Amplitudes and form factors from N=4 super Yang-Mills to QCD

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based on

Brandhuber, Kostacinska, Penante, GT, Young 1606.08682 [hep-th]
& earlier work also with Bill Spence, Congkao Wen and Gang Yang

Aspects of Amplitudes, NORDITA, 24th June 2016

- Scattering amplitudes
  - fully on-shell
- Form factors
  - partially on-shell
- Correlation functions
  - off-shell

progressively less on shell

Why form factors?

- They share the beautiful simplicity of amplitudes
  - calculation with textbook (i.e. Feynman diagrams) methods cumbersome, however final results are often strikingly simple
- Important applications
  - phenomenology
  - dilatation operator
- Work in N=4 SYM, with en eye on QCD....
  - we like models!
  - QCD has non-zero beta function, is not superconformal, (anti)quarks in (anti)-fundamental representation, no scalars

- Example: amplitudes from super Yang-Mills to QCD
  - one-loop amplitudes in SYM are expressed as linear combinations of boxes, triangles and bubbles only (just boxes in N=4 SYM)
  - devise special techniques to compute the corresponding coefficients (quadruple cuts, triple cuts, MHV diagrams…)
  - next, find methods to compute rational terms which are specific to non-supersymmetric amplitudes
- Apply these ideas to form factors
  - conceptual motivation: explore simplicity of off-shell quantities
  - practical motivation: surprising connection to Higgs + multi-gluon amplitudes in QCD (no supersymmetry!)

Plan

- Three form factor calculations, towards QCD

  1. Half-BPS quadratic operators $\mathrm{Tr}\,(\phi_{12})^2$ & connection to Higgs amplitudes
     - Leading term in the effective action for Higgs+multi-gluon processes
  2. Half-BPS operators of the form $\mathrm{Tr}\,(\phi_{12})^3$ (more in general $\mathrm{Tr}\,(\phi_{12})^k$)
  3. Non-BPS operators, operators of the form $\mathrm{Tr}(\,X[Y\,,Z])$ ($SU(2|3)$ sector)
     - subleading terms in $1/m^2{}_{\mathrm{top}}$ in the Higgs + multi-gluon effective action ?

- Long-term goal
  - Understand better the connection to Higgs+multi-gluon amplitudes
  - N=4 super Yang-Mills as a tool to compute Higgs amplitudes in QCD?
  - Dilatation operator, Yangian symmetry

What are form factors ?

- Less on-shell (i.e. partially off-shell) quantities

a gauge-invariant operator in the theory

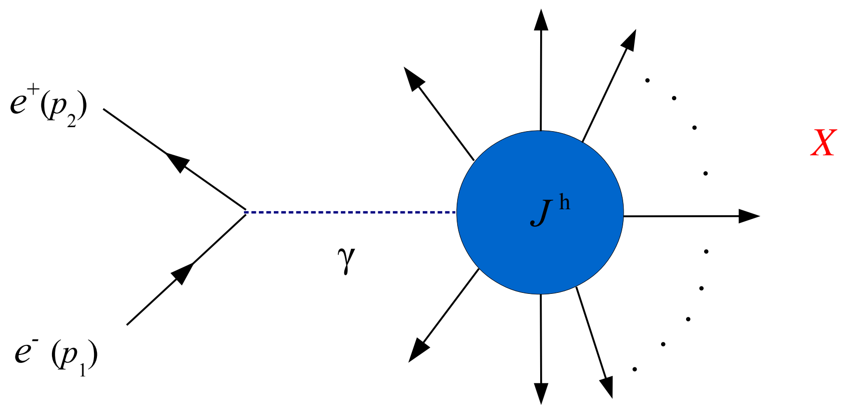
$$F_{\mathcal{O}} := \int d^4x\, e^{-iqx} \langle state|\, \mathcal{O}(x)\, |0\rangle = \delta^{(4)}(q - p_{state})\langle state|\, \mathcal{O}(0)\, |0\rangle$$

  - momentum q carried by the operator is off shell

- Form factors appear in many important contexts:
  - electromagnetic form factor, or g–2
  - deep inelastic scattering ($e^- + p \to e^-$ + hadrons)
  - $e^+ e^- \to$ hadrons (X)

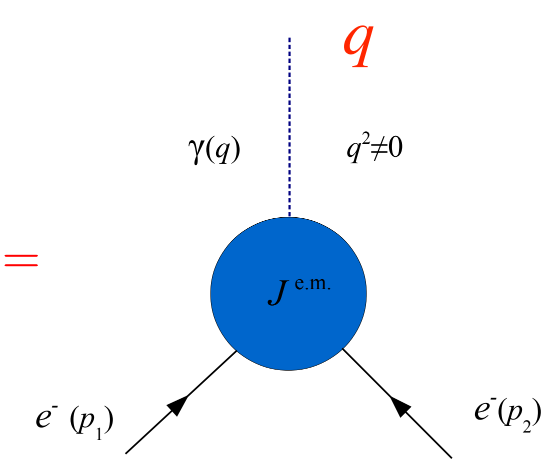
- $e^+ e^- \rightarrow$ hadrons (X), all orders in α_{strong}, first order in $\alpha_{\text{e.m.}}$

hadronic electromagnetic current



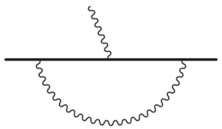
$$= \bar{v}(p_2)\gamma_\mu u(p_1)\frac{\eta^{\mu\nu}}{(p_1+p_2)^2}(-e)\langle X|\, J_\nu^{\text{h}}(0)\,|0\rangle$$

- electron g–2: $\langle e^-(p')|J_\mu^{\text{e.m.}}(0)|e^-(p)\rangle =$



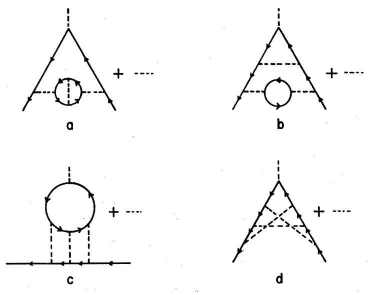
  - $J_\mu^{\text{e.m.}} = \bar{\psi}\gamma_\mu\psi$
  - $p^2 = m_e^2$ on shell, but $q = p - p'$ off shell

Simplicity of the g–2

- one loop: $$\frac{g-2}{2} = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2)$$  (Schwinger 1952)
  - $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \sim \frac{1}{137}$ fine structure constant

- Three loops:

72 diagrams like



$$= (1.181241456...)(\alpha/\pi)^3$$

(Cvitanovic & Kinoshita '74; Laporta & Remiddi '96)

  - numerical values of each diagram oscillate wildly…
  - … but final result is $O(1)$
  - an example of surprising simplicity outside amplitudes!

