Symmetries and Soft Theorems
in
Field and String Theories

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Based on:

P. Di Vecchia, R.M. , M. Mojaza, J. Nohle arXiv:1512.03316.

P. Di Vecchia, R.M. , M. Mojaza arXiv:1502.05258, 1507.00938, 1604.03355.

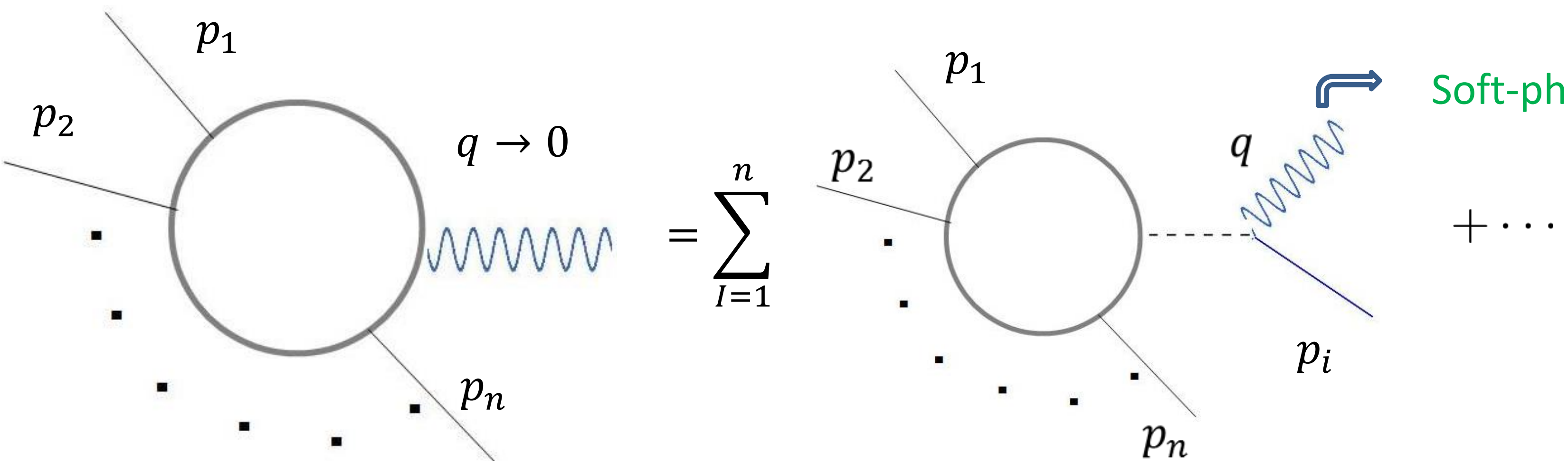
Summary

- Old-soft theorems: Low and Weinberg soft theorems. Adler's zero.
- New low energy -theorems.
- Soft-theorems in Bosonic and Superstring theories.
- New soft theorems for dilatons at sub-subleading order.
- Double soft behavior of scalar and Yang-Mills amplitudes.

Motivations

- It is well know that scattering amplitudes in the deep infrared region (or soft limit) satisfy interesting relations.

Low's theorem: Amplitudes with a soft photon are determined from the corresponding amplitudes without the soft particle



Soft-photon polarization

Angular momentum

$$\mathcal{M}_{n+1}(q,\, p_1,\, \ldots p_n) \sim \left[\sum_{i=1}^{n} e_i \frac{\epsilon_q \cdot p_i}{q \cdot p_i} - i \sum_{i=1}^{n} e_i \frac{\epsilon_{q\mu} q_\nu J_i^{\mu\nu}}{k_i \cdot q}\right] \mathcal{M}_n(p_1, \ldots p_n) + \mathcal{O}(q)$$

e_i charge of the particle i

Weinberg: Amplitudes involving gravitons and matter particles show an universal behavior when one graviton becomes soft.

$$\mathcal{M}^{\mu\nu}_{n+1}(q,\, p_1,\, \ldots p_n) = \kappa_D \left[\sum_{i=1}^{n} \frac{p_i^\mu p_i^\nu}{q\cdot p_i} - i \sum_{i=1}^{n} \frac{k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q} \right] \mathcal{M}_n(p_1, \ldots p_n) + \mathcal{O}(q)$$

$\Downarrow$

They were recognized to be a consequence of the gauge invariance

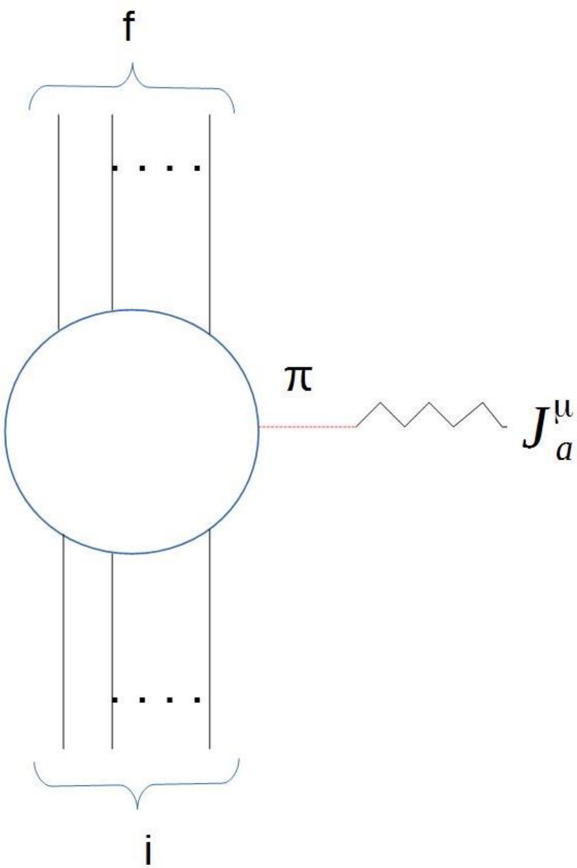
$$q^\mu M^\mu_{n+1}(q,\, p_1, \ldots p_n) = 0 \quad ; \quad q_\mu M^{\mu\nu}_{n+1}(q,\, p_1, \ldots p_n) = q_\nu M^{\mu\nu}_{n+1}(q,\, p_1, \ldots p_n) = 0$$

$\Downarrow$ Soft-Photon

$\Downarrow$ Soft-graviton

Adler's zero

(Weiberg, The Quatum Theory of Fields Vol .II.)



