrak{g}}^{n_{\mathfrak{g}}} \left(\frac{\eta^{n_{\mathfrak{g}}}(n_{\mathfrak{g}}\tau)}{\eta(\tau)} \right)^{\lambda_{\mathfrak{g}}} \right]^{1-\frac{1}{n_{\mathfrak{g}}}}$$

where $\lambda_{\mathfrak{g}} = 8, 3$ for $n_{\mathfrak{g}} = 2, 3$

Exploiting the recursion

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- In this case a few technical issues have to be addressed:
  - the S-duality transformations of $\{H_2, E_2, E_4, E_6\}$ are

$$H_2\big(-\tfrac{1}{n_{\mathfrak{g}}\tau}\big) = -\big(\sqrt{n_{\mathfrak{g}}}\,\tau\big)^2 H_2 \ ,$$

$$E_2\big(-\tfrac{1}{n_{\mathfrak{g}}\tau}\big) = \big(\sqrt{n_{\mathfrak{g}}}\,\tau\big)^2\Big[E_2 + (n_{\mathfrak{g}}-1)H_2 + \delta\Big] \ ,$$

$$E_4\big(-\tfrac{1}{n_{\mathfrak{g}}\tau}\big) = \big(\sqrt{n_{\mathfrak{g}}}\,\tau\big)^4\Big[E_4 + 5(n_{\mathfrak{g}}-1)H_2^2 + (n_{\mathfrak{g}}-1)(n_{\mathfrak{g}}-4)E_4\Big] \ ,$$

$$E_6\big(-\tfrac{1}{n_{\mathfrak{g}}\tau}\big) = \big(\sqrt{n_{\mathfrak{g}}}\,\tau\big)^6\Big[E_6 + \tfrac{7}{2}(n_{\mathfrak{g}}-1)(3n_{\mathfrak{g}}-4)H_2^3 - \tfrac{1}{2}(n_{\mathfrak{g}}-1)(n_{\mathfrak{g}}-2)(7E_4\,H_2 + 2E_6)\Big] \ ,$$

Exploiting the recursion

- The perturbative expression + the modular anomaly equation uniquely determine the exact result to all instantons !!

- This method can be generalized to all algebras, even the non-simply laced ones ($n_{\mathfrak{g}} = 2, 3$) (Billò et al '15)

- In this case a few technical issues have to be addressed:
  - the lattice sums over the roots are of two types, namely long sums and short sums:

$$L^{\mathfrak{g}}_{n;\,m_1\cdots\,m_\ell} = \sum_{\alpha\in\Psi^{\mathrm{L}}_{\mathfrak{g}}} \sum_{\beta_1\neq\cdots\beta_\ell\in\Psi_{\mathfrak{g}}(\alpha)} \frac{1}{(\alpha\cdot a)^n(\beta_1\cdot a)^{m_1}\cdots(\beta_\ell\cdot a)^{m_\ell}}\ ,$$

$$S^{\mathfrak{g}}_{n;\,m_1\cdots\,m_\ell} = \sum_{\alpha\in\Psi^{\mathrm{S}}_{\mathfrak{g}}} \sum_{\beta_1\neq\cdots\beta_\ell\in\Psi^{\vee}_{\mathfrak{g}}(\alpha)} \frac{1}{(\alpha\cdot a)^n(\beta^{\vee}_1\cdot a)^{m_1}\cdots(\beta^{\vee}_\ell\cdot a)^{m_\ell}}\ ,$$

Solving the recursion

$$f_1^{\mathfrak{g}} = \frac{m^2}{4} \sum_{\alpha \in \Psi_{\mathfrak{g}}} \log\left(\frac{\alpha \cdot a}{\Lambda}\right)^2 \ ,$$

$$f_2^{\mathfrak{g}} = -\frac{m^4}{24} E_2\, L_2^{\mathfrak{g}} - \frac{m^4}{24 n_{\mathfrak{g}}} \Big[E_2 + (n_{\mathfrak{g}} - 1) H_2\Big]\, S_2^{\mathfrak{g}} \ ,$$

$$\begin{aligned} f_3^{\mathfrak{g}} = &-\frac{m^6}{720}\Big[5E_2^2 + E_4\Big]\, L_4^{\mathfrak{g}} - \frac{m^4}{576}\Big[E_2^2 - E_4\Big]\, L_{2;11}^{\mathfrak{g}} \\ &- \frac{m^6}{720 n_{\mathfrak{g}}^2}\Big[5E_2^2 + E_4 + 10(n_{\mathfrak{g}} - 1) E_2 H_2 \\ &\qquad + 5 n_{\mathfrak{g}}(n_{\mathfrak{g}} - 1) H_2^2 + (n_{\mathfrak{g}} - 1)(n_{\mathfrak{g}} - 4) E_4\Big]\, S_4^{\mathfrak{g}} \\ &- \frac{m^6}{576 n_{\mathfrak{g}}^2}\Big[E_2^2 - E_4 + 2(n_{\mathfrak{g}} - 1) E_2 H_2 \\ &\qquad + (n_{\mathfrak{g}} - 1)(n_{\mathfrak{g}} - 6) H_2^2 - (n_{\mathfrak{g}} - 1)(n_{\mathfrak{g}} - 4) E_4\Big]\, S_{2;11}^{\mathfrak{g}} \ . \end{aligned}$$