- A side remark: from form factors to amplitudes

  - at $q \neq 0$: $$F_{\mathcal{O}} := \int d^4x\, e^{-iqx} \langle state| \mathcal{O}(x) |0\rangle$$

  - at $q = 0$: $$F_{\mathcal{O}}|_{q=0} = \int d^4x \langle state| \mathcal{O}(x) |0\rangle$$

  - this is the same as the correction to the amplitude $\langle state \,|\, 0 \,\rangle$ due to the addition of a new coupling to the action

    $$\delta S = g_{\mathcal{O}} \int d^4x\ \mathcal{O}(x)$$

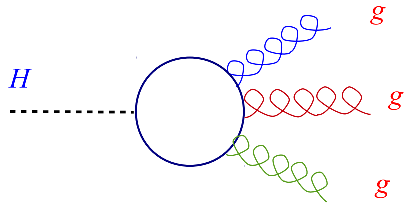
    to the first order in $g_{\mathcal{O}}$

  - a particular soft limit of the form factor…

One (more) reason SUSY is useful even if there is no SUSY…

Higgs amplitudes and form factors

- Higgs production at the LHC
  - dominant process at low $M_{\rm H}$ is gluon fusion
  - coupling to gluons through a fermion loop
    - proportional to the mass of the quark $\Rightarrow$ top quark dominates



- Effective Lagrangian description

  (Wilczek '77; Shifman, Vainshtein, Voloshin, Zakharov '79; Dawson '91; Djouadi, Graudenz, Spira, Zerwas '95)
  - for $M_{\rm H} < 2\, m_{\rm top}$, integrate out the top quark (shrink loop to a point-like effective interaction)
  - leading order: $\boxed{\mathcal{L}_{\rm eff}^{(0)} \sim H\,{\rm Tr}F^2}$, coupling independent of $m_{\rm top}$
  - efficient MHV rules (Dixon, Glover, Khoze; Badger, Glover & Risager; Boels, Schwinn)
  - How do we compute a process with one Higgs + gluons with $\mathcal{L}_{\rm eff}^{(0)}$?

- Higgs amplitudes are form factors of $\mathrm{Tr}\, F^2$!

$$\boxed{F_{\mathrm{Tr}F^2}(1,\ldots,n) \; = \; \int d^4x\, e^{-iqx} \, \langle state | \,\mathrm{Tr}\, F^2(x)\, |0\rangle} \qquad q^2 = M_{\mathrm{H}}^2$$

  - in N=4 super Yang-Mills, the form factor of $\mathrm{Tr}\, F_{\mathrm{SD}}{}^2$ (SD = self-dual) is related to that of $\mathrm{Tr}\, (\phi_{12})^2$ (simpler!)

$$F_{\mathrm{Tr}\phi_{12}^2}(1,\ldots,n) \; = \; \int d^4x \; e^{-iqx} \; \langle state' | \,\mathrm{Tr}\, \phi_{12}^2(x)\, |0\rangle$$

    - $\mathrm{Tr}\; \phi^2{}_{12}$ and $\mathrm{Tr}\, F_{\mathrm{SD}}{}^2$ part of the same half-BPS supermultiplet
    - supersymmetric form factor of the chiral part of the stress tensor multiplet (Brandhuber, Gurdogan, Mooney, GT, Yang)

  - Note: a priori no connection between QCD and N=4 SYM form factors, however comparing them will lead to a surprise…

Higgs → 3 gluons at 2 loops

(Brandhuber, GT, Yang)

- In N=4 SYM: 2 scalars, one gluon (MHV)

$$F_3(1,2,3) \;=\; \langle \phi_{12}(p_1)\, \phi_{12}(p_2) g^+(p_3)|\, \mathrm{Tr}(\phi_{12}\phi_{12})(0)\ |0\rangle$$

- A particularly simple form factor in N=4 super Yang-Mills
  - operator is protected from quantum corrections ("1/2 BPS")
- Loops: $F_3^{(L)} \;=\; F_3^{\mathrm{tree}}\, \mathcal{G}_3^{(L)}(1,2,3)$
  - $\mathcal{G}_3^{(L)}$ helicity-blind function, totally symmetric under legs exchange
  - one loop: IR divergences + sum of finite two-mass easy box
  - two loops: result encoded in finite remainder function

The form factor remainder

- Construct the ABDK/BDS finite remainder, $\mathcal{R}$

$$\boxed{\mathcal{R}_n^{(2)} := \mathcal{G}_n^{(2)} - \frac{1}{2}\big(\mathcal{G}_n^{(1)}(\epsilon)\big)^2 - f^{(2)}(\epsilon)\,\mathcal{G}_n^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)}$$

- introduced for amplitudes by Anastasiou, Bern Dixon & Kosower and Bern, Dixon & Smirnov

- Ingredients:

  - two-loop form factor $\mathcal{G}_n^{(2)}$, one-loop form factor $\mathcal{G}_n^{(1)}$ in dimensional regularisation ($D = 4 - 2\,\epsilon$)

  - $f^{(2)}(\epsilon) = -2\zeta_2 - 2\zeta_3\epsilon - 2\zeta_4\epsilon^2$ contains cusp and collinear anomalous dimensions (integrability!), $C^{(2)}(\epsilon) = 4\,\zeta_4$

- Key properties:

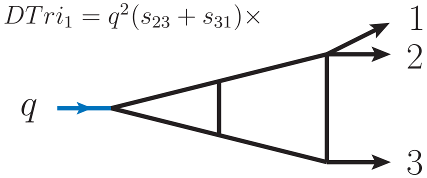
  1. finite: infrared divergences cancel (as in Bloch-Nordsiek)

  2. trivial collinear limits $\mathcal{R}_n^{(2)} \to \mathcal{R}_{n-1}^{(2)}$ (in particular: $\mathcal{R}_3^{(2)} \to 0$)

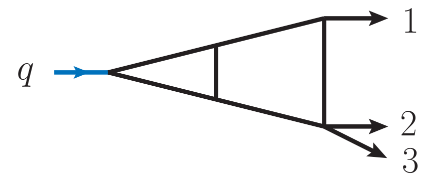
- Result of a unitarity-based two-loop calculation:

$$\boxed{\frac{F_3^{(2)}}{F_3^{\text{tree}}} = \sum_{i=1}^{2}(DTri_i + DBox_i) + TriPent + NBox + NTri + \text{cyclic}}$$

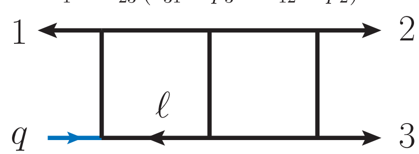
$DTri_1 = q^2(s_{23} + s_{31})\times$



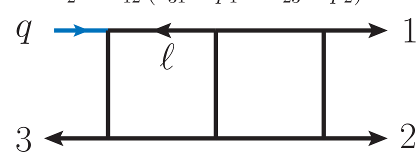
$DTri_2 = q^2(s_{12} + s_{31})\times$



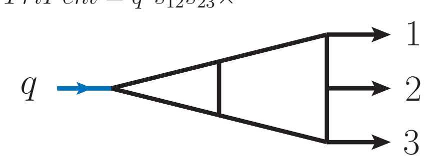
$DBox_1 = s_{23}\,(s_{31}\ell\cdot p_3 - s_{12}\ell\cdot p_2)\times$



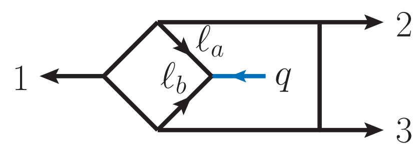
$DBox_2 = s_{12}\,(s_{31}\ell\cdot p_1 - s_{23}\ell\cdot p_2)\times$