Goldstone theorem: When a symmetry G is spontaneously broken to a sub-group H the spectrum of the theory contains as many Goldston bosons π^a, parametrizing the coset space $^{G}/_{H}$, as the number of the broken generators $J_\mu{}^a$.

The matrix elements of the currents $J_\mu{}^a$ associated to the broken generators are

$$\langle 0|J^a_\mu|\pi^b(p)\rangle = ip_\mu\, F_\pi\, \delta^{ab}$$

The current between arbitrary states has a pole at $q^2 \to 0$ then:

$$\langle f|J^a_\mu(0)|i\rangle = \frac{iF\, q^\mu}{q^2}\langle f+\pi|i\rangle + N^\mu_{if}$$

no-pole contribution to the matrix element of the current.

Goldstone pole dominance for $q^2 \to 0$

The conservation law $\partial_\mu J^\mu{}_a$=0 requires: $\langle f+\pi|i\rangle = i\frac{q_\mu\; N^\mu_{if}}{F}$

Unless $N^\mu{}_{if}$ has a pole at $q\to 0$, the matrix element $\langle f+\pi|i\rangle$ for emitting a Goldstone boson in a transition $i\to f$ vanishes as $q\to 0$. ⇨ Adler zero.

Double Soft Limit

The double soft limit contains information on the structure constants of the spontaneously broken symmetry group:.

$$[V_0^a, V_0^b] = if_H^{abc}V_0^c \quad ; \quad [V_0^a, J_0^b] = if^{abc}J_0^c \quad ; \quad [J_0^a, J_0^b] = iF^{abc}\,V_0^c$$

Generators of the unbroken group H

$$\frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\nu}\langle T\left\{J_a^\mu(x_1)J_b^\nu(x_2)\prod_{i=1}^n J_{a_i}^{\mu_i}\right\} = \frac{\partial}{\partial x_2^\nu}\langle T\left[J_a^0,\, J_b^\nu\right]\prod_{i=1}^n J_{a_i}^{\mu_i}(x_i)\rangle$$

$$+\sum_{i=1}^n \frac{\partial}{\partial x_2^\nu}\langle J_b^\mu(x_2)\prod_{j=i}^{i-1} J_{a_j}^{\nu_j}(x_j)\left[J_a^0,\, J_{a_i}^{\mu_i}\right]\prod_{j=i+1}^n J_{a_j}^{\mu_j}(x_j)\rangle$$

- LSZ reduction:

$$\langle\prod_{i=1}^n J_a^{\mu_1}(p_i)\rangle = \prod_{i=1}^n \frac{-F\,p_{\mu_i}}{p_i^2}\langle\pi_a(p_1)\ldots\pi_{a_n}(p_n)\rangle^t + \text{on-shell regular terms}$$

- Double soft behavior:

$$\langle\pi_a^\mu(\epsilon p)\pi_b^\nu(\epsilon q)\prod_{i=1}^n \pi_{a_i}^{\mu_i}(p_1\rangle_{\epsilon\to 0} = -\frac{1}{2}\sum_{i=1}^n \frac{(q-p)\cdot p_i}{p_i\cdot(p+q)}\frac{f^{ca_il}F_{abc}}{F^2}\langle\pi^{a_1}(p_1)\ldots\pi^l(p_i)\ldots\rangle + \ldots$$

S. Weinberg, Phys. Rew. Lett. 17, (1966), 336; N. Arkani-Hamed, F. Cachazo, J. Kaplan, arXiv:0808.1446; K. Kampf, J Novotny, J. Trnka, 1304.3048.

New Soft-Theorems

- Recently the interest in the subject has been renewed by a proposal of A. Strominger (arXiv:1312.2229) and T. He, V. Lysov, P. Mitra and A. Strominger (arXiv:1401.7026) asserting that soft-theorems are nothing but the Ward-identities of the BMS-symmetry of asymptotic flat metrics.

- Sketch of the idea: A general Lorentzian metric of an asymptotically flat space-time can be written in the form:

(Bondi, van der Burg, Metzner, Proc. Roy. Soc. Lond. A 269,21 (1962); Sachs, Proc.Roy. Soc. Lond. A270, 103(1962))

$(r, x^A) \equiv (r, \theta, \phi)$ sperical coordinates

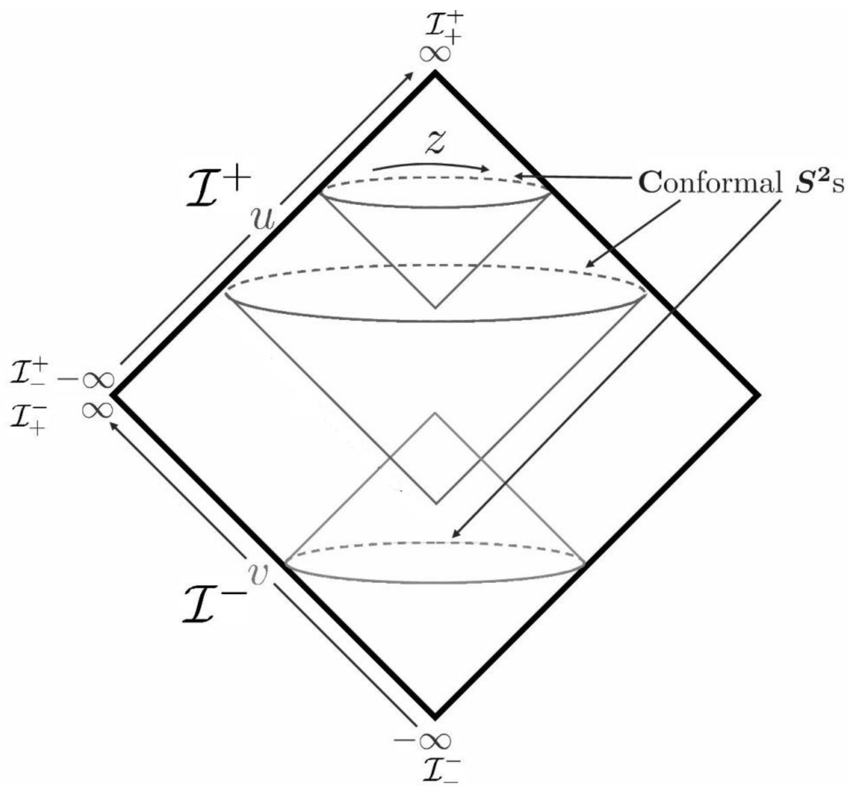
$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + r^2 g_{AB}\left(dx^A - U^A du\right)\left(dx^B - U^B du\right)$$