$$f_4^{\mathfrak{g}} = \ldots$$

Checks on the results

- For the classical algebras A, B, C and D
  - the ADHM construction of the *k* instanton moduli spaces is avaliable
  - the integration of the moduli action over the instanton moduli spaces can be performed à la Nekrasov using localization techniques
- In principle straightforward; in practice computationally rather intense. Not many explicit results for the N=2* theories in the literature.
- We worked it out:
  - for A_n and D_n with n<6, up to 5 instantons;
  - for C_n with n<6, up to 4 instantons;
  - for B_n with n<6, up to 2 instantons.
- The results match the q-expansion of those obtained above
- For the exceptional algebras our results are predictions!

One instanton terms

- Consider the q-expansion of the prepotential coefficients $f_n^{\mathfrak{g}}$ obtained from the recursion relation

- At order q only the sums of the type $L^{\mathfrak{g}}_{2;1\cdots 1}$ survive and we remain with

$$F^{\mathfrak{g}}_{k=1} = m^4 \sum_{\ell \geq 0} \frac{m^{2\ell}}{\ell!} \, L^{\mathfrak{g}}_{2;\underbrace{1\ldots1}_{\ell}}$$

One instanton terms

- Consider the *q*-expansion of the prepotential coefficients $f_n^{\mathfrak{g}}$ obtained from the recursion relation

- This can be given a closed form expression which is exact in *m*:

$$F_{k=1}^{\mathfrak{g}} = \sum_{\alpha \in \Psi_{\mathfrak{g}}^{\mathrm{L}}} \frac{m^4}{(\alpha \cdot a)^2} \prod_{\beta \in \Psi_{\mathfrak{g}}(\alpha)} \left(1 + \frac{m}{\beta \cdot a}\right)$$

- In the decoupling limit, taking into account that $|\Psi_{\mathfrak{g}}^{\mathrm{L}}| = 2\,h_{\mathfrak{g}}^{\vee} - 4$ we retain the highest power of *m*

$$\Lambda^{2h_{\mathfrak{g}}^{\vee}} \sum_{\alpha \in \Psi_{\mathfrak{g}}^{L}} \frac{1}{(\alpha \cdot a)^2} \prod_{\beta \in \Psi_{\mathfrak{g}}(\alpha)} \frac{1}{\beta \cdot a}$$

- This result for the pure *N*=2 SYM has been derived from completely different methods (5d realizations, Hilbert series,…)

(Benvenuti et al '10, Keller et al '11; Hanany et al '12, Cremonesi et al '14, …)

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- Our result generalizes this to the *N*=2* theory, even for the exceptional algebras

Generalizations

- These results can be extended to non-flat space-times by turning-on the so-called Ω background

$$\begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}$$

(Nekrasov '02)

which actually was already present in the localization calculations

- For $\epsilon_1, \epsilon_2 \neq 0$ one finds that the generalized prepotential

$$F^{\mathfrak{g}} = n_{\mathfrak{g}} i \pi \tau a^2 + f^{\mathfrak{g}}(a, \epsilon)$$

obeys a generalized modular anomaly equation

$$\frac{\partial f^{\mathfrak{g}}}{\partial E_2} + \frac{1}{24}\left(\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)^2 + \frac{\epsilon_1 \epsilon_2}{24}\frac{\partial^2 f^{\mathfrak{g}}}{\partial a^2} = 0$$

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$$\frac{\partial f^{\mathfrak{g}}}{\partial E_2} + \frac{1}{24}\left(\frac{\partial f^{\mathfrak{g}}}{\partial a}\right)^2 + \frac{\epsilon_1\epsilon_2}{24}\,\frac{\partial^2 f^{\mathfrak{g}}}{\partial a^2} = 0$$

Generalizations

- In the ADE case, this equation can be used to prove that S-duality acts on the prepotential as a Fourier transform

$$\exp\left(-\frac{S[F](a_D)}{\epsilon_1\epsilon_2}\right) = \left(\frac{i\,\tau}{\epsilon_1\epsilon_2}\right)^{n/2} \int d^n x \, \exp\left(\frac{2\pi\, i\, a_D \cdot x - F(x)}{\epsilon_1\epsilon_2}\right)$$

(Billo et al '13)