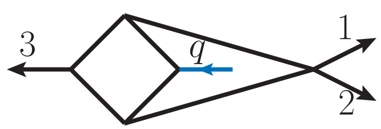
$TriPent = q^2 s_{12} s_{23}\times$



$NBox = s_{23}\left(\frac{1}{2}s_{12}s_{31} - s_{12}\ell_a\cdot p_2 - s_{31}\ell_b\cdot p_3\right)\times$



$NTri = \frac{1}{2}q^2(s_{23} + s_{31})\times$



- result expressed as rational coefficients × two-loop planar and non-planar integrals

- Some features of the result:

  - sum of transcendental functions, typically quite complicated: Goncharov's polylogarythms

  - defined recursively

$$G(a_1; z) \; := \; \int_0^z \frac{dt_1}{t_1 - a_1} \,, \qquad G(a_1, \vec{a}; z) \; := \; \int_0^z \frac{dt_1}{t_1 - a_1} G(\vec{a}; t_1)$$

  - compare to something simpler: classical polylogarithms

$$\mathrm{Li}_1(z) \; = \; -\log(1 - z) \,, \qquad \mathrm{Li}_n(z) \; = \; \int_0^z \frac{dt}{t} \, \mathrm{Li}_{n-1}(t)$$

  - key finding: our result is a sum of functions of homogeneous degree of "transcendentality". All terms have transcendentality 4 (this will change later…)

Strategy

- Compute the symbol of the finite remainder
  - either by taking the symbol of the known (but complicated answer)…
  - or by computing it directly using symmetry properties & analyticity
    - finite, trivial/understood collinear limits
    - analiticity
    - need to know the possible letters
- “lift” it to a function
  - result might be remarkably simple, and in particular much simpler than the original expression!
  - fix “beyond-the-symbol” terms

- The unique symbol satisfying these requirements:

☞

$$
\boxed{
\begin{aligned}
\mathcal{S}^{(2)} \;=\;& -2u \otimes (1-u) \otimes (1-u) \otimes \frac{1-u}{u} + u \otimes (1-u) \otimes u \otimes \frac{1-u}{u} \\
& -u \otimes (1-u) \otimes v \otimes \frac{1-v}{v} - u \otimes (1-u) \otimes w \otimes \frac{1-w}{w} \\
& -u \otimes v \otimes (1-u) \otimes \frac{1-v}{v} - u \otimes v \otimes (1-v) \otimes \frac{1-u}{u} \\
& +u \otimes v \otimes w \otimes \frac{1-u}{u} + u \otimes v \otimes w \otimes \frac{1-v}{v} \\
& +u \otimes v \otimes w \otimes \frac{1-w}{w} - u \otimes w \otimes (1-u) \otimes \frac{1-w}{w} \\
& +u \otimes w \otimes v \otimes \frac{1-u}{u} + u \otimes w \otimes v \otimes \frac{1-v}{v} \\
& +u \otimes w \otimes v \otimes \frac{1-w}{w} - u \otimes w \otimes (1-w) \otimes \frac{1-u}{u} \\
& + \text{cyclic permutations}\,.
\end{aligned}
}
$$

- four-fold tensor product ($2L$-fold at L loops, transcendentality $2L$)
- kinematic variables: $u_1 = u = s_{12}\,/\,q^2\,,\;\; u_2 = v = s_{23}\,/\,q^2\,,\;\; u_3 = w = s_{31}\,/\,q^2$
  where $s_{ij} \;:=\; (p_i + p_j)^2$ and $u_1 + u_2 + u_3 = 1$
- Note: coefficients ± 1, ± 2 (well... -2)

- How to "integrate" the symbol:
  - $\mathcal{S}^{(2)}$ satisfies a particular relation of Goncharov:

$$\mathcal{S}^{(2)}_{abcd} - \mathcal{S}^{(2)}_{bacd} - \mathcal{S}^{(2)}_{abdc} + \mathcal{S}^{(2)}_{badc} - (a \leftrightarrow c\,,\, b \leftrightarrow d) \;=\; 0$$

  - $\Rightarrow$ can re-express as a linear combination of classical polylogarithms only

$$\log x_1 \log x_2 \log x_3 \log x_4\ ,\ \mathrm{Li}_2(x_1) \log x_2 \log x_3\ ,\ \mathrm{Li}_2(x_1)\mathrm{Li}_2(x_2)\ ,\ \mathrm{Li}_3(x_1) \log x_2 \text{ and } \mathrm{Li}_4(x_i)$$

  - we find the following arguments:

$$\left(u, v, w, 1-u, 1-v, 1-w, 1-\frac{1}{u}, 1-\frac{1}{v}, 1-\frac{1}{w}, -\frac{uv}{w}, -\frac{vw}{u}, -\frac{wu}{v}\right)$$

- Final answer is very compact

- # Final answer: (Brandhuber, GT, Yang)

$$\boxed{\begin{aligned}\mathcal{R}_3^{(2)} \;=\; & -2\left[\mathrm{J}_4\left(-\frac{uv}{w}\right)+\mathrm{J}_4\left(-\frac{vw}{u}\right)+\mathrm{J}_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]\\ & -2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i^{-1})\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4\end{aligned}}$$

- $u_1 = u = s_{12} / q^2$, $u_2 = v = s_{23} / q^2$, $u_3 = w = s_{31} / q^2$ kinematic invariants
- $\mathrm{J}_4(z) := \mathrm{Li}_4(z) - \log(-z)\mathrm{Li}_3(z) + \frac{\log^2(-z)}{2!}\mathrm{Li}_2(z) - \frac{\log^3(-z)}{3!}\mathrm{Li}_1(z) - \frac{\log^4(-z)}{48}$.
- Block-Wigner-Ramakrishnan(-Zagier) polylogarithmic function
- Result is free of Goncharov polylogarithms

Next: QCD

Higgs amplitudes in QCD

- **Higgs + 3 partons** (Koukoutsakis 2003; Gehrmann, Glover, Jaquier & Koukoutsakis 2011)
  - $H\, g^+\, g^-\, g^-$ MHV
  - $H\, g^+\, g^+\, g^+$ maximally non-MHV
  - $H\, q\ \ \bar{q}\ \ g$ fundamental quarks

$$F^{\text{tree}}(H, g_1^-, g_2^-, g_3^+) = \frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle\,\langle 3\,1\rangle}$$

$$F^{\text{tree}}(H, g_1^+, g_2^+, g_3^+) = \frac{q^4}{[1\,2]\,[2\,3]\,[3\,1]}$$

$$q^2 = M_H^2$$

- **In N=4 SYM:**
  - $(H\, g^+\, g^-\, g^-)$ and $(H\, g^+\, g^+\, g^+)$ both derived from super form factor
  - from supersymmetric Ward identities: (Brandhuber, GT, Yang)

$$\frac{F^{(L)}(g_1^-, g_2^-, g_3^+)}{F^{\text{tree}}(g_1^-, g_2^-, g_3^+)} = \frac{F^{(L)}(g_1^+, g_2^+, g_3^+)}{F^{\text{tree}}(g_1^+, g_2^+, g_3^+)} = \mathcal{G}^{(L)}(u, v, w) \quad \leftarrow \text{what we computed}$$

- QCD answer from Gehrmann, Glover, Jaquier & Koukoutsakis
  - expressed in terms of several pages of Goncharov polylogarithms
  - transcendentality $4, 3, 2, 1$ and rational
  - entirely expected because of expansion as $\sum$ (coefficient x integral) !
    - each integral is separately quite complicated
- Next, compare N=4 form factors to Higgs amplitudes:
  - take maximally transcendental piece of $(H\, g^{+}\, g^{-}\, g^{-})$ and $(H\, g^{+}\, g^{+}\, g^{+})$

- We find a surprising connection...

$$\boxed{\mathcal{R}^{(2)}_{H\,g^- g^- g^+}\Big|_{\mathrm{MAX\,TRANS}} = \mathcal{R}^{(2)}_{H\,g^+ g^+ g^+}\Big|_{\mathrm{MAX\,TRANS}} = \mathcal{R}^{(2)}_{\mathcal{N}=4\,\mathrm{SYM}}}$$

!