Retarded Bondi's coordinates $u = t - r$

- Asymptotic behaviors $u \to -\infty$: $V \simeq O(r) \;\; ; \;\; \beta \simeq O(r^{-2}) \;\; ; \;\; U \simeq O(r^{-2})$

$$g_{AB}dx^A dx^B = (d\theta^2 + \sin^2\theta d\phi^2) + O(r^{-1})$$

- Solutions asymptotically flat for $v \to \infty$, are obtained changing $u \leftrightarrow v = t + r$



- The "conform mapping" of the asymptotic metric is regular at $r = u - v \to \infty$ (conformal infinity)

$$d\hat{s}^2 = \frac{ds^2}{r^2} = -\frac{dudv}{r^2} + (d\theta^2 + sin^2\theta d\phi^2)$$
$$= -\frac{dudv}{r^2} + \gamma_{z\bar{z}}dzd\bar{z}$$

With $z = e^{i\phi}\cot\frac{\theta}{2}$ and $\gamma_{z\bar{z}} = \frac{4}{(1+z\bar{z})^2}$

The boundaries (null infinities) $\mathcal{I}^+ \equiv (v \to \infty)$ and $\mathcal{I}^- \equiv (u \to -\infty)$ have topology $R \times S^2$.

- The BMS-transformation are coordinate transformations leaving invariant the form of the original metric and preserving its asymptotic behavior.

Conformal factor of S^2

$$z \to \frac{\alpha z + \beta}{\gamma z + \delta} \quad ; \quad u \to k[u + a(z,\, \bar{z})]$$

$$\alpha\delta - \beta\gamma = 1 \quad ; \quad k = \frac{1 + z\bar{z}}{|\alpha z + \beta|^2 + |\gamma z + \delta|^2}$$

- Supertranslations $\quad z \to z \quad ; \quad u \to u + a(z,\, \bar{z})$

- Strominger's conjecture: BMS invariance of the S-matrix:

$T^{\pm}$ are the supertranslation generators defined on $\mathcal{I}^{\pm}$.

$$\langle out|T^{+}\mathcal{S} - \mathcal{S}T^{-}|in\rangle = 0$$

Weinberg soft theorem for gravitons

- A. Strominger and F. Cachazo have proposed that 4D three level graviton amplitudes have a universal behavior through second subleading order in the soft –graviton momentum.
- (F. Cachazo, A. Strominger:arXiv:14044091)

- These considerations were extended to gluons through the subleading order. The soft-gluon theorem arises as the Ward identity of a two dimensional Kac-Moody type symmetry.
- (T. He, V. Lysonv, P. Mitra, A. Strominger, arXiv: 1409.7026)

- These theorems to subleading order for gluons and sub-subleading order for gravitons have been proved in arbitrary dimensions by using Poincaré and on-shell gauge invariance of the amplitudes.
- (J. Broedel, M. de Leeuw, J. Plefka, M. Bosso , arXiv: 1406.6574 and Z. Bern, S. Davies, P. Di Vecchia, and J. Nohle, arXiv:1506.6987)

- In these new soft theorems $n+1$-point amplitudes with a soft graviton or gluon are obtained acting on n-point amplitude with universal soft operators (Huge literature).

Soft-particle momentum ⇧

Only for soft- gravitons ⇧

$$A_{n+1}(q,\, p_1, \ldots p_n) = \left(S^{(0)} + S^{(1)} + S^{(2)}\right) A_n(p_1, \ldots p_n)$$

Graviton soft-operators

$$S^{(0)} = \epsilon_{\mu\nu} \sum_{i=1}^{n} \frac{k_i^\mu \cdot k_i^\nu}{k_i \cdot q} \qquad S^{(1)} = -i\epsilon_{\mu\nu} \sum_{i=1}^{n} \frac{q_\rho\, k_i^\mu\, J_i^{\nu\rho}}{k_i \cdot q} \qquad S^{(2)} = -\epsilon_{\mu\nu} \sum_{i=1}^{n} \frac{q_\rho\, J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{2k_i \cdot q}$$

Gluon soft-operators

$$S^{(0)} = \frac{k_1 \cdot \epsilon}{\sqrt{2}k_1 \cdot q} - \frac{k_n \cdot \epsilon}{\sqrt{2}k_n \cdot q} \qquad S^{(1)} = -i\epsilon_\mu p_\sigma \left(\frac{J_1^{\mu\sigma}}{\sqrt{2}k_1 \cdot q} - \frac{J_n^{\mu\sigma}}{\sqrt{2}k_n \cdot q} \right)$$

ϵ is the polarization of the soft particle and J_i total angular momentum (orbital+spin) of the matter particles.

- In these approach soft-theorems follows from the symmetries of the action. On the other hand, soft theorems associated to broken symmetries can be used to constrain effective lagrangians. (C. Cheung, K. Kampf, J. Novotny, C. H. Shen and J. Trnka, arXiv:1509.03309; I. Low and A. V. Manohar, 0110285; I. Low, arXiv:1412.2145; M. Bianchi, J. F. Morales and C. Wen, 1508.00554; Y. t. Huang and C. Wen, 1509.07840; H. Luo and C. Wen, JHEP 1603, 1512.06801.)

⇩

Infrared dynamic and symmetries are deeply connected

Soft Theorems in String Theories

In string theory the soft theorems have been investigated by:
(Ademollo et al (1975) and J. Shapiro (1975); B.U.W. Schwab, arXiv:1406.4172 and arXiv:1411.6661; M. Bianchi, Song He, Yu-tin Huang and Congkao Wen arXiv:1406.5155; P. Di Vecchia, R. M. and M. Mojaza, arXiv:1507.00938, 1512.0331 and 1604.03355.
M. Bianchi and A. Guerrieri, arXiv:1505.05854, 1512.00803 and 1601.03457)

- How much universal are the soft theorems? Naively they should be modified in any theory with a modified three point interaction. String theories is a good arena where to explore the universality of low energy theorems.