- This is consistent with viewing
    - a and a_D as canonically conjugate variables
    - S-duality as a canonical transformation and

$$\mathcal{Z}(a,\epsilon) = \exp\left(-\frac{F(a,\epsilon)}{\epsilon_1\epsilon_2}\right)$$

as a wave-function in this space with $\epsilon_1\epsilon_2$ as Planck's constant, in agreement with the topological string

(BCOV '93, Witten '93,Aganagic et al '06, Gunaydin et al '06 …)

$\mathcal{N}$=2 SQCD

N=2 SQCD

- Consider N=2 SYM with N_f fundamental flavours
- If $N_f = 2N_c$, the β−function vanishes (SCFT)
- One can repeat the previous analysis of S-duality by turning on masses for the flavours
- Even in the massless case, there are quantum corrections to the classical prepotential which show that the bare-coupling τ_0 is not the good modular parameter for the duality group
- The effective theory is described by a matrix of couplings

$$\tau_{ij} \sim \frac{\partial^2 F}{\partial a_i \partial a_j}$$

which are functions of τ_0

N=2 SQCD

- Consider *N*=2 SYM with N_f fundamental flavours
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- Even in the massless case, there are quantum corrections to the classical prepotential which show that the bare-coupling τ_0 is not the good modular parameter for the duality group
- For SU(2) the single effective coupling reads

$$2\pi\, i\, \tau = 2\pi\, i\, \tau_0 + i\,\pi - \log 16 + \frac{1}{2} q_0 + \frac{13}{64} q_0^2 + \cdots$$

which can be inverted to give

$$q_0 = e^{2\pi i\, \tau_0} = -16 \Big(\frac{\eta(4\tau)}{\eta(\tau)} \Big)^8$$

N=2 SQCD

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- For SU(3) in a "special vacuum" the coupling matrix is proportional to the Cartan matrix $\tau_{ij} = \tau\, C_{ij}$ with

$$2\pi\, i\, \tau = 2\pi\, i\, \tau_0 + i\, \pi - \log 27 + \frac{4}{9} q_0 + \frac{14}{81} q_0^2 + \cdots$$

which can be inverted to give

$$q_0 = e^{2\pi i\, \tau_0} = -27 \Big(\frac{\eta(3\tau)}{\eta(\tau)} \Big)^{12}$$

$\mathcal{N}$=2 SQCD

- Consider N=2 SYM with N_f fundamental flavours
- If $N_{f=}2N_{c\,,}$ the β−function vanishes (SCFT)
- One can repeat the previous analysis of S-duality by turning on masses for the flavours
- Even in the massless case, there are quantum corrections to the classical prepotential which show that the bare-coupling τ_0 is not the good modular parameter for the duality group
- These results can be generalized to SU(N) in terms of the absolute-invariants (j-invariants) of the modular group(s), at least in the so-called special vacuum

(Ashoke et al. ’16)

Applications

- Using Pestun's localization formula

$$Z_{S^4} = \int d^n x \left| \exp\left(- \frac{F(a,\epsilon)}{\epsilon_1 \epsilon_2} \right) \right|^2 \Bigg|_{a = i\,x; \epsilon_1 = \epsilon_2 = \frac{1}{R}}$$

and our modular anomaly equation, one can easily prove that the partition function on the sphere Z_{S^4} is modular invariant (a result that was expected on general grounds)

- From Z_{S^4} one can compute (by simply doing gaussian integrations) several interesting observables
  - Wilson loops
  - Zamolodchikov metric
  - Correlation functions
  - …

(Pestun '07, … ,
Baggio, Papadpdimas et al '14,
Fiol et al '15,
Gerchkovitz, Gomis, Komagordki et al '16)

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and our modular anomaly equation, one can easily prove that the partition function on the sphere Z_{S^4} is modular invariant (a result that was expected on general grounds)

- From Z_{S^4} one can compute (by simply doing gaussian integrations) several interesting observables

- Our S-duality results could be used to promote these calculations to the fully non-perturbative regime

Conclusions

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- The requirement that the duality group acts simply as in the N=4 theories also in the mass-deformed cases leads to a modular anomaly equation

- This allows one to efficiently reconstruct the mass-expansion of the prepotential resumming all instanton corrections into (quasi-)modular forms of the duality group

- A similar pattern (although a bit more intricate) arises in N=2 SQCD theories with $N_f=2N_c$ fundamental flavours

Conclusions

- This approach can be profitably used in other contexts to study the consequences of S-duality on:

  - theories formulated in curved spaces (*e.g.* S^4)
  - correlation functions of chiral and anti-chiral operators
  - other observables (e.g. Wilson loops, cusp anomaly, …)
  - more general extended observables (surface operators, ...
  - …

  with the goal of studying the strong-coupling regime

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  with the goal of studying the strong-coupling regime.

Thank you for your attention