  - N=4 result is a particular part of the QCD result - in fact it is the "most complicated part"
  - all Goncharov polylogarithms in QCD results can be eliminated in favour of classical polylogarithms

- Nothing similar seems to hold for the form factor $(H, q, \bar{q}, g)$ (see also Duhr '12)

  - maximally transcendental part does not satisfy Goncharov et al criterion

Comments

- Typical presentation of the result of a calculation:
    - result = $\sum$ (coefficient x integral)
    - integrals are separately complicated, but final result is strikingly simple
    - there must be better way to present the result than Σ(coefficient x integral)

- Supersymmetry is a very useful organisational principle!
    - even if there is no supersymmetry…

What next?

- Obvious (but nontrivial) extensions:
  - different operators, more legs (Penante, Spence, GT, Wen; Brandhuber, Penante, GT, Wen)
  - further potential connections to phenomenology, e.g. in Higgs + 4 gluons

- Corrections due to the finiteness of the top mass
  - leading order term (infinite top mass limit) is the dimension-5 coupling studied earlier

$$\mathcal{L}_{\rm eff}^{(0)} \sim H \, {\rm Tr} F^2$$

  - next corrections from four dimension-7 operators, suppressed by powers of $1/m^2{}_{\rm top}$ (Buchmüller & Wyler; Neill; Harlander & Neumann)

- Look at this question with the N=4 SYM microscope…
  - identify couplings which are present also in N=4 SYM. Just two:

$$\mathcal{L}_{\rm eff}^{(1)} \sim H\,{\rm Tr} F^3 \qquad\qquad \mathcal{L}_{\rm eff}^{(2)} \sim H\,{\rm Tr}(D_\mu F_{\rho\sigma})(D^\mu F^{\rho\sigma})$$

  - compute in N=4 SYM
  - use Ward identities to connect to operators in the same multiplet but containing less derivatives / more scalars
  - compare to QCD

- Key questions & conjectures:
  - does the "maximal-transcendental connection" still holds?
  - any other interesting connection?

- Perform simpler “toy” calculations
  - Form factors of operators containing three fields in N=4 SYM
  - simpler than $\mathrm{Tr}\, F^3$. Operators with scalars!
  - Naturally leads to the $SU(2|3)$ sector studied by Beisert
  - Several possibilities, two broad classes: unprotected and protected operators (with and without UV divergences)
  - interesting, unexpected connections between the two classes!

The two classes of operators:

- Protected
  - $\mathrm{Tr}\,(\phi_{12})^3$ half-BPS, form factors free of UV divergences
  - Generalisation: $\mathrm{Tr}\,(\phi_{12})^k$, also half-BPS $\forall\, k$
- Non-protected
  - Length 3: $\mathcal{O}_{\mathrm{B}} := \mathrm{Tr}\,(X\,[Y\,,Z])$ where $X = \phi_{12}\,,\ Y = \phi_{23},\ \ Z = \phi_{31}$
    - same one-loop anomalous dimension as $\mathrm{Tr}\,F^3$
  - Carries along a few dimension-three friends via operator mixing…
    - $\mathcal{O}_{\mathrm{BPS}} := \mathrm{Tr}\,(X\,\{Y\,,Z\})$, which is BPS (symmetric traceless)
    - $\mathcal{O}_{\mathrm{F}} := (1/2)\,\mathrm{Tr}\,(\psi\psi)$, which mixes with $\mathcal{O}_{\mathrm{B}}$ (and $\psi := \psi_{123}$)
  - This is the $SU(2|3)$ sector! The $SU(2|3)$ "dynamic" spin chain (Beisert '03)
    - key features: 1. closed sector, 2. length changing $(\psi\psi \leftrightarrow XYZ)$

- **Two distinguished combinations:**
  (Bianchi, Kovacs, Rossi, Stanev; Eden; …)

  1. an additional BPS operator $\mathcal{O}'_{\mathrm{BPS}} = (1/2)\,\mathrm{Tr}\,(\psi\psi) + g\,\mathrm{Tr}\,(X\,[Y\,,Z])$
     - can also be obtained by acting with 2 susy transformations on $\mathrm{Tr}\,(\phi_{12})^2$
  2. A descendant of the Konishi operator

$$\mathcal{O}_K \;=\; \mathrm{Tr}(X[Y,Z]) - \frac{gN}{8\pi^2}\mathrm{Tr}(\psi\psi)$$

- **Four interesting calculations to carry out:**

$\langle\, X\,Y\,Z \,|\; \mathrm{Tr}\,(X\,[Y\,,Z]) \;|\, 0 \,\rangle$	minimal	harder
$\langle\, X\,Y\,Z \,|\; \mathrm{Tr}\,(\psi\psi) \;|\, 0 \,\rangle$	non-minimal	v. easy
$\langle\, \psi\psi \,|\; \mathrm{Tr}\,(X\,[Y\,,Z]) \;|\, 0 \,\rangle$	sub-minimal	easy
$\langle\, \psi\psi \,|\; \mathrm{Tr}\,(\psi\psi) \;|\, 0 \,\rangle$	minimal (“Sudakov”)	♻

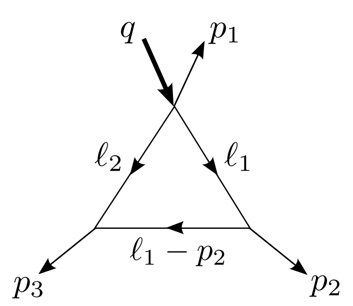
Protected operators

3-point form factor of $\mathrm{Tr}\phi^3$ at 2 loops

(Brandhuber, Penante, GT, Wen)

$$F_3(1,2,3) \;:=\; \langle \phi_{12}(p_1),\, \phi_{12}(p_2),\, \phi_{12}(p_3)\,|\,\mathrm{Tr}\,[(\phi_{12})^3](0)\;|0\rangle$$