- String effective action up to $O(\alpha')$:

$$S = \int d^D x \sqrt{G} \Bigg\{ \frac{1}{2\kappa_D} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{24\kappa_D^2} e^{-\frac{4\kappa_D \phi}{\sqrt{D-2}}} H^2 + \frac{\lambda_0 \alpha'}{2\kappa_D^2} e^{-\frac{2\kappa_D \phi}{\sqrt{D-2}}} \left(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 \right) + \dots \Bigg\}$$

(Zwiebach, Phys. Lett. B(156) (1985)315; Metasaev, Tseytlin, Nucl. Phys. B293(1987), 385.)

$\lambda_0 = 0, \frac{1}{4}, \frac{1}{8}$ respectively for superstring , bosonic and Heterotic string

- Massless closed strings include the graviton, dilaton and Kalb Ramond. ⇨ Soft Theorems for Dilaton and Kalb-Ramond states

- String theory is also a powerful tool to get field theory amplitudes. There are few diagrams at each order of the perturbative expansions that are represented as complex integrals on the string moduli space.

- Single and double soft limit can be “easily” studied on amplitudes involving n arbitrary external states.

Amplitudes in Bosonic and Superstring theories

- In closed bosonic and superstring theory amplitudes with a graviton or a dilaton or a Kalb-Ramond field with soft momentum q and n hard particles with momentum k_i, are obtained from the same two index tensor $M^{\mu\nu}_{n+1}(q,\, k_1, \ldots k_n)$

z_l are complex coordinates parametrizing the insertion of the string world-sheet of the hard state vertex operators

$F^{\mu\nu}$ is a function of all Koba-Nielsen variables having pole for $z \sim z_i$

$$M^{\mu\nu}_{n+1}(q,\, k_1, \ldots k_n) = M_n(\epsilon_i,\, k_i) * \int dz \prod_{l=1}^{n} |z - z_i|^{\alpha' q\cdot k_i} \mathcal{F}^{\mu\nu}(q,\, \epsilon_i,\, k_i;\, z,\, z_i)$$

Model dependent n points amplitude

z Koba-Nielsen variable of the soft particle

- All integrals necessary for the result can be reduced to the general form:

$$I^{j_1 j_2 \cdots}_{i_1 i_2 \cdots} = \frac{1}{2\pi}\int d^2z \, \frac{\prod_{l=1}^{n} |z-z_l|^{\alpha' k_l q}}{(z-z_{i_1})(z-z_{i_2})\cdots(\bar{z}-\bar{z}_{j_1})(\bar{z}-\bar{z}_{j_2})\cdots} \, .$$

- However all the higher index integrals can all be obtained with some "tricks" from two of them:

$$I^i_i = \frac{2}{\alpha' k_i q}\left(1 + \alpha' \sum_{j\neq i}(k_j q)\log|z_i - z_j| + \frac{(\alpha')^2}{2}\sum_{j\neq i}\sum_{k\neq i}(k_j q)(k_k q)\log|z_i-z_j|\log|z_i-z_k|\right)$$
$$+(\alpha')^2\sum_{j\neq i}(k_j q)\log^2|z_i-z_j| + \log\Lambda^2 + \mathcal{O}(q^2)$$

and

$$I^j_i = \sum_{m\neq i,j}\frac{\alpha' q k_m}{2}\left(\mathrm{Li}_2\left(\frac{\bar{z}_i-\bar{z}_m}{\bar{z}_i-\bar{z}_j}\right) - \mathrm{Li}_2\left(\frac{z_i-z_m}{z_i-z_j}\right) - 2\log\frac{\bar{z}_m-\bar{z}_j}{\bar{z}_i-\bar{z}_j}\log\frac{|z_i-z_j|}{|z_i-z_m|}\right)$$
$$-\log|z_i-z_j|^2 + \log\Lambda^2 + \mathcal{O}(q^2)$$

Dilogarithmic function

Λ is a cut-off on large $|z|$

One soft-graviton and n-hard particles

Soft-graviton amplitude is obtained by saturating the n+1-amplitude with the polarization:

$$\epsilon^g_{\mu\nu} = \frac{1}{2}\left(\epsilon_{\mu\nu} + \epsilon_{\nu\mu}\right) \quad ; \quad \eta^{\mu\nu}\epsilon^g_{\mu\nu} = 0$$

$$\epsilon^g_{\mu\nu} M^{\mu\nu}_{n+1}(q,\, k_1 \dots k_n) = \left(S^{(0)} + S^{(1)} + S^{(2)}\right) M_n(k_1 \, \dots k_n)$$

- $S^{(0)}$ is the standard Weinberg leading soft behavior.
- $S^{(1)}$ in bosonic and superstring theory is:

$$S^{(1)} = -i\epsilon^g_{\mu\nu} \sum_{i=1}^{n} \frac{q_\rho \, k_i^\mu \, J_i^{\nu\rho}}{k_i \cdot q} \qquad J_i^{\nu\rho} = L_i^{\mu\rho} + \mathcal{S}_i^{\mu\rho} + \bar{\mathcal{S}}_i^{\mu\rho}$$

$$L_i^{\mu\rho} = k_i^\mu \frac{\partial}{\partial k_{i\rho}} - k_i^\rho \frac{\partial}{\partial k_{i\mu}} \quad ; \quad \mathcal{S}_i^{\mu\rho} = \epsilon_i^\mu \frac{\partial}{\partial \epsilon_{i\mu}} - \epsilon_i^\rho \frac{\partial}{\partial \epsilon_{i\mu}}$$

$$\boxed{\epsilon_{\mu\nu} = \epsilon_\mu \; \bar{\epsilon}_\nu}$$

Sub-subleading order is different in bosonic and superstring amplitudes.

$$S^{(2)} = -\frac{\epsilon^g_{\mu\nu}}{2}\left[\frac{q_\rho J_i^{\mu\rho}\, q_\sigma J_i^{\nu\sigma}}{k_i\cdot q} - \left(\frac{q_\rho q_\sigma \eta_{\mu\nu} - q_\sigma q_\nu \eta_{\rho\mu} - q_\sigma q_\mu \eta_{\sigma\nu}}{k_i\cdot q}\right)\left(k_i^\rho \frac{\partial}{\partial k_{i\sigma}} + \Pi^{\rho\sigma}\right)\right.$$
$$\left. -\alpha'\left(q_\sigma k_{i\nu}\eta_{\rho\mu} + q_\rho k_{i\mu}\eta_{\sigma\nu} - \eta_{\rho\mu}\eta_{\sigma\nu}(k_i\cdot q) - q_\rho q_\sigma \frac{k_{i\nu}k_{i\nu}}{k_i\cdot q}\right)\right]\Pi^{\rho\sigma}$$

String correction.