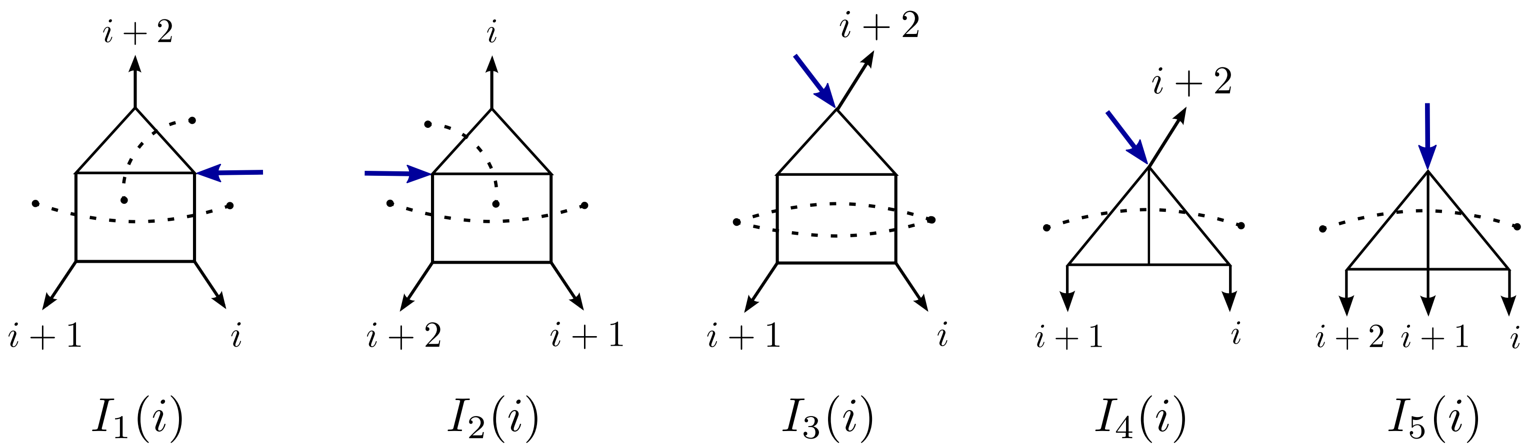
- "minimal form factor": as many particles as fields
- Tree: $F_3^{(0)}(1,2,3) \,=\, 1$
- One loop: sum of three "one-mass" triangles



+ 2 cyclic perms

- Result at two loops:

$$F^{(2)}_{\mathcal{T}_3,3} = \sum_{i=1}^{3} \Big[I_1(i) + I_2(i) + I_3(i) + I_4(i) - I_5(i) \Big]$$



- Result expressed in terms of two-loop planar integrals
- No sub-triangle and -bubble topologies on the amplitude side (no triangle theorem for N=4 SYM amplitudes)
- All integrals known from work of Gehrmann & Remiddi except 1 (and 2), decompose remaining ones using FIRE/LiteRed (Smirnov/Lee)
- Compute the symbol and lift it to a function

- The symbol of $\mathcal{R}_3$ is very simple!

☞

$$\boxed{\begin{aligned} \mathcal{S}_3^{(2)}(u,v,w) \;=\; & -\frac{3}{2}\,u\otimes(1-u)\otimes\frac{v}{w}\otimes\frac{v}{w}+\frac{1}{2}\,u\otimes u\otimes\frac{v}{w}\otimes\frac{v}{w} \\ & +u\otimes v\otimes\left(\frac{u}{w}\otimes\frac{v}{w}+\frac{v}{w}\otimes\frac{u}{w}\right)+\text{perms}\,(u,v,w) \end{aligned}}$$

- transcendentality four function ⇨ rank-four tensor
- entries: $(u, v, w, 1\text{-}u, 1\text{-}v, 1\text{-}w)$ $\quad u := \frac{s_{12}}{q^2}\,,\ v := \frac{s_{23}}{q^2}\,,\ w := \frac{s_{31}}{q^2}\,,$
- first entry: (u, v, w) for correct branch cuts (Gaiotto, Maldacena, Sever, Vieira)
  - $\mathcal{S}[\mathcal{R}^{(2)}] = \sum_{i,j} P_{i,j}^2 \otimes \mathcal{S}[\mathrm{disc}_{i,j}\mathcal{R}^{(2)}]$ with $P_{i,j} := p_i + \ldots + p_j$
- unusual second entry condition
- last entry condition: ratios of simple ratios only
- satisfies Goncharov, Spradlin, Vergu & Volovich's criterion, thus can be re-expressed in terms of classical polylogarithms only

- Table of symmetry properties from Goncharov, Spradlin, Vergu & Volovich:

Function	$A \otimes A$	$S \otimes A$	$A \otimes S$	$S \otimes S$
$\mathrm{Li}_4(x)$	$\times$	$\times$	$\checkmark$	$\checkmark$
$\mathrm{Li}_3(x)\ \log(y)$	$\times$	$\times$	$\checkmark$	$\checkmark$
$\mathrm{Li}_2(x)\,\mathrm{Li}_2(y)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$\mathrm{Li}_2(x)\ \log(y)\ \log(z)$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$\log(x)\ \log(y)\ \log(z)\ \log(w)$	$\times$	$\times$	$\times$	$\checkmark$

- Two more stringent properties of our symbol: AA[$\mathcal{S}^{(2)}$] = SA[$\mathcal{S}^{(2)}$] = 0

- Need: $\mathrm{Li}_4\,(x)$, $\mathrm{Li}_3\,(x)\log(x)$, $\log(x)\log(y)\log(z)\log(w)$ but no Li_2 !

- Entries: $\left\{u, v, w, 1-u, 1-v, 1-w, -\frac{u}{v}, -\frac{u}{w}, -\frac{v}{u}, -\frac{v}{w}, -\frac{w}{u}, -\frac{w}{v}, -\frac{uv}{w}, -\frac{uw}{v}, -\frac{vw}{u}\right\}$

- Final answer fits on a couple of lines…

- Final answer (including beyond the symbol terms):

$$
\begin{aligned}
\mathcal{R}^{(2)}_{3,3} \;:=\; & -\frac{3}{2}\,\mathrm{Li}_4(u) + \frac{3}{4}\,\mathrm{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{2}\log(w)\mathrm{Li}_3\left(-\frac{u}{v}\right) + \frac{1}{16}\log^2(u)\log^2(v) \\
& + \frac{\log^2(u)}{32}\Big[\log^2(u) - 4\log(v)\log(w)\Big] + \frac{\zeta_2}{8}\log(u)[5\log(u) - 2\log(v)] \\
& + \frac{\zeta_3}{2}\log(u) + \frac{7}{16}\zeta_4 + \text{permutations}\,(u,v,w)
\end{aligned}
$$

  - beyond the symbol terms: fixed using numerics (with GiNaC)
  - no Goncharov polylogarithms, no Li_2's

Non-BPS operators

Form factors in the SU(2|3) sector

(Brandhuber, Kostacinska, Penante, GT, Young)

- Strategy:
  - compute the four form-factors in terms of two-loop integrals, using unitarity (two- and three-particle cuts)
  - compute the remainder functions
    - remainders are free of IR divergences; UV divergences still present
  - simplify the remainders using symbols, lift back to (simpler) functions
  - renormalise the operators, and resolve the mixing
    - eigenvalues of the mixing matrix: anomalous dimensions
    - eigenvectors: operators that diagonalise the dilatation operator

- The most interesting/complicated
  - minimal form factor $\langle\, X\, Y\, Z \,|\, \mathrm{Tr}\,(X\,[Y\,,Z])\,(0)\,|\,0\,\rangle$
- Key observation (slightly embarassing…)