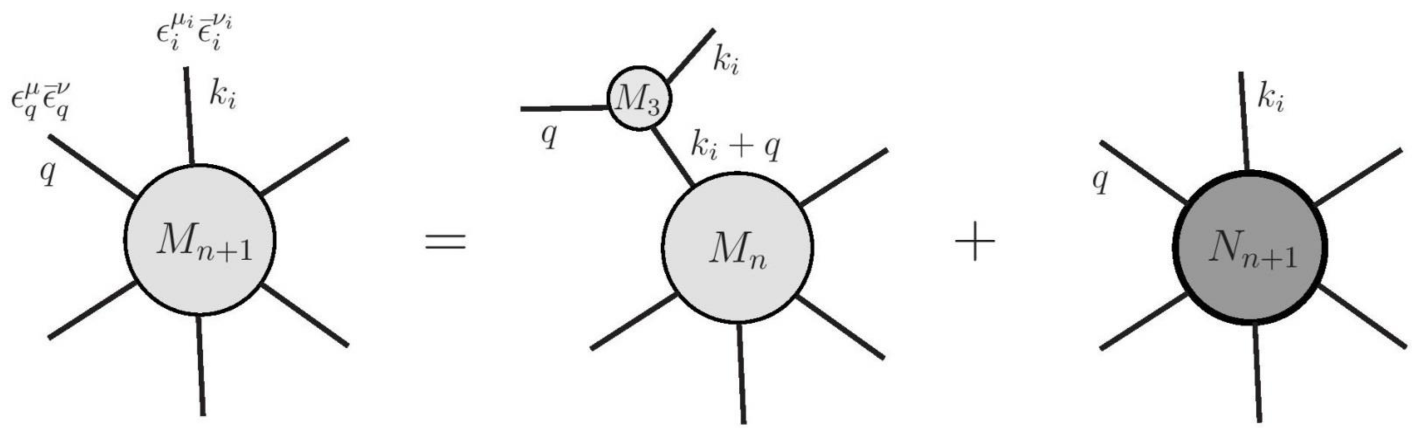
$$\Pi^{\rho\sigma} = \epsilon_i^\rho \frac{\partial}{\partial \epsilon_{i\sigma}} + \bar{\epsilon}_i^\rho \frac{\partial}{\partial \bar{\epsilon}_{i\sigma}}$$

- The expression in the square bracket is the same both for soft graviton and dilaton.
- String corrections are present only in bosonic string amplitudes. They are due to coupling between the dilaton and the Gauss-Bonnet term which is present in the bosonic string effective action but not in its supersymmetric extension.

The complete three point amplitude with massless states (graviton + dilaton + Kalb-Ramond) in bosonic string theory is:

$$M_3^{\mu\nu;\mu_i\nu_i;\alpha\beta} = 2\kappa_D \left(\eta^{\mu\mu_i} q^\alpha - \eta^{\mu\alpha} q^{\mu_i} + \eta^{\mu_i\alpha} k_i - \frac{\alpha'}{2} k_i^\mu q^{\mu_i} q^\alpha\right) \times \left(\eta^{\nu\nu_i} q^\beta - \eta^{\nu\beta} q^{\nu_i} + \eta^{\nu_i\beta} k_i^\nu - \frac{\alpha'}{2} k_i^\nu q^{\nu_i} q^\beta\right)$$

The generic $n+1$ amplitude is decomposed in the exchange diagram and a finite contribution in the soft expansion.



The only three point amplitude proportional to α' that contributes at order q is the graviton, dilaton, dilaton , vertex.

By using the explicit form of the of the three point string amplitude and focusing only on the term liner in α'.

$$\mathcal{M}_{n+1}^{\mu\nu}\Big|_{\alpha'} = -\frac{\alpha'}{2}\sum_{i=1}^{n}\frac{k_i^{\mu}\,k_i^{\nu}}{k_i\cdot q}q_\rho\, q_\sigma \Pi^{\{\rho,\sigma\}}M_n(k_i) + N^{\mu\nu}(q,\,k_i)\Big|_{\alpha'} + O(q^2)$$

$$\Pi^{\rho\sigma} = \epsilon_i^{\rho}\frac{\partial}{\partial\epsilon_{i\sigma}} + \bar{\epsilon}_i^{\rho}\frac{\partial}{\partial\bar{\epsilon}_{i\sigma}}$$

Now, imposing gauge invariance $q_\mu M_{n+1}^{\mu\nu} = 0$ on the previous expression and Taylor expanding in q , one gets:

$$N^{\mu\nu}(q=0,k_i)\Big|_{\alpha'} = 0 \;\; ; \;\; \frac{1}{2}\left(\frac{\partial N^{\mu\nu}}{\partial q_\rho} + \frac{\partial N^{\rho\nu}}{\partial q_\mu}\right)\Big|_{\alpha',\,q=0} = \alpha'\sum_{i=1}^{n}k_i^{\nu}\Pi_i^{\{\rho,\mu\}}M_n(k_i$$

Inserting this back into the starting expression, and imposing also $q_\nu M_{n+1}^{\mu\nu} = 0$ one obtains again the string correction to the sub-subleading soft-graviton operator.

Kalb-Ramond soft theorem

- Soft-Kalb-Ramond amplitude is obtained by saturating the n+1-amplitude with the polarization:

$$\epsilon^{B}_{\mu\nu} = \frac{1}{2}\left(\epsilon_{\mu\nu} - \epsilon_{\nu\mu}\right)$$

- The leading divergent term doesn't contribute because it is symmetric in the two free indices and one gets:

$$\mathcal{M}_{n+1} \simeq -i\epsilon^{B}_{\mu\nu} \sum_{i=1}^{n} \frac{k_i^{\nu}\, q_{\rho}}{q \cdot k_i} \left[(L_i + S_i)^{\mu\rho} - (\bar{L}_i + \bar{S}_i)^{\mu\rho} \right] M_n(k_i, \epsilon_i, \bar{k}_i, \bar{\epsilon}_i) \Big|_{k=\bar{k}} + \mathcal{O}(q^1)$$

- It is gauge invariant $q_{\mu} M^{\mu\nu}_{n+1} = q_{\nu} M^{\mu\nu}_{n+1} = 0$.
- It reproduces the soft behavior of the antisymmetric tensor, but it is not a real soft theorem as in the case of the graviton and dilaton because, due to the separation of k and $\bar{k}$, the amplitude is not a physical amplitude.