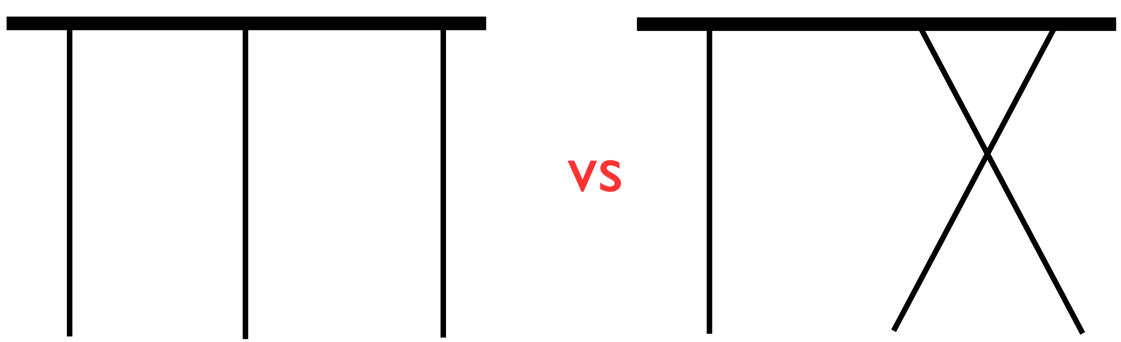
$$\boxed{\mathrm{Tr}(X\,[Y\,,Z]) = \mathrm{Tr}(X\,\{Y\,,Z\}) - 2\,\mathrm{Tr}(X\,Z\,Y) := \mathcal{O}_{\mathrm{BPS}} + \mathcal{O}_{\mathrm{offset}}}$$

half-BPS

"one shuffling",
hence simpler

operator $X \quad Y \quad Z$ $\quad\quad$ $X \quad Z \quad Y$

compare



vs

state $X \quad Y \quad Z$ $\quad\quad$ $X \quad Y \quad Z$

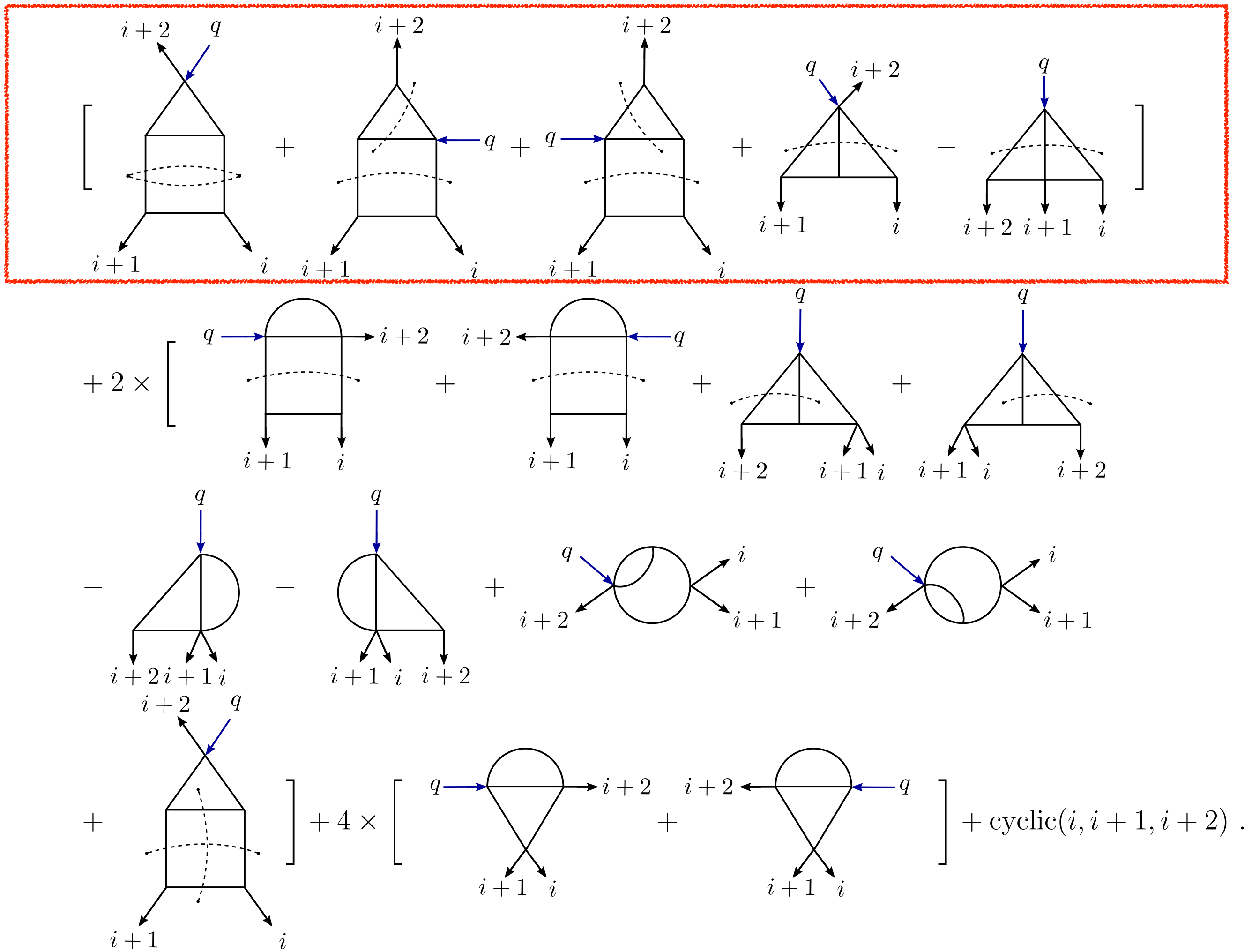
- What does "simple" mean:
  - $\langle\, X\,Y\,Z \mid \mathrm{Tr}\,(X\,\{Y\,,Z\}) \;\mid 0\,\rangle$ (half-BPS) is maximally transcendental
  - equal to $\langle\, X\,X\,X \mid \mathrm{Tr}\,(X^3) \mid 0\,\rangle$ (discussed earlier)
  - $\langle\, X\,Y\,Z \mid \mathrm{Tr}\,(X\,Z\,Y) \mid 0\,\rangle$ has NO maximally transcendental piece
    - transcendentality equal to $3, 2, 1$ and 0 (rational terms) only

- A cute observation in the $SU(2)$ spin chain

(Loebbert, Nandan, Sieg, Wilhelm, Yang)

  - highest transcendentality of a "term" is $4 - s$ where $s = \#$ of shufflings
  - same happens here for $\langle\, X\,Y\,Z \mid \mathrm{Tr}\,(X\,Z\,Y) \;\mid 0\,\rangle$
  - one shuffling, hence transcendentality $3, 2, 1$ and rational

- # Result for the remainder in terms of integral functions:



- first line corresponds to the half-BPS form factor
- dotted lines correspond to numerators in the integral functions
- presence of sub-bubbles points at UV divergences