Soft-theorem for the gravity dilaton.

Soft-dilaton amplitude is obtained by saturating the n+1- string amplitude with the dilaton projector:

$$\epsilon_d^{\mu\nu} = \frac{1}{\sqrt{D-2}} \left(\eta^{\mu\nu} - q^\mu \bar{q}^\nu - q^\nu \bar{q}^\mu \right) \quad ; \quad \bar{q} \text{ lightlike vector } q \cdot \bar{q} = 1$$

D is the space-time dimension

Similarly, the soft behavior of a dilaton in an amplitude with hard gravitons and/or dilatons

$$\epsilon_d^{\mu\nu} M_{\mu\nu}(q; k_i) = \frac{\kappa_D}{\sqrt{D-2}} \Big\{ 2 - \sum_{i=1}^{n-1} k_i^\mu \frac{\partial}{\partial k_i^\mu} + \frac{q^\rho}{2} \sum_{i=1}^{n-1} \left[\left(2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho} \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} \right) - i S_{\mu\rho}^{(i)} \frac{\partial}{\partial k_{i\mu}} \right] + \sum_{i=1}^{n-1} \frac{q^\rho q^\sigma}{2 k_i \cdot q} \left((S_{\rho\mu}^{(i)}) \eta^{\mu\nu} (S_{\nu\sigma}^{(i)}) + D \epsilon_{i\rho} \frac{\partial}{\partial \epsilon_i^\sigma} \right) \Big\} M_n + \mathcal{O}(q^2) \,.$$

with: $k_n = \sum_{i=1}^{n} k_i + q$.

Remarkably, in the gravity dilaton soft theorem the dependence on α', disappears and appear the generators of the conformal transformations acting in momentum space.

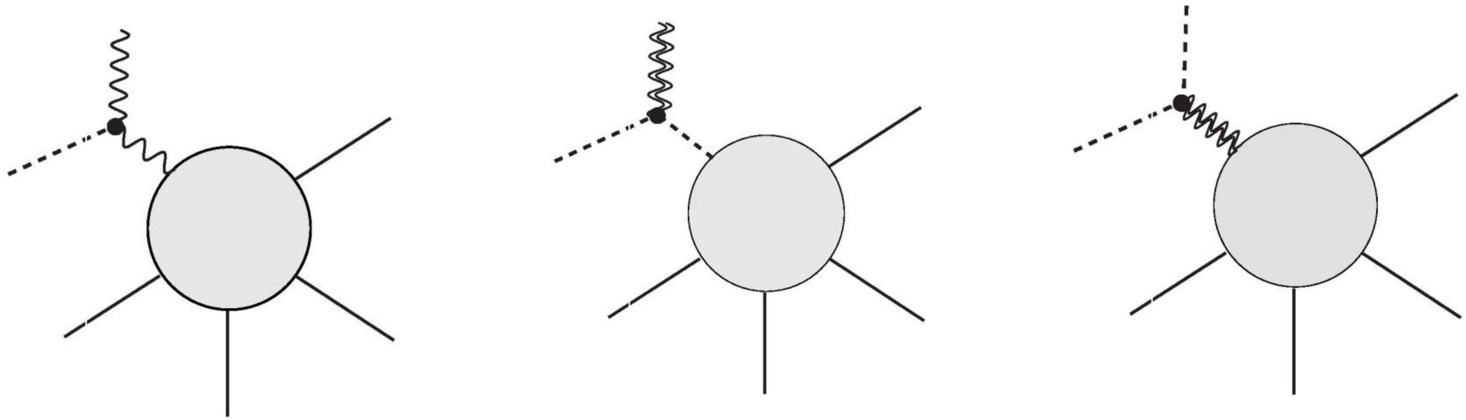
$$\mathcal{D} \equiv k_\mu \frac{\partial}{\partial k_\mu}$$

Generator of the scale transformations

$$\mathcal{K}_\rho = \left(2\, k^\mu \frac{\partial^2}{\partial k^\mu \partial k^\rho} - k_\rho \frac{\partial^2}{\partial k^\mu \partial k_\mu}\right) - i S_{\mu\rho} \frac{\partial}{\partial k_\mu}$$

Generator of the special conformal transformations

The singular terms present to $O(q)$ are due to the exchange diagrams where a soft-dilaton interacts with an hard graviton or an hard Kalb-Ramond.



Dilaton , Antisymmetric, Atisymmetric Vertex

Dilaton , Dilaton, Graviton, Vertex

Dilaton , Dilaton, Graviton, Vertex

- The n-point amplitude may involve other external hard states, than those specified in the $n+1$-point amplitude.
- The dilaton, graviton, graviton vertex is subleading in the soft expansion.
- The same result has been obtained also in field theory via gauge invariance.

Soft theorem for massive scalar matter

Bosonic string theory is a framework where the universality of soft graviton theorem, through the sub-subleading order, can be easily studied in the case of scalar massive matter.

According to an old standard trick, the tachyon can be seen as a scalar particle with $m^2 = -\frac{1}{\alpha'}$ and the amplitude with a soft graviton (and dilaton) and n-hard scalar particles Is:

z parametrizes the insertion on the word sheet of the soft particle vertex.

$$\mathcal{T}^{\mu\nu}_{n+1} = \mathcal{T}_n * \frac{\alpha'}{2} \int d^2 z \prod_{i=1}^{n} |z - z_l|^{\alpha' k_l q} \sum_{i,j=1}^{n} \frac{k_i^\mu \, k_j^\nu}{(z - z_i)(\bar{z} - \bar{z}_j)}$$

The integral over z is done up to the sub-subleading order

$$\mathcal{T}^{\mu\nu}_{n+1} \simeq \sum_{i=1}^{n} \left[\frac{k_i^\mu \, k_i^\nu}{k_i \cdot q} - i \frac{k_i^\nu \, q_\rho J_i^{\mu\rho}}{q \cdot k_i} - \frac{q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{2 k_i \cdot q} \right] T_n + \mathcal{O}(q^2)$$

It has the same form as for massless hard particles.