- Remainder can be decomposed as

$$\boxed{\mathcal{R}^{(2)}{}_{X[Y,Z]} = \mathcal{R}^{(2)}{}_{\text{BPS}} + \mathcal{R}^{(2)}{}_{\text{non-BPS}}}$$ where

$$\mathcal{R}^{(2)}_{\text{BPS}} = F^{(2)}_{\mathcal{O}_{\text{BPS}}}(\epsilon) - \frac{1}{2}\left(F^{(1)}_{\mathcal{O}_{\text{BPS}}}(\epsilon)\right)^2 - f^{(2)}(\epsilon)\; F^{(1)}_{\mathcal{O}_{\text{BPS}}}(2\epsilon) - C^{(2)}\,,$$

$$\mathcal{R}^{(2)}_{\text{non-BPS}} = F^{(2)}_{\mathcal{O}_{\text{offset}}}(\epsilon) - F^{(1)}_{\mathcal{O}_{\text{offset}}}\left(\frac{1}{2}F^{(1)}_{\mathcal{O}_{\text{offset}}} + F^{(1)}_{\text{BPS}}\right)(\epsilon) - f^{(2)}(\epsilon)\; F^{(1)}_{\mathcal{O}_{\text{offset}}}(2\epsilon)$$

- recall that $\mathcal{O}_{\text{BPS}} := \ \mathrm{Tr}\,(X\{Y\,,Z\})\,,\qquad \mathcal{O}_{\text{offset}} := \ -2\,\mathrm{Tr}\,(X\,Z\,Y)$
- BDS remainder free of IR but not UV divergences
- $\mathcal{R}^{(2)}{}_{\text{BPS}}$ computed earlier, transcendentality-4 function

$$\begin{aligned}\mathcal{R}^{(2)}_{\text{BPS}} &= \frac{3}{2}\,\mathrm{Li}_4(u) - \frac{3}{4}\,\mathrm{Li}_4\left(-\frac{uv}{w}\right) + \frac{3}{2}\log(w)\,\mathrm{Li}_3\left(-\frac{u}{v}\right) - \frac{1}{16}\log^2(u)\log^2(v)\\ &- \frac{\log^2(u)}{32}\left[\log^2(u) - 4\log(v)\log(w)\right] - \frac{\zeta_2}{8}\log(u)[5\log(u) - 2\log(v)]\\ &- \frac{\zeta_3}{2}\log(u) - \frac{7}{16}\zeta_4 + \text{perms}\,(u,v,w)\end{aligned}$$

- Focus now on the new part, i.e. $\mathcal{R}^{(2)}_{\text{non-BPS}}$

$$\mathcal{R}^{(2)}_{\text{non-BPS}} = \frac{c}{\epsilon} + \sum_{i=0}^{3} \mathcal{R}^{(2)}_{\text{non-BPS};3-i}$$

  - $c = 18 - \pi^2$ this is the UV pole, π^2 "spurious" $-f^{(2)}(\epsilon)\ F^{(1)}_{\mathcal{O}_{\text{offset}}}(2\epsilon)$
  - "18" will enter the mixing matrix

$$\mathcal{R}^{(2)}_{\text{non-BPS};3} = 2\Big[\text{Li}_3(u) + \text{Li}_3(1-u)\Big] - \frac{1}{2}\log^2(u)\log\frac{vw}{(1-u)^2} + \frac{2}{3}\log(u)\log(v)\log(w) + \frac{2}{3}\zeta_3 + \text{perms}\,(u,v,w)$$

$$\mathcal{R}^{(2)}_{\text{non-BPS};2} = -12\Big[\text{Li}_2(1-u) + \text{Li}_2(1-v) + \text{Li}_2(1-w)\Big] - 2\log^2(uvw) + 36\,\zeta_2$$

$$\mathcal{R}^{(2)}_{\text{non-BPS};1} = -12\ \log(uvw)\ ,$$

$$\mathcal{R}^{(2)}_{\text{non-BPS};0} = 126$$

  - transcendentality < 4, hence only classical polylogarithms

- Summary so far:
  - leading transcendental part of $\langle\, X\, Y\, Z \mid \mathrm{Tr}\,(X\ [Y, Z]) \mid 0\, \rangle$ same as for the half-BPS case $\langle\, X\, X\, X \mid \mathrm{Tr}\,(X^3) \mid 0\, \rangle$

- Goal for the future: compare to $\langle\, g\; g\; g \mid\ \mathrm{Tr}\, F^3\ |0\rangle$
  - conjecture: maximally transcendental part computed by the form factor of the half-BPS operator $\mathrm{Tr}\,(X^3)$? This would parallel the situation for $\mathrm{Tr}\, F^2$ in QCD vs $\mathrm{Tr}\,(\phi_{12})^2$ in N=4 SYM…
  - if conjecture is true, then half-BPS operators in N=4 SYM have a prominent role in QCD!
  - Understand multiplet structure for $\mathrm{Tr}\, F^3$
  - Same one-loop anomalous dimension of $\mathrm{Tr}\,(X[Y\,, Z])$

An $SU(2) \Leftrightarrow SU(2|3)$ sector connection

or are we missing a trivial Ward identity?

- An intriguing connection with the remainder densities in the $SU(2)$ spin chain (Loebbert, Nandan, Sieg, Wilhelm, Yang)
- Contrast the two sectors:
  - $SU(2)$: two bosons, X and Y (scalars). Closed, no length change
  - $SU(2|3)$: $\phi_{12}=X$, $\phi_{23}=Y$, $\phi_{31}=Z$ and $\psi_{123;\alpha}$, $\alpha=1, 2$. Closed, length change
- LNSWY computed the two-loop spin-chain Hamiltonian
  - "open", equivalent to removing the trace (form factor of a product of fields, without the trace)
  - involves three sites at two loops
  - finite parts expressed in terms of remainder densities

- Interaction range 2 and 3 processes:
  - Range 2: 1. $XX \rightarrow XX$, 2. $XY \rightarrow XY$, 3. $XY \rightarrow YX$
  - Range 3: 1. $XXX \rightarrow XXX$, 2. $XXY \rightarrow XXY$, 3. $XYX \rightarrow XYX$,
    4. $XXY \rightarrow XYX$, 5. $XYX \rightarrow XXY$, 6. $XXY \rightarrow YXX$

- Focus on range 3
  - there are only 3 independent processes/remainder densities

$$\left(R_i^{(2)}\right)^{XXX}_{XXX}, \quad \left(R_i^{(2)}\right)^{XYX}_{XXY}, \quad \left(R_i^{(2)}\right)^{YXX}_{XXY}$$

  - i denotes the site
  - each remainder depends on $u_i = \dfrac{s_{ii+1}}{s_{ii+1i+2}}\,, v_i = \dfrac{s_{i+1i+2}}{s_{ii+1i+2}}\,, w_i = \dfrac{s_{ii+2}}{s_{ii+1i+2}}$
  - no particular symmetry in the u_i, v_i and w_i

- We find the following relations:

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};3} = -\sum_{S_3} \left(R_i^{(2)}\right)^{XYX}_{XXY}\Big|_3 \ + \ 6\,\zeta_3 \ ,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};2} = -\sum_{S_3} \left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_2 \ + \ 5\pi^2 \ ,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};1} = -\sum_{S_3} \left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_1 \ ,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};0} = -\sum_{S_3} \left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_0$$

  - $(R_i)|_m$ indicates the transcendentality-m part
  - S_3 denotes sum over all six permutations of $(u\ v,\ w)$

- Universality of form factors across different sectors?
  - or is there a trivial explanation for this result?