- The amplitude of a soft dilaton and n-scalars then reads

$$\epsilon_d^{\mu\nu} T_{\mu\nu}(q;k_i) = \frac{\kappa_D}{\sqrt{D-2}} \Bigg\{ -\sum_{i=1}^{n} \frac{m_i^2}{k_i \cdot q} \left(1 + q^\rho \frac{\partial}{\partial k_i^\rho} + \frac{q^\rho q^\sigma}{2} \frac{\partial^2}{\partial k_i^\rho \partial k_i^\sigma} \right)$$

$$+2 - \sum_{i=1}^{n} k_i^\mu \frac{\partial}{\partial k_i^\mu} + \frac{q^\rho}{2} \sum_{i=1}^{n} \left(2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho} \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} \right) \Bigg\} \mathcal{T}_n + \mathcal{O}(q^2)\,,$$

- The soft- behavior is the same both for Bosonic and Superstring amplitudes.
- The presence of the conformal generators seems universal. They appear both for hard-massless particles and for massive scalar matter.
- Is there any (broken) symmetry behind this behavior?

String dilaton vs field theory dilaton

- The low energy behavior of string dilaton involves the generators of the conformal group.
- String theories are not conformal invariant because they contain a dimensional parameter, α', which gives mass to all the string states.
- Could the string Dilaton be the Nabu-Goldston boson of a spontanuosly broken conformal symmetry?
- The Nabu-Goldstone boson of the spontaneously broken conformal symmetry is again named dilaton, to distinguish them we refer to this latter as field theory dilaton.

Dilaton of broken conformal scale invariance

A field theory whose action is invariant under some trasformation, with corresponding Noether current J^{μ} satisfies the following Ward-identity:

$$\int d^D x\, \mathrm{e}^{-iq\cdot x}\Big[-\partial_\mu\, T^*\langle 0|j^\mu(x)\phi(x_1)\ldots\phi(x_n)|0\rangle + T^*\langle 0|\partial_\mu j^\mu(x)\phi(x_1)\ldots\phi(x_n)|0\rangle\Big]$$
$$= -\sum_{i=1}^{n} \mathrm{e}^{-iq\cdot x_i}\, T^*\langle 0|\phi(x_1)\ldots\delta\phi(x_i)\ldots\phi(x_n)|0\rangle\,,$$

- T^* denotes the T-product with the derivatives placed outside of the time-ordering symbols. $\delta\phi$ is the infinitesimal transformation of the filed under the generators of the supesymmetry.

- We consider the Ward-identity associated to the conformal transformations to derive the low-energy theorem.

- The starting point is a theory invariant under conformal transformations.

➢ Example (R.Boels, W. Woermsbecker, arXiv:1507.08162):

$$L = -\frac{1}{2}\partial_\mu\xi\partial^\mu\xi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left(\frac{\lambda\xi}{a}\right)^{\frac{4}{D-2}}\phi^2 a^{\frac{4}{D-2}}$$

Expanding around the flat direction $\xi = a + r$, the field ϕ acquire mass $m^2 = (\lambda a)^{\frac{4}{D-2}}$ and the conformal invariance is spontaneously broken. The field r remains massless (dilaton).

- The generator of the conformal group are the traslations ($P_\mu = i\partial_\mu$), Lorentz transformations ($M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$), dilatations ($D = ix^\mu\partial_\mu$) and special conformal transformations ($K^\mu = i(2\,x^\mu x^\nu - \eta^{\mu\nu}x^2)\partial_\nu$).
- The Noether current associated to dilatations satisfies:

$$J^\mu_{\mathcal{D}} = x_\nu T^{\mu\nu} \quad ; \quad \partial_\mu J^\mu_{\mathcal{D}} = T^\mu_{\ \mu}$$

being $T^{\mu\nu}$ the energy momentum tensor.

The action of the generator of scale transformations on a scalar field is:

$$\delta\phi(x) = [\mathcal{D}, \phi(x)] = i\,(d + x^\mu \partial_\mu)\,\phi(x)\,,$$

In a theory with spontaneously broken conformal symmetry, the Goldoston boson ξ (NG -Dilaton) is by its equation of motion related to the trace of the energy momentum tensor:

$$T_\mu{}^\mu(x) = -v\,\partial^2\,\xi(x)\,,$$

v is the vev of the field ξ. We have found in different conformal models with scalar fields that:

$$v = \frac{D-2}{2}\langle\xi\rangle$$

Amplitudes are obtained from the Ward-identity via the LSZ-opertor

$$\left[\mathrm{LSZ}\right] \equiv i^n \left(\prod_{j=1}^{n} \lim_{k_j^2 \to -m_j^2} \int d^D x_j \, \mathrm{e}^{-ik_j \cdot x_j} (-\partial_j^2 + m_j^2) \right) ,$$

where the limits $k_j^2 \to -m^2$ put the external states on shell, which has to be performed only at the end.

We can neglect the first term in the Ward-identity by keeping terms of $O(q^0)$ provided that there is no pole.

$$(-i)\, v\, (2\pi)^D \delta^{(D)}(\sum_{j=1}^{n} k_j + q) \mathcal{T}_{n+1}(q; k_1, \ldots, k_n)$$

$$= - \sum_{i=1}^{n} \left[\lim_{k_i^2 \to -m_i^2} (k_i^2 + m_i^2)\, i \left(d - D - (k_i + q)^\mu \frac{\partial}{\partial k_i^\mu} \right) \right.$$

$$\left. \times \frac{(2\pi)^D \delta^{(D)} \left(\sum_{j=1}^{n} k_j + q \right)}{(k_i + q)^2 + m_i^2} \mathcal{T}_n(k_1, \ldots, k_i + q, \ldots, k_n) \right]$$

- The next step consists in commuting the delta function with the differential operator. This operation requires to enforce the momentum conservation.