$SU(2|3)$ dilatation operator

- Resolve mixing
$$\begin{pmatrix} \mathcal{O}_F^{\rm ren} \\ \mathcal{O}_B^{\rm ren} \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_F{}^F & \mathcal{Z}_F{}^B \\ \mathcal{Z}_B{}^F & \mathcal{Z}_B{}^B \end{pmatrix} \begin{pmatrix} \mathcal{O}_F \\ \mathcal{O}_B \end{pmatrix}$$
  - $\mathcal{O}_{\rm B} := {\rm Tr}\,(X\,[Y\,,Z])$ and $\mathcal{O}_{\rm F} := (1/2)\,{\rm Tr}\,(\psi\psi)$

- Extract mixing matrix from requesting finiteness of the renormalised form factors
  - $\langle\, X\,Y\,Z\,|\;\; {\rm Tr}\,(X\,[Y\,,Z]) \;\;|\,0\,\rangle$
  - $\langle\, X\,Y\,Z\,|\;\; {\rm Tr}\,(\psi\psi) \;\;|\,0\,\rangle$ (IR finite, starts at one loop)
  - $\langle\, \psi\psi\,|\;\; {\rm Tr}\,(X\,[Y\,,Z]) \;\;|\,0\,\rangle$ (IR finite, starts at two loops)
  - $\langle\, \psi\psi\,|\;\; {\rm Tr}\,(\psi\psi) \;\;|\,0\,\rangle$

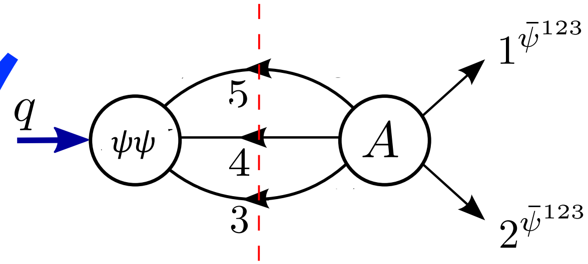
- Dilatation operator $\delta\mathcal{D} = -\mu_R \dfrac{\partial}{\partial\mu_R} \log \mathcal{Z}$

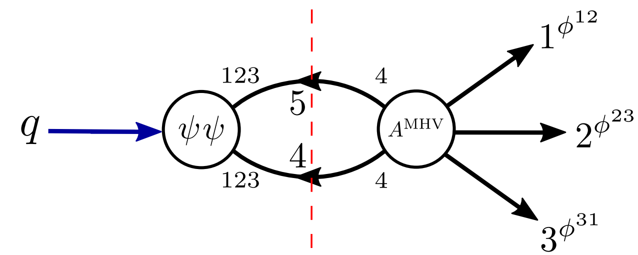
Schematically:

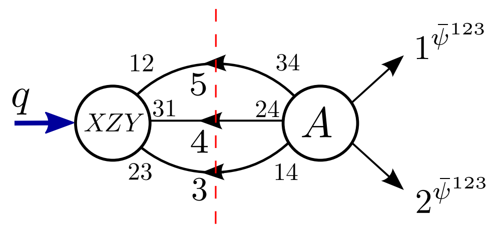
$g \times g^3 \times N^2 \sim a^2$

$g^3 \times N \sim a\, g$

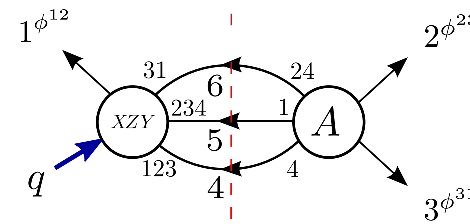
$\log \mathcal{Z} \sim$







1 loop +



$g^3 \times N^2 \sim a^2/g$

$g^2 \times N \sim a$

$g \times g^3 \times N^2 \sim a^2$

$(a \sim g^2 N)$

$$\sim \frac{1}{\epsilon}\begin{pmatrix} \mathcal{O}(a^2) & \mathcal{O}(a\,g) \\ \mathcal{O}(a^2/g) & \mathcal{O}(a)+\mathcal{O}(a^2) \end{pmatrix}$$

- Result for $\log(\mathcal{Z})$:

$$\log(\mathcal{Z}) = \begin{pmatrix} a^2(\mu_R)\,\frac{6}{\epsilon} & -a(\mu_R)\cdot g\,\frac{6}{\epsilon} \\ -\frac{a^2(\mu_R)}{g}\cdot\frac{6}{\epsilon} & a(\mu_R)\cdot\frac{6}{\epsilon} - a^2(\mu_R)\cdot\frac{18}{\epsilon} \end{pmatrix}$$

  - running 't Hooft coupling: $a(\mu_R) := \frac{g^2 N e^{-\epsilon\gamma}}{(4\pi)^{2-\epsilon}}\left(\frac{\mu_R}{\mu}\right)^{-2\epsilon}$

- two-loop dilatation operator:

$$\delta\mathcal{D} = \lim_{\epsilon\to 0}\left[-\mu_R\frac{\partial}{\partial\mu_R}\log(\mathcal{Z})\right] = 12\times\begin{pmatrix} 2a^2 & -a\,g \\ -2\,\frac{a^2}{g} & a - 6\,a^2 \end{pmatrix}$$

  - 't Hooft coupling $a := \frac{g^2 N}{(4\pi)^2}$

- Next: eigenvalues and eigenvector

- Eigenvalues:
  - $\gamma_{\mathrm{BPS}'} = 0, \qquad \gamma_{\mathcal{K}'} = 12\,a - 48\,a^2 + \ldots.$
  - one further BPS combination, one descendent of the Konishi. Results in agreement with Beisert '03

- Eigenvectors:

$$\begin{cases} \mathcal{O}_{\mathrm{BPS}'} = \mathcal{O}_F + g\,\mathcal{O}_B \\ \mathcal{O}_{\mathcal{K}'} = \mathcal{O}_B - \dfrac{gN}{8\pi^2}\,\mathcal{O}_F \end{cases}$$

  - recall that $\mathcal{O}_\mathrm{B} := \mathrm{Tr}\,(X\,[Y\,,Z])$ and $\mathcal{O}_\mathrm{F} := (1/2)\,\mathrm{Tr}\,(\psi\psi)$
  - $X = \phi_{12}\,,\ Y = \phi_{23},\quad Z = \phi_{31}\quad \psi := \psi_{123}$
  - agrees with Bianchi et al, Eden
  - BPS combination can also be obtained by explicitly acting with supersymmetry generators on $\mathrm{Tr}\,(\phi_{12}\,\phi_{12})$ (Intriligator & Skiba)

- Other research direction: derive the dilatation operator from amplitudes techniques (no time to discuss this!)
  - complete two-loop dilatation operator still not known
  - amplitudes symmetries (Yangian) could play an important role
  - one-loop approach in Brandhuber, Heslop, GT, Young '15

Summary

- Form factors in N=4 SYM appear in several interesting contexts
  - connection to Higgs amplitudes in QCD
  - possibly true also for higher-dimensional operators describing the corrections to the infinite top-mass approximation
  - can be used to compute the dilatation operator of the theory
- Can the connection between Higgs amplitudes in QCD and form factors in N=4 SYM be made (more) systematic?
- Universality of form factors across different sectors?