- Expanding the amplitude in soft momentum we get:

$$v\mathcal{T}_{n+1}(q;k_1,\ldots k_{n-1}) = \left\{ D - n\,d - \sum_{i=1}^{n-1} k_i^\mu \frac{\partial}{\partial k_i^\mu} - \sum_{i=1}^{n-1} \frac{m_i^2}{k_i\cdot q}\left(1 + q^\mu \frac{\partial}{\partial k_i^\mu}\right)\right\}\mathcal{T}_n(k_1,\ldots,k_{n-1})$$

$d = \frac{D-2}{2}$ is the scale dimension of the scalar fields

We can repeat the same calculation with the current associated to the special conformal transformation.

$$j^\mu{}_{(\lambda)} = T^{\mu\nu}(2x_\nu x_\lambda - \eta_{\nu\lambda}x^2) \;\; ; \;\; \partial_\mu\, j^\mu{}_{(\lambda)} = 2\,x_\lambda\, T^\mu{}_\mu = 2\,v\,x_\lambda(-\partial^2)\,\xi(x)$$

whose action on the scalar fields is:

$$\delta_{(\lambda)}\phi(x) = [\mathcal{K}_\lambda, \phi(x)] = i\left((2x_\lambda x_\nu - \eta_{\lambda\nu}x^2)\partial^\nu + 2\,d\,x_\lambda\right)\phi(x)\,.$$

- Ward identity associated to the special conformal transformation:

$$v\, q^\lambda \frac{\partial}{\partial q^\lambda} \mathcal{T}_{n+1}(q; k_1, \ldots, k_{n-1}) = \sum_{i=1}^{n-1} \left\{ \frac{m_i^2}{k_i \cdot q} \left(1 - \frac{1}{2} q^\mu q^\lambda \frac{\partial^2}{\partial k^\mu \partial k^\lambda} \right) \right.$$

$$\left. - q^\lambda \left[k_i^\mu \left(\frac{\partial^2}{\partial k_i^\mu \partial k_i^\lambda} - \frac{1}{2} \eta_{\mu\lambda} \frac{\partial^2}{\partial k_{i\nu} \partial k_i^\nu} \right) + d \frac{\partial}{\partial k_i^\lambda} \right] \right\} \times \mathcal{T}_n(k_1, \ldots, k_{n-1}) + \mathcal{O}(q^2)\,.$$

$$k_n = -q - \sum_{i=1}^{n} k_{n-1}$$

- The most general decomposition of the $n+1$ amplitude is:

$$\mathcal{T}_{n+1} = \left[\sum_{i=1}^{n} \frac{\mathbf{S}_i^{(-1)}(q)}{k_i \cdot q} + \mathbf{S}^{(0)} + q^\mu \mathbf{S}_\mu^{(1)} \right] \times \mathcal{T}_n(k_1, \ldots, k_{n-1}) + \mathcal{O}(q^2)\,,$$

getting:

$$v\,\mathcal{T}_{n+1}(q;k_1,\dots,k_{n-1}) = \Bigg\{ -\sum_{i=1}^{n-1}\frac{m_i^2}{k_i\cdot q}\left(1+q^\mu\frac{\partial}{\partial k_i^\mu}+\frac{1}{2}q^\mu q^\nu\frac{\partial^2}{\partial k_i^\mu\partial k_i^\nu}\right)+D-nd-\sum_{i=1}^{n-1}k_i^\mu\frac{\partial}{\partial k_i^\mu}$$

$$-q^\lambda\sum_{i=1}^{n-1}\left[\frac{1}{2}\left(2\,k_i^\mu\frac{\partial^2}{\partial k_i^\mu\partial k_i^\lambda}-k_{i\,\lambda}\frac{\partial^2}{\partial k_{i\nu}\partial k_i^\nu}\right)+d\,\frac{\partial}{\partial k_i^\lambda}\right]\Bigg\}\mathcal{T}_n(k_1,\dots,k_{n-1})+\mathcal{O}(q^2)$$

The soft-behavior of the NG-dilaton is quite similar but not equal to the one of the gravity dilaton.

- Differences:

$$D-nd \longleftrightarrow 2$$

In field theory a scale transformation rescale all the dimensional quantities. The amplitude T^n has the following mass dimension

$$\mathcal{T}_n\left(m;k_i\right)=m^{D-n\frac{D-2}{2}}g^{n-2}G_n\left(k_i/m\right)\ ,$$

Mass dimension of the amplitude.

In string theory the amplitudes have the general structure

$$\kappa_D = \frac{1}{2^{\frac{D-10}{4}}} \frac{g_s}{\sqrt{2}} (2\pi)^{\frac{D-3}{2}} (\sqrt{\alpha'})^{\frac{D-2}{2}} .$$

$$M_n = \frac{4\pi}{\alpha'} \left(\frac{\kappa_D}{\pi}\right)^{n-2} F_n\left(\sqrt{\alpha'} k_i\right) = C_n m_s^{D-n\frac{D-2}{2}} g_s^{n-2} F_n\left(k_i/m_s\right) ,$$

$$m_s = \frac{1}{\sqrt{\alpha'}}$$

F_n is dimensionless and satisfies:

$$\sum_i k_i \frac{\partial}{\partial k_i} F_n(\sqrt{\alpha'} k_i) = \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} F_n(\sqrt{\alpha'} k_i)$$

$$\kappa_D \left(2 - \sum_{i=1}^{n} k_i^{\mu} \frac{\partial}{\partial k_i^{\mu}}\right) M_n = \kappa_D \left(\frac{D-2}{2} g_s \frac{\partial}{\partial g_s} - \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}}\right) M_n$$

This operator leaves invariant k_D

In string theory a scale transformation change $\frac{1}{\alpha'}$ leaving k_D invariant.

The terms of $O(q)$ are equivalent up to a single piece; in the case of the NG dilaton, there is a term with a single derivative, which is not present in the case of the gravity dilaton.

Conclusions

- String theory is a useful laboratory to study the low energy behaviour of amplitudes in field theory.
- We have computed the behaviour of the scattering amplitudes in bosonic and superstring theory involving massless states; i.e. gravitons , dilatons and Kalb-Ramond antisymmetric tensors (only to sub-leading order).
- In bosonic string sub-subleading behaviour with hard massive scalar matter has also been obtained.
- The string soft operators for the graviton coincide with the field theory ones except for the amplitudes in bosonic string where there are string corrections.
- We have found the sub-subleading soft-behaviour for the string/gravity dilaton. It does not receive α' corrections and it is the same both in bosonic and in superstring theory and for massless and massive hard states.

- We have then considered a spontaneous broken conformal theory and we have shown that the Ward identities of scale and special conformal transformations fix the three leading terms of the soft behaviour of the Nambu-Goldstone boson, that we call field theory dilaton.

- The soft behaviour of the two dilatons are similar but not exactly equal.

- Is the soft behaviour of the gravity dilaton the Ward-identity of some symmetry?

- Double soft-limit of the two dilatons and loop analyses may shed some light on these problems.

- Low energy behaviour in theories with simultaneous breaking of space-time and internal symmetries.
- (M. Bianchi, A. Guerrieri, Y.t. Huang, C. J. Lee, C. Wen, 1605.08697)